Math 396. Vector fields and local coordinates

1. Motivation

Let M be a smooth manifold and let $\vec{v}_1, \dots, \vec{v}_n$ be pointwise linearly independent smooth vector fields on an open subset $U \subseteq M$ ($n \geq 1$). One simple example of such vector fields is $\partial_{x_1}, \dots, \partial_{x_n}$ on a coordinate domain for local smooth coordinates $\{x_1, \dots, x_N\}$ on an open set U in M. Can all examples be described in this way (locally) for suitable smooth coordinates?

Choose a point $m_0 \in U$. It is very natural (e.g., to simplify local calculations) to ask if there exists a local C^∞ coordinate system $\{x_1, \dots, x_N\}$ on an open subset $U_0 \subseteq U$ around m_0 such that $\vec{v}_i|_{U_0} = \partial_{x_i}$ in $\mathrm{Vec}_M(U_0) = (TM)(U_0)$ for $1 \leq i \leq n$. The crux of the matter is to have such an identity across an entire open neighborhood of m_0. If we only work in the tangent space *at* the point m_0, which is to say we inquire about the identity $\vec{v}_i(m_0) = \partial_{x_i}|_{m_0}$ in $\mathrm{T}_{m_0}(U_0) = \mathrm{T}_{m_0}(M)$ for $1 \leq i \leq n$, then the answer is trivial (and not particularly useful): we choose local C^∞ coordinates $\{y_1, \dots, y_N\}$ near m_0 and write $\vec{v}_i(m_0) = \sum c_{ij} \partial_{y_j}|_{m_0}$, so the $N \times n$ matrix (c_{ij}) has independent columns. We extend this to an invertible $N \times N$ matrix, and then make a *constant* linear change of coordinates on the y_j's via the inverse matrix to get to the case $c_{ij} = \delta_{ij}$. Of course, such new coordinates are only adapted to the situation at m_0. If we try to do the same construction by considering the matrix of functions (h_{ij}) with $\vec{v}_i = \sum h_{ij} \partial_{y_j}$ near m_0, the change of coordinates will now typically be *non-constant* and so in this new cooordinate system the coefficient functions for our vector fields will involve partials of the h_{ij}'s (which were 0 in the constant case), thereby leading to a big mess.

There is a very good reason why the problem over an open set (as opposed to at a single point) is complicated: usually no such coordinates exist! Indeed, if $n \geq 2$ then the question generally has a negative answer because there is an obstruction that is often non-trivial: since the commutator vector field $[\partial_{x_i}, \partial_{x_j}]$ vanishes for any i, j, if such coordinates are to exist around m_0 then the commutator vector fields $[\vec{v}_i, \vec{v}_j]$ must vanish near m_0. (Note that the concept of commutator of vector fields is meaningless when working on a single tangent space; it only has meaning when working with vector fields over open sets. This is why we had no difficulties when working at a single point m_0.)

For $n \geq 2$, the necessary condition of vanishing of commutators for pointwise independent vector fields usually fails. For example, on an open set $U \subseteq \mathbf{R}^3$ consider a pair of smooth vector fields

$$\vec{v} = \partial_x + f\partial_z, \ \ \vec{w} = \partial_y + g\partial_z$$

for smooth functions f and g on U. These are visibly pointwise independent vector fields, but

$$[\vec{v}, \vec{w}] = (\partial_x g - \partial_y f)\partial_z,$$

and so a necessary condition to have $\vec{v} = \partial_{x_1}$ and $\vec{w} = \partial_{x_2}$ for local C^∞ coordinates $\{x_1, x_2, x_3\}$ near $m_0 \in U$ is $\partial_x g = \partial_y f$ near m_0. We shall see later (as part of the proof of the Frobenius Integrability Theorem) that such vanishing conditions on commutators are in fact necessary and sufficient for an affirmative answer to our question. The proof requires as input a special case in which the commutator condition is vacuous: $n = 1$. Indeed, since $[\vec{v}, \vec{v}] = 0$ for any smooth vector field, in the case $n = 1$ we see no obvious reason why our question cannot always have an affirmative answer. The aim of this handout is to show how to use the theory of vector flow along integral curves to prove such a result.

In the case $n = 1$, pointwise-independence for the singleton $\{\vec{v}_1\}$ amounts to pointwise non-vanishing. Hence, we may restate the goal we have: if $\vec{v}$ is a smooth vector field on an open set $U \subseteq M$ and $\vec{v}(m_0) \neq 0$ for some $m_0 \in U$ (so $\vec{v}(m) \neq 0$ for m near m_0, by continuity of $\vec{v} : U \to TM$), then there exists a local C^∞ coordinate system $\{x_1, \dots, x_N\}$ near m_0 in U such that $\vec{v} = \partial_{x_1}$ near m_0.

Example 1.1. Consider the circular vector field $\vec{v} = -y\partial_x + x\partial_y$ on $M = \mathbf{R}^2$ with constant speed $r \geq 0$ on the circle of radius r centered at the origin. This vector field vanishes at the origin, but for $m_0 \neq (0,0)$ we have $\vec{v}(m_0) \neq 0$. Let $U_0 = \mathbf{R}^2 - L$ for a closed half-line L emanating from the origin and not passing through m_0. For a suitable θ_0, trigonometry provides a C^∞ parameterization $(0,\infty) \times (\theta_0, \theta_0 + 2\pi) \simeq U_0$ given by $(r,\theta) \mapsto (r\cos\theta, r\sin\theta)$, and $\partial_\theta = \vec{v}|_{U_0}$. Thus, in this special case we get lucky: we already "know" the right coordinate system to solve the problem. But what if we didn't already know trigonometry? How would we have been able to figure out the answer in this simple special case?

Example 1.2. In order to appreciate the non-trivial nature of the general assertion we are trying to prove, let us try to prove it in general "by hand" (i.e., using just basic definitions, and no substantial theoretical input such as the theory of vector flow along integral curves). We shrink U around m_0 so that there exist local C^∞ coordinates $\{y_1, \dots, y_N\}$ on U. Hence, $\vec{v} = \sum h_j \partial_{y_j}$, and since $\vec{v}(m_0) = \sum h_j(m_0)\partial_{y_j}|_{m_0}$ is nonzero, we have $h_j(m_0) \neq 0$ for some j. By relabelling, we may assume $h_1(m_0) \neq 0$. By shrinking U around m_0, we may assume h_1 is non-vanishing on U (so $\vec{v}$ is non-vanishing on U). We wish to find a C^∞ coordinate system $\{x_1, \dots, x_N\}$ near m_0 inside of U such that $\vec{v} = \partial_{x_1}$ near m_0.

What conditions are imposed on the x_i's in terms of the y_j's? For any smooth coordinate system $\{x_i\}$ near m_0, $\partial_{y_j} = \sum(\partial_{y_j} x_i)\partial_{x_i}$ near m_0, so near m_0 we have

$$\vec{v} = \sum_j h_j \sum_i (\partial_{y_j} x_i)\partial_{x_i} = \sum_i (\sum_j h_j \partial_{y_j}(x_i))\partial_{x_i}.$$

Thus, the necessary and sufficient conditions are two-fold: $x_1, \dots, x_N$ are smooth functions near m_0 such that $\det((\partial_{y_j} x_i)(m_0)) \neq 0$ (this ensures that the x_i's are local smooth coordinates near m_0, by the inverse function theorem) and

$$\sum_j h_j \partial_{y_j}(x_i) = \delta_{ij}$$

for $1 \leq i \leq N$. This is a system of linear first-order PDE's in the N unknown functions $x_i = x_i(y_1, \dots, y_N)$ near m_0. We have already seen that the theory of first-order linear ODE's is quite substantial, and here were are faced with a PDE problem. Hence, our task now looks to be considerably less straightforward than it may have seemed to be at the outset.

The apparent complications are an illusion: it is because we have written out the explicit PDE's in local coordinates that things look complicated. It will be seen in the proof in the next section that when we restate our problem in *geometric* language, the idea for how to solve the problem essentially drops into our lap without any pain at all. This is reminiscent of a basic principle we learned in linear algebra: geometric language is very effective at cutting through apparent difficulties in coordinatized problems.

2. Main result

The fundamental theorem is this:

Theorem 2.1. *Let M be a smooth manifold and $\vec{v}$ a smooth vector field on an open set $U \subseteq M$. Let $m_0 \in U$ be a point such that $\vec{v}(m_0) \neq 0$. There exists a local C^∞ coordinate system $\{x_1, \dots, x_N\}$ on an open set $U_0 \subseteq U$ containing m_0 such that $\vec{v}|_{U_0} = \partial_{x_1}$.*

This theorem is proved in the course text as Theorem 7 in Chapter 5. You may like the picture there, and perhaps you may also prefer the proof there. (It is the same proof as we give, except we include some more details and geometric explanation.)

Proof. What is the geometric meaning of what we are trying to do? We are trying to find local coordinates $\{x_i\}$ an open open U_0 in U around m_0 so that the integral curves for $\vec{v}|_{U_0}$ are exactly flow along the x_1-direction at unit speed. That is, in this coordinate system for any point ξ near m_0 the integral curve for $\vec{v}$ through ξ is coordinatized as $c_\xi(t) = (t + x_1(\xi), x_2(\xi), \dots, x_N(\xi))$ for t near 0. This suggests that we try to find a local coordinate system around m_0 such that the first coordinate is "time of vector flow". Recall from our study of openness of the domain of definition for vector flow along integral curves in manifolds that for a sufficiently small open $U_0 \subseteq U$ around m_0 there exists $\varepsilon > 0$ such that for all $\xi \in U_0$ the maximal interval of definition for the integral curve c_ξ contains $(-\varepsilon, \varepsilon)$. More specifically, we proved that the vector-flow mapping

$$X_{\vec{v}} : \mathscr{D}(\vec{v}) \to M$$

defined by $(t, \xi) \mapsto c_\xi(t)$ has *open* domain of definition in $\mathbf{R} \times M$ and is a smooth mapping. Thus, for small $\varepsilon > 0$ and small $U_0 \subseteq U$ around m_0, we have that $(-\varepsilon, \varepsilon) \times U_0$ is contained in $\mathscr{D}(\vec{v})$ (as $\{0\} \times M \subseteq \mathscr{D}(\vec{v})$). The mapping $X_{\vec{v}}$, restricted to $(-\varepsilon, \varepsilon) \times U_0$, will be the key to creating a coordinate system on M near m_0 such that the time-of-flow parameter t is the first coordinate.

Here is the construction. We first choose an arbitrary smooth coordinate system $\phi : W \to \mathbf{R}^N$ on an open around m_0 that "solves the problem at m_0". That is, if $\{y_1, \dots, y_N\}$ are the component functions of ϕ, then $\partial_{y_1}|_{m_0} = \vec{v}(m_0)$. This is the trivial pointwise version of the problem that we considered at the beginning of this handout (and it has an affirmative answer precisely because the singleton $\{\vec{v}(m_0)\}$ in $\mathrm{T}_{m_0}(M)$ is an independent set; i.e., $\vec{v}(m_0) \neq 0$). Making a constant translation (for ease of notation), we may assume $y_j(m_0) = 0$ for all j. In general this coordinate system will fail to "work" at any other points, and we use vector flow to fix it. Consider points on the slice $W \cap \{y_1 = 0\}$ in M near m_0. In terms of y-coordinates, these are points $(0, a_2, \dots, a_N)$ with small $|a_j|$'s. By openness of the domain of flow $\mathscr{D}(\vec{v}) \subseteq \mathbf{R} \times M$, there exists $\varepsilon > 0$ such that, after perhaps shrinking W around m_0, $(-\varepsilon, \varepsilon) \times W \subseteq \mathscr{D}(\vec{v})$.

By the definition of the y_i's in terms of ϕ, $\phi(W \cap \{y_1 = 0\})$ is an open subset in $\{0\} \times \mathbf{R}^{N-1} = \mathbf{R}^{N-1}$, and ϕ restricts to a C^∞ isomorphism from the smooth hypersurface $W \cap \{y_1 = 0\}$ onto $\phi(W \cap \{y_1 = 0\})$. Consider the vector-flow mapping

$$\Psi : (-\varepsilon, \varepsilon) \times \phi(W \cap \{y_1 = 0\}) \to M$$

defined by

$$(t, a_2, \dots, a_N) \mapsto X_{\vec{v}}(t, \phi^{-1}(0, a_2, \dots, a_N)) = c_{\phi^{-1}(0, a_2, \dots, a_N)}(t).$$

By the theory of vector flow, this is a *smooth* mapping. (This is the family of solutions to a first-order initial-value problem with varying initial parameters $a_2, \dots, a_N$ near 0. Thus, the smoothness of the map is an instance of smooth dependence on varying initial conditions for solutions to first-order ODE's.) Geometrically, we are trying to parameterize M near m_0 by starting on the hypersurface $H = \{y_1 = 0\}$ in W (with coordinates given by the restrictions $y'_2, \dots, y'_N$ of $y_2, \dots, y_N$ to H) and flowing away from H along the vector field $\vec{v}$; the time t of flow provides the first parameter in our attempted parameterization of M near m_0.

Note that $\Psi(0, 0, \dots, 0) = c_{m_0}(0) = m_0$. Is Ψ a parameterization of M near m_0? That is, is Ψ a local C^∞ isomorphism near the origin? If so, then its local inverse near m_0 provides a C^∞ coordinate system $\{x_1, \dots, x_N\}$ with $x_1 = t$ measuring time of flow along integral curves for $\vec{v}$ with their *canonical* parameterization (as integral curves). Thus, it is "geometrically obvious" that in such a coordinate system we will have $\vec{v} = \partial_{x_1}$ (but we will also derive this by direct calculation below).

To check the local isomorphism property for Ψ near the origin, we use the inverse function theorem: we have to check $\mathrm{d}\Psi(0,\dots,0)$ is invertible. In terms of the local C^∞ coordinates $\{t, y'_2, \dots, y'_N\}$ near the origin on the source of Ψ and $\{y_1, \dots, y_N\}$ near $m_0 = \Psi(0,\dots,0)$ on the target of Ψ, the $N \times N$ Jacobian matrix for $\mathrm{d}\Psi(0,\dots,0)$ has lower $(N-1)\times(N-1)$ block given by the identity matrix (i.e., $(\partial_{y'_j} y_i)(0,\dots,0) = \delta_{ij}$) because $\partial_{y'_j} y_i = \delta_{ij}$ at points on $W \cap \{y_1 = 0\}$ (check! It is *not* true at most other points of W).

What is the left column of the Jacobian matrixat $(0,\dots,0)$? Rather generally, if ξ is the point with y-coordinates $(t_0, a_2, \dots, a_N)$ then the t-partials $(\partial_t y_i)(t_0, a_2, \dots, a_N)$ are the coefficients of the velocity vector $c'_\xi(t_0)$ to the integral curve c_ξ of $\vec{v}$ at time t_0, and such a velocity vector is equal to $\vec{v}(c_\xi(t_0))$ by the *definition* of the concept of integral curve. Hence, setting $t_0 = 0$, $c'_\xi(0) = \vec{v}(c_\xi(0)) = \vec{v}(\xi)$, so taking $\xi = m_0 = \Psi(0,\dots,0)$ gives that $(\partial_t y_i)(0,\dots,0)$ is the coefficient of $\partial_{y_i}|_{m_0}$ in $\vec{v}(m_0)$. Aha, but recall that we *chose* $\{y_1,\dots,y_N\}$ at the outset so that $\vec{v}(m_0) = \partial_{y_1}|_{m_0}$. Hence, the left column of the Jacobian matrix at the origin has $(1,1)$ entry 1 and all other entries equal to 0. Since the lower right $(N-1)\times(N-1)$ block of the Jacobian matrix is the identity, this finishes the verification of invertibility of $\mathrm{d}\Psi(0,\dots,0)$, so Ψ gives a local C^∞ isomorphism between opens around $(0,\dots,0)$ and m_0.

Let $\{x_1,\dots,x_N\}$ be the C^∞ coordinate system near m_0 on M given by the local inverse to Ψ. We claim that $\vec{v} = \partial_{x_1}$ near m_0. By definition of the x-coordinate system, $(a_1,\dots,a_n)$ is the tuple of x-coordinates of the point $X_{\vec{v}}(a_1, \phi^{-1}(0, a_2, \dots, a_n)) \in M$. Thus, ∂_{x_1} is the field of velocity vectors along the parameteric curves $X_{\vec{v}}(t, \phi^{-1}(0, a_2, \dots, a_n)) = c_{\phi^{-1}(0,a_2,\dots,a_n)}(t)$ that are the integral curves for the smooth vector field $\vec{v}$ with initial positions (time 0) at points $\phi^{-1}(0, a_2, \dots, a_n) \in W \cap \{y_1 = 0\}$ near m_0. Thus, the velocity vectors along these parametric curves are exactly the vectors from the smooth vector field $\vec{v}$! This shows that the smooth vector fields ∂_{x_1} and $\vec{v}$ coincide near m_0. ■