1 Weil Sheaves: Notes by Dan Dore

1.1 Definition and Classification of Weil Sheaves

In the course of Deligne's proof of the purity theorem, he makes certain monodromy constructions which do not a priori yield legitimate ℓ-adic sheaves. In order to make the proof work, it therefore becomes necessary to slightly enlarge the category of sheaves considered from ℓ-adic (constructible) sheaves to *Weil sheaves*. These satisfy very similar formal properties to ordinary ℓ-adic sheaves, and the Grothendieck-Lefschetz trace formula remains valid in this context. Then, using this formalism, it becomes possible to analyze the sheaves that are actually relevant to the proof, and to demonstrate that they are in fact ℓ-adic sheaves in the ordinary sense.

Now, fix a scheme X_0 of finite type[1] over a finite field $k = \mathbf{F}_q$ and an algebraic closure $\overline{k}$[2], and let $X = X_0 \times_k \overline{k}$ with $\pi : X \to X_0$ the projection morphism. Let $\mathrm{F} : \alpha \mapsto \alpha^{1/q}$ be the *geometric* Frobenius automorphism of $\overline{k}$, which topologically generates $\mathrm{Gal}(\overline{k}/k)$.

Definition 1.1.1. The base change $\mathrm{F}_X := \mathrm{id}\, X_0 \times_k \mathrm{F}$ acts as an automorphism of X. I will call this the *Galois-theoretic geometric Frobenius automorphism* (or G-Frobenius for short) of X to emphasize that this is the Frobenius morphism coming from $\mathrm{Gal}(\overline{k}/k)$.

There are several other Frobenius morphisms floating around:

Definition 1.1.2. The *absolute Frobenius endomorphism* of an $\mathbf{F}_q$-scheme Y is the morphism $\sigma_Y : Y \to Y$ which is the identity on the underlying topological space $|Y|$ and which is the map $\alpha \mapsto \alpha^q$ on the structure sheaf.

Definition 1.1.3. The *relative Frobenius endomorphism* of X is the morphism $\mathrm{Fr}_X : \sigma_{X_0} \times_k \mathrm{id}_{\overline{k}}$.

Note that this is $\overline{k}$-linear.[3] These endomorphisms are related to each other:

Proposition 1.1.4. $\sigma_X \circ \mathrm{F}_X = \mathrm{Fr}_X$

This is because $\sigma_X = \sigma_{X_0} \times_k \sigma_{\mathrm{Spec}\, k} = \sigma_{X_0} \times_k \mathrm{F}^{-1}$.

Let's recall a few facts about Frobenius maps: details are given in Nicollo's notes from the previous lecture.

Proposition 1.1.5. For any constructible $\overline{\mathbf{Q}_\ell}$-sheaf $\mathscr{G}_0$ on X_0 with pullback $\mathscr{G} = \pi^*\mathscr{G}_0$, there is a canonical isomorphism $\mathrm{Fr}_X^*\mathscr{G} \xrightarrow{\sim} \mathscr{G}$.

[1] The finite type hypothesis is not necessary for most of these basic definitions, but certainly we will use it once we start discussing constructible sheaves, etc.

[2] The notion of a Weil sheaf depends on the choice of algebraic closure, and so $\overline{k}$ will be fixed once and for all.

[3] For this reason, one might reasonably refer to this as a "geometric" Frobenius, which is why I emphasized the Galois-theoretic nature of the 'G-Frobenius' F_X. I will avoid using the phrase "geometric Frobenius," which I think is unduly confusing.

Since Fr_X is proper, it induces a pullback map from $\mathrm{H}^i_c(X, \mathscr{G})$ to $\mathrm{H}^i_c(X, \mathrm{Fr}^*_X\mathscr{G})$. Composing this with the above canonical isomorphism $\mathrm{Fr}^*_X\mathscr{G} \xrightarrow{\sim} \mathscr{G}$, we obtain an endomorphism $\mathrm{Fr}^*_{\mathscr{G}} : \mathrm{H}^i_c(X, \mathscr{G}) \to \mathrm{H}^i_c(X, \mathscr{G})$.

Due to Proposition 1.1.4 and the fact, proven in Nicollo's notes, that σ_X acts on cohomology by the identity map, we have the following pleasant fact:

Proposition 1.1.6. The action on cohomology $\mathrm{Fr}^*_{\mathscr{G}} : \mathrm{H}^i_c(X, \mathscr{G}) \to \mathrm{H}^i_c(X, \mathscr{G})$ agrees with the Galois action of the automorphism F_X.

Let's recall how F_X acts on cohomology. Since it is an automorphism, it induces a pullback map $\mathrm{H}^i_c(X, \mathscr{G}) \to \mathrm{H}^i_c(X, \mathrm{F}^*_X\mathscr{G})$. Then, since F_X is an automorphism of X *over* X_0 and because $\mathscr{G} = \pi^*\mathscr{G}_0$, there is a canonical isomorphism $\mathrm{F}^*_X\mathscr{G} \simeq \left(\pi \circ (\mathrm{F}_X)\right)^* \mathscr{G}_0 = \mathscr{G}$. So, as with $\mathrm{Fr}^*_{\mathscr{G}}$, we obtain an endomorphism on $\mathrm{H}^i_c(X, \mathscr{G})$.

Now, consider the following situation. Let $\mathscr{G}$ be a constructible $\overline{\mathbf{Q}_\ell}$-sheaf on X; what additional information do we need to be able to descend $\mathscr{G}$ to a $\overline{\mathbf{Q}_\ell}$-sheaf on X_0? Galois descent theory gives a complete answer: the correspondence $\mathscr{G}_0 \mapsto \pi^*\mathscr{G}_0$ gives an equivalence of categories between constructible $\overline{\mathbf{Q}_\ell}$-sheaves on X_0 and constructible $\overline{\mathbf{Q}_\ell}$-sheaves on X *with a specified action of* $\mathrm{Gal}(X/X_0) = \mathrm{Gal}(\overline{k}/k)$ on $\mathscr{G}$. This action takes the form of isomorphisms $\varphi^*\mathscr{G} \xrightarrow{\sim} \mathscr{G}$ for all $\varphi \in \mathrm{Gal}(X/X_0)$ which respect the composition in $\mathrm{Gal}(X/X_0)$.

Now, we have an isomorphism $\widehat{\mathbf{Z}} \xrightarrow{\sim} \mathrm{Gal}(X/X_0)$ which sends 1 to F_X, so this Galois action amounts to an isomorphism $\mathrm{F}^*_X\mathscr{G} \to \mathscr{G}$ *which extends continuously to* $\widehat{\mathbf{Z}}$. If we are in a setting where we know that such an isomorphism exists, but we cannot immediately verify that it is continuous, we are led to the following definition:

Definition 1.1.7. A *Weil sheaf*[4] $\mathscr{G}_0$ on X_0 consists of a constructible $\overline{\mathbf{Q}_\ell}$-sheaf $\mathscr{G}$ on X, plus a specified isomorphism $\mathrm{F}_{\mathscr{G}_0} : \mathrm{F}^*_X\mathscr{G} \to \mathscr{G}$. A *lisse Weil sheaf* on X_0 is a Weil sheaf $\mathscr{G}_0$ such that the corresponding constructible $\overline{\mathbf{Q}_\ell}$-sheaf $\mathscr{G}$ on X is lisse.

Note that every constructible $\overline{\mathbf{Q}_\ell}$-sheaf $\mathscr{G}_0$ is canonically a Weil sheaf, via the canonical isomorphism $\mathrm{F}^*_X\pi^*\mathscr{G}_0 \xrightarrow{\sim} (\pi \circ \mathrm{F}_X)^*\mathscr{G}_0 = \pi^*\mathscr{G}_0$. Most notions which make sense for constructible (resp. lisse) $\overline{\mathbf{Q}_\ell}$-sheaves on X_0 can be extended immediately to the category of Weil sheaves (resp. lisse Weil sheaves) on X_0. Most importantly, we can define the functors $\mathrm{R}^p f_*, \mathrm{R}^p f_!, f^*$ for maps $f : X_0 \to Y_0$ of $\mathbf{F}_q$-schemes by applying the corresponding functors for the base change $f \times_k \overline{k} : X \to Y$ to $\mathscr{G}$ and using functoriality to obtain the structural isomorphism. Of course, these specialize to functors $\mathrm{H}^i, \mathrm{H}^i_c$ for the morphism $X \to \mathrm{Spec}\, k$. Using the specified isomorphism $\mathrm{F}^*_X\mathscr{G} \to \mathscr{G}$, we obtain a Frobenius action $\mathrm{F}^*_{\mathscr{G}_0}$ on cohomology.

In particular, we can define both sides of the Grothendieck-Lefschetz trace formula:

Theorem 1.1.8 (Grothendieck-Lefschetz trace formula for Weil sheaves)**.** *For any Weil sheaf $\mathscr{G}_0$ on the finite type separated $k = \mathbf{F}_q$-scheme X_0, the following formula holds:*

$$\prod_{x \in |X_0|} \det(1 - t^{d(x)}\mathrm{F}^*_{(\mathscr{G}_0)_x} \mid (\mathscr{G}_0)_{\overline{x}})^{-1} = \prod_{i=0}^{2\dim X_0} \det(1 - t\mathrm{F}^*_{\mathscr{G}_0} \mid \mathrm{H}^i_c(X, \mathscr{G}))^{(-1)^i}$$

[4]For ease of notation, the adjective 'constructible' will be omitted for Weil sheaves; we will never consider anything which is not constructible.

In order to deduce this theorem from the trace formula for ordinary $\overline{\mathbf{Q}_\ell}$-sheaves, we would like to characterize "how far" a Weil sheaf is from an ordinary sheaf. To do this, it is convenient to adopt the representation-theoretic (monodromy) perspective to sheaves. Recall the following correspondences:

Proposition 1.1.9. Assume that X_0 is geometrically connected. Then, for a geometric point $\overline{x}$ of X_0, the functor $\mathscr{F} \mapsto \mathscr{F}_{\overline{x}}$ defines an equivalence of categories from the category of lisse $\overline{\mathbf{Q}_\ell}$-sheaves to the category of continuous finite-dimensional representations of $\pi_1(X_0, \overline{x})$ over $\overline{\mathbf{Q}_\ell}$ such that:

- For a finite extension E of $\mathbf{Q}_\ell$, lisse E-sheaves correspond to representations which are defined over E. Note that every continuous representation of the profinite group $\pi_1(X_0, \overline{x})$ into a finite-dimensional $\overline{\mathbf{Q}_\ell}$ vector space is defined over some such finite-dimensional extension E. This corresponds to the fact that a $\overline{\mathbf{Q}_\ell}$-sheaf is an E-sheaf for some E, and it follows essentially from the Baire category theorem applied to the metrizable topological space $\mathrm{GL}_n(\overline{\mathbf{Q}_\ell}) = \cup_E \mathrm{GL}_n(E)$, using the fact that there are *countably many* finite extensions E of $\mathbf{Q}_\ell$.

- Any lisse E-sheaf can be regarded as a lisse $\mathscr{O}_E$-sheaf, and this corresponds to the fact that any continuous representation of the profinite group $\pi_1(X_0, \overline{x})$ stabilizes some lattice $\Lambda : \mathscr{O}_E^n \longrightarrow E^n$. (To see this, note that for any lattice Λ_0, $\mathrm{GL}(\Lambda_0) \simeq \mathrm{GL}_n(\mathscr{O}_E) \subseteq \mathrm{GL}_n(E)$ is an open subgroup, so its stabilizer has finite index in the profinite group $\pi_1(X_0, \overline{x})$. Then by adding the finitely many translates of Λ_0, we get a new lattice Λ which is $\pi_1(X_0, \overline{x})$-stable by construction).

- Choosing a strictly $\mathfrak{m}_E$-adic representation $\mathscr{F} = (\mathscr{F}_\bullet)$ of a lisse $\mathscr{O}_E$-sheaf, the lcc sheaves $\mathscr{F}_m$ correspond to the finite $\pi_1(X_0, \overline{x})$-modules $\Lambda/\mathfrak{m}_E^m \Lambda$.

- The *rank* of a $\overline{\mathbf{Q}_\ell}$ sheaf, i.e. the $\overline{\mathbf{Q}_\ell}$-dimension of a stalk, corresponds to the dimension of the corresponding representation.

- A lisse $\overline{\mathbf{Q}_\ell}$-sheaf is *irreducible* in the sense of having no proper non-zero sub-objects iff the corresponding representation is irreducible. A lisse $\overline{\mathbf{Q}_\ell}$-sheaf is *semisimple*, i.e. a direct sum of irreducibles, iff the corresponding representation is. If we regard a $\overline{\mathbf{Q}_\ell}$-sheaf as a E-sheaf for some (sufficiently large) E, then the condition of irreducibility means that the E-representation is *absolutely irreducible*. In other words, even if an E-sheaf is irreducible in the category of E-sheaves (i.e. has no non-trivial proper subsheaves which are E-sheaves), it may not be irreducible when considered as a $\overline{\mathbf{Q}_\ell}$-sheaf. Whenever we use the adjectives "irreducible" or "semisimple", it always means in this stronger absolute sense.

- A lisse $\overline{\mathbf{Q}_\ell}$-sheaf $\mathscr{F}_0$ on X_0 is *geometrically irreducible* (resp. *geometrically semisimple*) iff $\mathscr{F} = (\mathscr{F}_0)_{\overline{k}}$ is irreducible (resp. semisimple). In terms of representations, this says that $\mathscr{F}_{\overline{x}}$ is irreducible (resp. semisimple) as a representation of the normal subgroup $\pi_1(X, \overline{x}) \longrightarrow \pi_1(X_0, \overline{x})$. Clearly geometric irreducibility (resp. geometric semisimplicity) implies ordinary irreducibility (resp. semisimplicity).

Now, we may rephrase the definition of a Weil sheaf in terms of the *Weil group* of X_0. Recall that there is an exact sequence of topological groups:

$$1 \to \pi_1(X, \overline{x}) \to \pi_1(X_0, \overline{x}) \to \mathrm{Gal}(\overline{k}/k) \to 1$$

Now, $\mathrm{Gal}(\overline{k}/k) \simeq \widehat{\mathbf{Z}}$ with topological generator F. The *Weil group* of k, $\mathrm{W}(\overline{k}/k)$ is defined to be the infinite cyclic subgroup generated by F. We consider this as a topological group with the *discrete* topology. Then, we define:

Definition 1.1.10. The *Weil group* of X_0, denoted $\mathrm{W}(X_0, \overline{x})$, is the inverse image of $\mathrm{W}(\overline{k}/k)$ under the above exact sequence. It is generated by $\pi_1(X, \overline{x})$ together with any element $\sigma \in \pi_1(X_0, \overline{x})$ which restricts to F in $\mathrm{Gal}(\overline{k}/k)$. We give it a topology by considering it as the fiber product of $\mathrm{W}(\overline{k}/k)$ with $\pi_1(X_0, \overline{x})$ over $\mathrm{Gal}(\overline{k}/k)$. This makes $\pi_1(X, \overline{x})$ an open and closed subgroup. (Note: this is not the same as the subspace topology).

We call the map $\mathrm{W}(X_0, \overline{x}) \to \mathrm{W}(\overline{k}/k) \simeq \mathbf{Z}$, with the latter isomorphism sending F to 1, "degree". We will use σ to denote any degree one element. This should be thought of as "F_X", although there may not be a canonical lift of F_X to an element of $\pi_1(X_0, \overline{x})$ (without specifying a chosen rational point). However, any element of $\pi_1(X_0, \overline{x})$ induces an automorphism of the ind-étale cover X/X_0, and $\pi_1(X, \overline{x})$ acts trivially. Thus, the action of $\pi_1(X_0, \overline{x})$ on X/X_0 factors through the quotient $\pi_1(X_0, \overline{x}) \to \mathrm{Gal}(\overline{k}/k)$. In fact, identifying $\overline{k} = k(\overline{x})$ allows us to realize this action of $\mathrm{Gal}(\overline{k}/k)$ on X/X_0 as $\sigma \mapsto \mathrm{id}_{X_0} \times \sigma$. In particular, an element of $\mathrm{W}(X_0, \overline{x})$ of degree 1 acts on X by $\mathrm{id}_{X_0} \times \mathrm{F} = \mathrm{F}_X$.

Now, we have:

Proposition 1.1.11. When X_0 is geometrically connected, the functor $\mathscr{G}_0 \mapsto (\mathscr{G}_0)_{\overline{x}}$ defines an equivalence of categories between the category of lisse Weil sheaves on X_0 and the category of continuous $\overline{\mathbf{Q}_\ell}$-representations of $\mathrm{W}(X_0, \overline{x})$. The compatibilities in Proposition 1.1.9 continue to hold, with the fundamental group replaced everywhere by the Weil group.

The data of the ℓ-adic sheaf $\mathscr{G}$ on X corresponds by the above dictionary to a continuous $\overline{\mathbf{Q}_\ell}$-representation $(\rho_{\mathscr{G}}, V)$ of $\pi_1(X, \overline{x})$. Then, since $\mathrm{W}(X_0, \overline{x})$ is a semi-direct product of this group with an infinite cyclic group generated by any degree 1-element σ, a continuous representation of $\mathrm{W}(X_0, \overline{x})$ on V restricting to $\rho_{\mathscr{G}}$ is determined entirely by the action of σ on V. Thus, continuous representations of $\mathrm{W}(X_0, \overline{x})$ on V may be identified with lisse ℓ-adic sheaves on X with stalk V at $\overline{x}$, along with an element $\rho(\sigma) \in \mathrm{GL}(V)$ such that conjugation by $\rho(\sigma)$ on $\rho(\pi_1(X, \overline{x})) \subseteq \mathrm{GL}(V)$ is compatible with conjugation by σ on $\pi_1(X, \overline{x})$. But we know that σ acts as F_X on X, so a map $\rho(\sigma)$ which is compatible with this conjugation amounts to an isomorphism $\mathrm{Fr}_X^* \mathscr{G} \to \mathscr{G}$.

Using this proposition, we obtain a criterion for a lisse Weil sheaf to be an ordinary $\overline{\mathbf{Q}_\ell}$-sheaf:

Proposition 1.1.12. A lisse Weil sheaf $\mathscr{G}_0$ on a geometrically connected finite type k-scheme X_0 is an ordinary $\overline{\mathbf{Q}_\ell}$-sheaf if and only if some (equivalently, any) degree-1 element $f \in \mathrm{W}(X_0, \overline{x})$ acts on $\mathscr{G}_{0\overline{x}}$ with eigenvalues which are ℓ-adic units (i.e. units of $\mathscr{O}_E$).

Proof. By the previous proposition, the question is whether the $\overline{\mathbf{Q}_\ell}$-representation (ρ, V) of $\mathrm{W}(X_0, \overline{x})$ given by the action of $W(X_0, \overline{x})$ on $\mathscr{G}_{\overline{x}}$ extends to a representation of $\pi_1(X_0, \overline{x})$. Since the representation restricted to $\pi_1(X, \overline{x})$ is continuous, the image of $\pi_1(X, \overline{x})$ is contained in $\mathrm{GL}_n(E_0)$ for some finite extension E_0 of $\mathbf{Q}_\ell$ by Proposition 1.1.9. Then, the image of $\mathrm{W}(X_0, \overline{x})$ is generated by the images of $\pi_1(X, \overline{x})$ and of f, so increasing E_0 to include the matrix coefficients of $\sigma = \rho(f)$ as well as its eigenvalues, we see that the representation is defined over some finite extension E of $\mathbf{Q}_\ell$.

If the representation does extend to $\pi_1(X_0, \overline{x})$, by Proposition 1.1.9, there is a $\pi_1(X_0, \overline{x})$-stable lattice $\Lambda \simeq \mathscr{O}_E^n$, so $\pi_1(X_0, \overline{x})$ factors through $\mathrm{GL}_n(\mathscr{O}_E)$, and the eigenvalues of an element of $\mathrm{GL}_n(\mathscr{O}_E)$ are ℓ-adic units.

To see the converse, note that $\pi_1(X_0, \overline{x})$ is the profinite completion of $\mathrm{W}(X_0, \overline{x})$, so any morphism from $W(X_0, \overline{x})$ to a profinite group extends to $\pi_1(X_0, \overline{x})$. If $\mathrm{W}(X_0, \overline{x})$ stabilizes a lattice Λ, then as above the map $\mathrm{W}(X_0, \overline{x}) \to \mathrm{GL}_n(E)$ factors through the profinite group $\mathrm{GL}_n(\mathscr{O}_E)$. So we need to show that if the eigenvalues of σ are ℓ-adic units, then there is a stable lattice.

To do this, we use the multiplicative Jordan decomposition theorem for σ to write

$$\sigma = (\sigma)_{\mathrm{ss}} \cdot (\sigma)_{\mathrm{un}}$$

with $(\sigma)_{\mathrm{ss}}$ semisimple and $(\sigma)_{\mathrm{un}}$ unipotent, such that the eigenspaces of $(\sigma)_{\mathrm{ss}}$ are invariant subspaces for $(\sigma)_{\mathrm{un}}$. By splitting V into the eigenspaces of $(\sigma)_{\mathrm{ss}}$ and constructing a stable lattice in each subspace, we may assume that $(\sigma)_{\mathrm{ss}}$ acts by multiplication by an ℓ-adic unit on V. Then, *any* lattice is stable for $(\sigma)_{\mathrm{ss}}$. Now, $(\sigma)_{\mathrm{un}} = 1 + N$, with $N^k = 0$ for some $k > 0$, so the powers of $(\sigma)_{\mathrm{un}}$ are spanned inside $\mathrm{M}_n(E)$ by $1, (\sigma)_{\mathrm{un}}, \dots, (\sigma)_{\mathrm{un}}^k$. Thus, the lattice spanned by the images of the basis under the first k powers of $(\sigma)_{\mathrm{un}}$ is preserved by $(\sigma)_{\mathrm{un}}$ and therefore also by σ. □

Now, we can study the monodromy to obtain a precise parametrization of irreducible lisse Weil sheaves:

Theorem 1.1.13. *Suppose that X_0 is normal and geometrically connected. Then, an irreducible lisse Weil sheaf $\mathscr{G}_0$ of rank n is an actual $\overline{\mathbf{Q}_\ell}$-sheaf if and only if its determinant $\wedge^n \mathscr{G}_0$ is.*

Corollary 1.1.14. If X_0 is normal and geometrically connected, and $\mathscr{G}_0$ is an irreducible lisse Weil sheaf, then there is some $b \in \overline{\mathbf{Q}_\ell}^\times$ and a lisse $\overline{\mathbf{Q}_\ell}$-sheaf $\mathscr{F}_0$ such that

$$\mathscr{G}_0 \simeq \mathscr{F}_0 \otimes \chi_b$$

Here χ_b is the character $\mathrm{W}(X_0, \overline{x}) \to \overline{\mathbf{Q}_\ell}^\times$ which is trivial on $\pi_1(X, \overline{x})$ and maps some (equivalently, any) degree-1 element to b.

The fact that the theorem implies the corollary is immediate from Proposition 1.1.12; choose any degree-1 element σ, and then

$$\wedge^n \left(\mathscr{G}_0 \otimes \chi_{\det(\sigma)^{-1/n}} \right) = \wedge^n (\mathscr{G}_0) \otimes \chi_{\det(\sigma)^{-1}}$$

Then, the right hand side clearly maps σ to a unit, so Proposition 1.1.12 implies that it is a lisse $\overline{\mathbf{Q}_\ell}$-sheaf.

This result implies more generally, by an easy induction on rank:

Corollary 1.1.15. If X_0 is normal and geometrically connected, and $\mathscr{G}_0$ is a lisse Weil sheaf, there is a filtration $0 = \mathscr{G}_0^{(0)} \subseteq \mathscr{G}_0^{(1)} \subseteq \mathscr{G}_0^{(2)} \subseteq \cdots \subseteq \mathscr{G}_0^{(k)} = \mathscr{G}_0$ such that the quotients $\mathscr{G}_0^{(m)}/\mathscr{G}_0^{(m-1)}$ are of the form $\mathscr{F}_0^{(m)} \otimes \chi_{b_m}$ for some lisse $\overline{\mathbf{Q}_\ell}$-sheaf $\mathscr{F}_0^{(m)}$ and some $b_m \in \overline{\mathbf{Q}_\ell}^\times$.

Finally, we can use this to prove the trace formula, since the substitution $t \mapsto tb$ fixes twisting by χ_b, and both sides of the formula are additive in exact sequences.

Now let's prove the theorem:

Proof. Let (V, ρ) be the representation of $\mathrm{W}(X_0, \overline{x})$ associated to $\mathscr{G}_0$, and let σ be the image of a degree-1 element. We know that the representation is defined over E for some finite extension $E/\mathbf{Q}_p$. Thus, ρ is an *absolutely* irreducible representation of $\mathrm{W}(X_0, \overline{x})$ over E. We will assume that E is sufficiently large so that the characteristic polynomial of σ splits into linear factors. By Proposition 1.1.12, it suffices to show that the eigenvalues of σ are units in $\mathscr{O}_E$. We already know that the eigenvalues of $\rho(g)$ are units for $g \in \pi_1(X, \overline{x})$, and the hypothesis on $\wedge^n \mathscr{G}_0$ implies that $\det(\sigma)$ is a unit in $\mathscr{O}_E$. Furthermore, it clearly suffices to show that the eigenvalues of σ^m for some power m are units.

In order to study this representation, we will use the following notion:

Definition 1.1.16. For a lisse Weil sheaf $\mathscr{G}_0$ of rank n, the *geometric monodromy group* $G_g(\mathscr{G}_0)$ is the Zariski closure of the image of $\pi_1(X, \overline{x})$ under the associated representation $\mathrm{W}(X_0, \overline{x}) \to \mathrm{GL}_n(\overline{\mathbf{Q}_\ell})$. The *arithmetic monodromy group* $G(\mathscr{G}_0)$ is the semi-direct product of $G_g(\mathscr{G}_0)$ with the cyclic group generated by σ. Its $\overline{\mathbf{Q}_\ell}$-points coincide with the subgroup of $\mathrm{GL}_n(\overline{\mathbf{Q}_\ell})$ generated by $G_g(\mathscr{G}_0)$ and σ.

In a later lecture, we will prove the following:

Theorem 1.1.17. *Suppose that X_0 is geometrically connected and normal and that $\mathscr{G}_0$ is* geometrically semisimple. *Then:*

- *$G_g(\mathscr{G}_0)$ is a semisimple algebraic group defined over E.*
- *For some m, $\sigma^m = g \cdot z$ for some $g \in G_g(\mathscr{G}_0)(\overline{\mathbf{Q}_\ell})$ and some z in the center of $G(\mathscr{G}_0)(\overline{\mathbf{Q}_\ell})$.*

Furthermore, if $\mathscr{G}_0$ is an arbitrary lisse Weil sheaf (not supposed to be geometrically semisimple), it has geometrically semisimple irreducible constituents. In particular, a semisimple lisse Weil sheaf is geometrically semisimple.

Remark 1.1.18. Let $\mathscr{G}_0$ be irreducible but not necessarily geometrically irreducible. Then it is at least geometrically semisimple, so the Theorem applies. This means that there is $g \in G_g(\mathscr{G}_0)(\overline{\mathbf{Q}_\ell}) = \rho(\pi_1(X, \overline{x}))$ and some $m \geq 0$ such that $\sigma' := g^{-1}\sigma^m$ is in the center of $\rho(\mathrm{W}(X_0, \overline{x}))$. Now, σ' and $\pi_1(X, \overline{x})$ generate $\rho(W(X_0', \overline{x}))$, with $X_0' = X_0 \times_k k'$ and k'/k the degree m extension. Now, let $\mathscr{F}_0$ be an irreducible component of the geometrically semisimple Weil sheaf $\mathscr{F} \times_k k'$ on $X_0 \times_k k'$. Since σ' is in the center of $\rho(\mathrm{W}(X_0, \overline{x}))$, by Schur's lemma it acts by a scalar on $(\mathscr{F}_0)_{\overline{x}}$. Thus, any $\pi_1(X, \overline{x})$-invariant subspace of $(\mathscr{F}_0)_{\overline{x}}$ is automatically σ'-invariant as well, so it is $\rho(\mathrm{W}(X_0 \times_k k', \overline{x}))$-invariant. Thus, $\mathscr{F}_0$ is *geometrically irreducible*.

Now, assume for the moment that (V, ρ) is irreducible as a representation of $\pi_1(X, \overline{x})$, i.e. that $\mathscr{G}_0$ is geometrically irreducible. The above theorem then implies that there is a power m such that $\sigma^m = g \cdot z$ with $g \in G_g(\mathscr{G}_0)$ and z in the center of $G(\mathscr{G}_0)$. Since $G_g(\mathscr{G}_0)$ is a semisimple algebraic group, any algebraic character from $G_g(\mathscr{G}_0)$ to $\mathbf{G}_m$ must have *finite* image. This is because *connected* semisimple groups have no non-trivial algebraic characters (i.e. because they are equal to their own derived subgroups, which is the commutator on the level of points over some algebraically closed field). Thus, an algebraic character on the finite-type $\overline{\mathbf{Q}_\ell}$-group scheme $G_g(\mathscr{G}_0)$ factors through its *finite* component group. In particular, the determinant of g has finite order, so by passing to a further power m if necessary we may assume that the determinant of g is 1, so $\det(\sigma^m) = \det(z)$ is an ℓ-adic unit. Now, by irreducibility, z must be a scalar (using Schur's lemma). Thus, we are left with the task of showing that for $g \in G_g(\mathscr{G}_0)$ with $\det g = 1$, the eigenvalues of g are units.

Now, note that $\rho(\pi_1(X, \overline{x})) \subseteq \operatorname{End}(V)$ is a compact subset. This generates a $\mathscr{O}_E$-submodule A of $\operatorname{End}(V)$, and because ρ is an absolutely irreducible $\pi_1(X, \overline{x})$-representation, this must be a full-rank lattice in the n^2-dimensional E-vector space $\operatorname{End}(V)$. This follows from the Jacobson density theorem; see Corollary XVII.3.4 in Lang's *Algebra*. Now, note that $g = \sigma^m z^{-1}$ normalizes the image of $\pi_1(X, \overline{x})$, because σ^m does and because z is in the center. Because g normalizes $\rho(\pi_1(X, \overline{x}))$, conjugation by g stabilizes A. This means that the eigenvalues of the adjoint action of g on $\operatorname{End}(V)$ are units, and these eigenvalues are of the form λ_i/λ_j^{-1}, $1 \leq i, j \leq n$ for λ_i the eigenvalues of g. Thus, since their product $\prod_i \lambda_i = \det(g)$ is a unit, each one must be as well.

Now, let's reduce the general case to the case where $\mathscr{G}_0$ is geometrically irreducible. Since it is irreducible by hypothesis, Remark 1.1.18 applies and thus there is a finite extension k'/k such that the irreducible constituents of the semisimple Weil sheaf $\mathscr{G}_0' := \mathscr{G}_0 \times_k k'$ are geometrically irreducible. Then, we have:

$$\rho(\mathrm{W}(X_0, \overline{x})) = \bigcup_{j=0}^{m-1} \sigma^j \rho(\mathrm{W}(X_0', \overline{x}))$$

Let $U \subseteq V$ be an irreducible constituent of V for the action of $\mathrm{W}(X_0', \overline{x})$. Since V is irreducible for $\mathrm{W}(X_0, \overline{x})$, we have $V = \sum_j \sigma^j U$, and the isomorphism $\sigma^j : U \to \sigma^j U$ intertwines the action of σ^m (i.e. $\sigma^m \sigma^j = \sigma^j \sigma^m$), so σ^m acts on each space $\sigma^j U$ with $\det(\sigma^m | \sigma^j U) = \det(\sigma^m)$. Since U is an irreducible constituent of $\mathscr{G}_0'$, it is geometrically irreducible. Thus, the previous case applies to show that σ^m acts with ℓ-adic unit eigenvalues on U, and hence on each $\sigma^j U$. Thus, the eigenvalues of σ^m, and hence of σ, are ℓ-adic units on V. □

1.2 Properties of $\overline{\mathbf{Q}_\ell}$-sheaves and Weil sheaves

Finally, we will record various formal properties of Weil sheaves, as well as ordinary $\overline{\mathbf{Q}_\ell}$-sheaves, which can be proved using the monodromy point of view. These are not automatic, and the hypothesis of normality is crucial for making sure the fundamental group is well-behaved with respect to open covers. However, the proofs will be omitted.

Proposition 1.2.1. If X is a normal noetherian scheme, lisse $\overline{\mathbf{Q}_\ell}$-sheaves satisfy étale descent. This means that for any étale cover $\sqcup U_i \to X$, there is an equivalence of categories between the category of lisse $\overline{\mathbf{Q}_\ell}$-sheaves on X and the category of lisse $\overline{\mathbf{Q}_\ell}$-sheaves on U_i equipped with descent data.

The proof in the crucial special case of a Zariski covering $X = U_1 \cup U_2$ (so the descent data is just an isomorphism over $U_1 \cap U_2$) uses an étale version of the Seifert-van Kampen theorem. This says that for $i = 1, 2$ and any $\overline{x} \in U_1 \cap U_2$, there are *surjections* $\pi_1(U_1 \cap U_2, \overline{x}) \to \pi_1(U_i, \overline{x})$ and $\pi_1(U_i, \overline{x}) \to \pi_1(X, \overline{x})$ such that the kernel of $\pi_1(U_1, \overline{x}) \to \pi_1(X, \overline{x})$ is the image in $\pi_1(U_1, \overline{x})$ of the kernel of $\pi_1(U_1 \cap U_2, \overline{x}) \to \pi_1(U_2, \overline{x})$.

Corollary 1.2.2. On a *normal* noetherian scheme, the property that a given constructible $\overline{\mathbf{Q}_\ell}$-sheaf is lisse may be checked étale-locally. Furthermore, the *specialization criterion* holds: a constructible $\overline{\mathbf{Q}_\ell}$ sheaf is lisse iff for every specialization of geometric points $s \in \overline{\eta}$, $\mathscr{F}_s \xrightarrow{\sim} \mathscr{F}_\eta$.

Remark 1.2.3. In the case that X is of finite type over a field or a Dedekind base scheme, we can extend the descent result to constructible $\overline{\mathbf{Q}_\ell}$-sheaves, even without normality assumptions. The reasons for this additional hypothesis is that we need it to ensure that constructible sheaves are preserved under pushforward j_* (i.e. not just $j_!$) by open immersions j, and also to ensure that all normalizations are finite.

Next, with more care, some of the statements proved above for lisse Weil sheaves can be extended to the constructible case:

Proposition 1.2.4. A (constructible) Weil sheaf $\mathscr{G}_0$ on X_0 comes from an honest constructible $\overline{\mathbf{Q}_\ell}$ sheaf on X_0 iff it is "continuous", meaning that there is a stratification of X_0 such that the restriction of $\mathscr{G}_0$ to each stratum is an honest lisse $\overline{\mathbf{Q}_\ell}$ sheaf.

Note that in order to prove this, we must formulate Proposition 1.1.11 without connectivity assumptions. One application of these considerations is to verify that pullback (for example, enlarging the base field k to an extension) behaves as expected. In particular, the pullback of a constructible $\overline{\mathbf{Q}_\ell}$ sheaf considered as a Weil sheaf agrees with its pullback as a constructible sheaf.

2 A Semicontinuity Theorem for Weights: Notes by David Benjamin Lim

2.1 Weights

We fix the following notation. We will denote by $k = \mathbf{F}_q$ the finite field with q elements, $\overline{k}$ an algebraic closure of k and k_n a degree n extension of k (necessarily isomorphic to $\mathbf{F}_{q^n}$). In this article, we will refer to a finite type scheme X_0/k as simply a scheme. Furthermore, we will refer to a Weil sheaf $\mathscr{G}_0$ on X_0 as simply a sheaf. By convention, such a sheaf is always constructible, viz isomorphic in the Artin-Rees category to a strictly constructible sheaf. We will fix an isomorphism $\tau : \overline{\mathbf{Q}}_\ell \to \mathbf{C}$. For a closed point $x \in |X_0|$, we will denote by $d(x)$ the degree of the field extension $k(x)/k$, and $N(x)$ the cardinality of $k(x)$. Recall the following definition of purity:

Definition 2.1.0.1. Let β be a real number.

1. Choose a $\overline{k}$-point $\overline{x} \in X$ lying over $x \in |X_0|$. The Weil group $W(\overline{k}/k(x))$ acts on the stalk at $\mathscr{G}_{0\overline{x}}$ via the geometric Frobenius $F_x : \mathscr{G}_{0\overline{x}} \to \mathscr{G}_{0\overline{x}}$. We say that $\mathscr{G}_0$ is **τ-pure** of weight β if for every $x \in |X_0|$, and all eigenvalues $\alpha \in \overline{\mathbf{Q}}_\ell$ of F_x, we have
$$|\tau(\alpha)| = N(x)^{\beta/2}.$$

If $\mathscr{G}_0$ is a general Weil sheaf, not necessarily pure, we would still like to have the notion of the weight of $\mathscr{G}_0$. This brings us to the following definition.

Definition 2.1.0.2. For a scheme X_0/k and sheaf $\mathscr{G}_0$ on X_0, we define the **maximal weight** of $\mathscr{G}_0$ (with respect to τ) as
$$w(\mathscr{G}_0) := \sup_{x \in |X_0|} \sup_{\alpha \text{ eigenvalue}} \frac{\log(|\tau(\alpha)|^2)}{\log N(x)}.$$
For reasons of convention, we define the weight of the zero sheaf to be $-\infty$.

2.2 Convergence of the L-function

We show in this section that the weight of a Weil sheaf controls the convergence of its L-function.

Lemma 2.2.0.1. Let X_0/k be a scheme. Then we have the estimate
$$|X_0(k_n)| = O(q^{n \dim X_0})$$
as $n \to \infty$.

Proof. We have $|X_0(k_n)| = |X_{0\text{red}}(k_n)|$ and so we can reduce to the case that X_0 is reduced. By the principle of inclusion-exclusion, we can reduce to the case where X_0 is integral. Then by Noether normalization, there is an open dense subset $U_0 \subseteq X_0$ with a finite morphism $f : U_0 \to \mathbf{A}_{k_n}^{\dim X_0}$. Hence we obtain
$$|U(k_n)| \leq (\deg f)(\#k_n)^{\dim X_0} = (\deg f) q^{n \dim X_0}.$$
The result follows by induction on dimension, since $\dim(X_0 \setminus U_0) < \dim X_0$.

□

Lemma 2.2.0.2. Let V be a finite dimensional vector space and F an endomorphism of V, and $d \in \mathbf{N}$ a non-negative integer. Then

$$\frac{d}{dt}\log\det(1-t^dF|V)^{-1} = \sum_{n\geq 1}\mathrm{Tr}(F^n)dt^{dn-1}.$$

Proof. Recall that Niccolò introduced the formula

$$\det(1-t^dF|V)^{-1} = \exp\left(\sum_{n\geq 1}\mathrm{Tr}(F^n)\frac{t^{dn}}{n}\right)$$

in his talk. Taking derivatives, we get

$$\begin{aligned}\frac{d}{dt}\det(1-t^dF|V)^{-1} &= \left(\sum_{n\geq 1}\mathrm{Tr}(F^n)(dn)\frac{t^{dn-1}}{n}\right)\cdot\exp\left(\sum_{n\geq 1}\mathrm{Tr}(F^n)\frac{t^{dn}}{n}\right)\\ &= \left(\sum_{n\geq 1}\mathrm{Tr}(F^n)dt^{dn-1}\right)\det(1-t^dF|V)^{-1}\end{aligned}$$

and hence the result.

□

Proposition 2.2.0.1. Let $\mathscr{G}_0$ be a sheaf on X_0 and β a real number such that $w(\mathscr{G}_0)\leq\beta$. Then the L-function

$$\tau L(X_0,\mathscr{G}_0,t) = \prod_{x\in|X_0|}\tau\det(1-t^{d(x)}F_x,\mathscr{G}_{0\overline{x}})^{-1}$$

converges for all $|t| < q^{-\beta/2-\dim X_0}$ and has no zeroes or poles in this region.

Proof. The idea is that we can detect the poles an zeroes of the L-function by looking at its logarithmic derivative. This is because the logarithmic derivative of a complex valued function has poles precisely where the original function has poles or zeroes. We will suppress the isomorphism $\tau : \overline{\mathbf{Q}}_\ell \to \mathbf{C}$ in the following for brevity.

$$\begin{aligned}\frac{d}{dt}\log L(X_0,\mathscr{G}_0,t) &= \sum_{x\in|X_0|}\frac{d}{dt}\log\left(\det(1-t^{d(x)}F_x|\mathscr{G}_{0\overline{x}})^{-1}\right)\\ &= \sum_{x\in|X_0|}\sum_{n\geq 1}d(x)(\mathrm{Tr}(F_x^n))t^{d(x)n-1}\\ &= \sum_{n\geq 1}\left(\sum_{x\in|X_0|:d(x)|n}d(x)(\mathrm{Tr}(F_x^{n/d(x)}))\right)t^{n-1}.\end{aligned}$$

We passed from the first to second line using Lemma 2.2.0.2. By assumption on the bound of the Frobenius eigenvalues, we have

$$|\operatorname{Tr}(F_x^{n/d(x)})| \le rq^{n\beta/2}$$

where

$$r := \max_{x\in|X_0|} \dim_{\overline{\mathbf{Q}_\ell}} \mathscr{G}_{0\overline{x}}.$$

Hence

$$\begin{aligned} \frac{d}{dt}\log L(X_0,\mathscr{G}_0,t) &= \sum_{n\ge 1}\left(\sum_{x\in|X_0|:d(x)|n} d(x)(\operatorname{Tr}(F_x^{n/d(x)}))\right)t^{n-1} \\ &\le \sum_{n\ge 1}\left(\sum_{x\in|X_0|:d(x)|n} d(x)\cdot(rq^{n\beta/2})\right)t^{n-1} \\ &= \sum_{n\ge 1}|X_0(k_n)|\cdot(rq^{n\beta/2})t^{n-1}. \end{aligned}$$

By Lemma 2.2.0.1, we see that the logarithmic derivative converges for all $|t| < q^{-\beta/2-\dim X_0}$. Therefore $L(X_0,\mathscr{G}_0,t)$ also converges for $|t| < q^{-\beta/2-\dim X_0}$.

□

2.3 Semicontinuity of Weights

The motivating question is the following. Suppose $\mathscr{G}_0$ is a sheaf on a scheme X_0/k. Given an open dense $j_0 : U_0 \to X_0$, how does the weight of $\mathscr{G}_0$ compare to that of $j_0^*\mathscr{G}_0$? It turns out that under certain hypotheses on $\mathscr{G}_0$ (made precise below), this is always true. This result will be used in future arguments involving Noetherian induction on X_0. First, we consider the case of curves.

Given a Weil sheaf $\mathscr{G}_0$ on a smooth curve X_0/k, we recall the following facts concerning $H_c^0(X,\mathscr{G})$ and $H_c^2(X,\mathscr{G})$.

1. If $\mathscr{G}_0$ is lisse, corresponding to some representation V of $\pi_1(X,x)$, then

$$H^0(X,\mathscr{G}) = V^{\pi_1(X,x)}.$$

2. If X_0 is geometrically irreducible and $U_0 \subseteq X_0$ is an open dense subset, we have

$$H_c^2(X,\mathscr{G}) = H_c^2(U,\mathscr{G}).$$

   Indeed, consider the excision sequence

$$0 \to j_!j^*\mathscr{G} \to \mathscr{G} \to i_*i^*\mathscr{G} \to 0$$

associated to the inclusion of spaces

$$\begin{array}{ccc} & & X \setminus U \\ & & \downarrow{\scriptstyle i} \\ U & \xrightarrow{j} & X \end{array}$$

Choose a compactification $j' : X \to \overline{X}$ and apply $j'_!$ to the exact sequence above to get

$$0 \to j'_! j_! j^* \mathscr{G} \to j'_! \mathscr{G} \to j'_! i_* i^* \mathscr{G} \to 0.$$

Since the sheaf $j'_! i_* i^* \mathscr{G}$ is supported on the finite set of closed points $X \setminus U$, it is enough to prove that

$$H^i(\overline{X}, j'_! i_* i^* \mathscr{G}) = 0$$

for $i = 1, 2$. This follows from the fact that the higher étale cohomology of a separably closed field is zero.

Lemma 2.3.0.1. Let X_0/k be a geometrically irreducible affine curve, $j_0 : U_0 \to X_0$ an open immersion of an open subscheme and $\mathscr{G}_0$ a sheaf on X_0 such that the canonical adjunction map $\mathscr{G}_0 \to j_{0*} j_0^* \mathscr{G}_0$ is an isomorphism. Assume further that $j_0^* \mathscr{G}_0$ is lisse. Then

$$H^0_c(X, \mathscr{G}) = 0.$$

Proof. Let $Z \subseteq X$ be a complete subvariety and define $V := X \setminus Z$. Then $V \neq \emptyset$ because X is not complete. We have to show that

$$H^0_Z(X, \mathscr{G}) := \ker \left(H^0(X, \mathscr{G}) \to H^0(V, \mathscr{G}|_V) \right)$$

is zero. Since $\mathscr{G}_0 \to j_{0*} j_0^* \mathscr{G}_0$ is an isomorphism, we may rewrite this as

$$H^0_Z(X, \mathscr{G}) = \ker \left(H^0(U, \mathscr{G}|_U) \to H^0(V \cap U, \mathscr{G}|_{U \cap V}) \right).$$

The intersection $U \cap V$ is non-empty for X is irreducible. Let η denote the generic point of U. Since $\mathscr{G}|_U$ is lisse, for any $u \in U$ the specialization map $\mathscr{G}_{0\overline{u}} \to \mathscr{G}_{0\overline{\eta}}$ is an isomorphism. This implies that any section that vanishes on $V \cap U$ also vanishes on U, consequently $H^0_Z(X, \mathscr{G}) = 0$ as desired.

□

Proposition 2.3.0.1 (Semicontinuity of Weights for Curves)**.** Let X_0/k be a geometrically irreducible smooth curve. Let $j_0 : U_0 \hookrightarrow X_0$ be an affine open and $\mathscr{G}_0$ a lisse sheaf on U_0. We define $S_0 := X_0 \setminus U_0$. Suppose $\mathscr{G}_0$ is a sheaf on X_0 that satisfies the following conditions:

1. $j_0^* \mathscr{G}_0$ is lisse.
2. $H^0_S(X, \mathscr{G}) = 0$.

Then

$$w(j_0^*\mathscr{G}_0) \leq \beta \implies w(\mathscr{G}_0) \leq \beta.$$

The idea here is the following. We first reduce to the case where $H_c^0(X, \mathscr{G}) = 0$ by the previous lemma. Therefore by the Grothendieck-Lefschetz trace formula, the only contribution to the poles of the L-function will come from $H_c^2(X, \mathscr{G})$. This allows us to show that $L(X_0, \mathscr{G}_0, t)$ has no poles outside of the disk $|t| < q^{-\beta/2-1}$. On the other hand, by assumption the L-function on U_0 converges and has no zeroes in the same region. So writing

$$L(X_0) = L(U_0)L(S_0),$$

and noticing that $L(S_0)$ has only finitely many factors, we deduce immediately bounds on the frobenius eigenvalues of $\mathscr{G}_{0\overline{s}}$ for every $s \in |S_0|$. The result follows by a trick of Deligne of considering higher tensor powers of $\mathscr{G}_0$.

Proof. By removing a point from U_0, we may assume that X_0 is affine. The assumption $H_S^0(X, \mathscr{G}) = 0$ implies $\mathscr{G}_0 \hookrightarrow j_{0*}j_0^*\mathscr{G}_0$. In this case

$$w(\mathscr{G}_0) \leq w(j_{0*}j_0^*\mathscr{G}_0)$$

and so we can reduce to the case where $\mathscr{G}_0 \to j_{0*}j_0^*\mathscr{G}_0$ is an isomorphism. Then by Lemma 2.3.0.1 and the Grothendieck-Lefschetz trace formula, we have

$$L(X_0, \mathscr{G}_0, t) = \frac{\det(1 - Ft|H_c^1(X, \mathscr{G}))}{\det(1 - Ft|H_c^2(X, \mathscr{G}))}.$$

Define $\mathscr{F}_0 := j_0^*\mathscr{G}_0$. For $u \in |U_0|$, this corresponds to a representation V of $\pi_1(U, \overline{u})$. Then

$$\begin{aligned} H_c^2(X, \mathscr{G}) &= H_c^2(U, \mathscr{F}) \\ &= H^0(U, \check{\mathscr{F}}(1))^\vee && \text{(Poincaré duality)} \\ &= H^0(U, \check{\mathscr{F}} \otimes \overline{\mathbf{Q}}_\ell(1))^\vee \\ &= H^0(U, \check{\mathscr{F}})^\vee \otimes \overline{\mathbf{Q}}_\ell(-1) && \text{(Künneth formula)} \\ &= (V^{\pi_1(U,\overline{u})})^\vee(-1) \\ &= (V_{\pi_1(U,\overline{u})})(-1). \end{aligned}$$

It follows that the poles of $L(X_0, \mathscr{G}_0, t)$ are of the form $1/\alpha q$ where α is an eigenvalue of F_u on $V_{\pi_1(U,\overline{u})}$ (recall geometric Frobenius acts by q^{-1} on $\overline{\mathbf{Q}}_\ell(1)$). Now from the definition of coinvariance, $\alpha^{d(u)}$ lifts to an eigenvalue on V. Therefore by the assumption $w(\mathscr{F}_0) \leq \beta$, we have that $|\tau(\alpha^{d(u)})| \leq q^{d(u)\beta/2}$, i.e.

$$\left|\tau\left(\frac{1}{\alpha q}\right)\right| > q^{-\beta/2-1}$$

and so $L(X_0, \mathscr{G}_0, t)$ converges for $|t| < q^{-\beta/2-1}$.

On the other hand, we may write

$$L(X_0, \mathscr{G}_0, t) = L(U_0, j_0^*\mathscr{G}_0, t) \prod_{s_0 \in |S_0|} \det(1 - F_s t^{d(s)}, \mathscr{G}_{0\overline{s}})^{-1}.$$

The asumption $w(j_0^*\mathscr{G}) \leq \beta$ implies that the factor $L(U_0, j_0^*\mathscr{G}, t)$ converges and has no zeroes for $|t| < q^{-\beta/2-1}$. Therefore since $|S_0|$ is finite it follows that none of the factors

$$\det(1 - F_s t^{d(s)}, \mathscr{G}_{0\bar{s}})^{-1}$$

has poles for $|t| < q^{-\beta/2-1}$, which implies the estimate

$$|\tau(\tilde{\alpha})| \leq q^{-\beta/2-1}$$

for $\tilde{\alpha}$ an eigenvalue of $F_s : \mathscr{G}_{0\bar{s}} \to \mathscr{G}_{0,\bar{s}}$. Finally, by considering the sheaves $j_{0*}\mathscr{F}^{\otimes k}$ we get the estimate $|\tau(\tilde{\alpha})| \leq q^{-\beta/2-1/k}$. Since this is true for every k, we are done.

□

Corollary 2.3.0.1 (Semicontinuity of Weights for general X_0)**.** Let $\mathscr{G}_0$ be a lisse sheaf on a geometrically irreducible scheme X_0/k and let $j_0 : U_0 \to X_0$ be the inclusion of an open dense subscheme of X_0. Then

$$w(\mathscr{G}_0) = w(j_0^*\mathscr{G}_0).$$

Proof. By taking the normalization of $X_{0\text{red}}$ we can reduce to the case where X_0 is a normal geometrically integral scheme. If $\dim X_0 = 1$ we are done by the semicontinuity theorem for curves above. If $\dim X_0 > 1$, we may connect any point of $X_0 \setminus U_0$ to a point of U_0 by a curve, and conclude again using the semicontinuity theorem for curves above. The assumption that $\mathscr{G}_0$ is lisse on X_0 is used to say that $H^0_{X\setminus U}(X, \mathscr{G}) = 0$ in order to apply the previous lemma.

□

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