A torsion Jacquet–Langlands correspondence

Frank Calegari and Akshay Venkatesh

Abstract. We prove a numerical form of a Jacquet–Langlands correspondence for torsion classes on arithmetic hyperbolic 3-manifolds.

Contents

Acknowledgments

The first author (F.C.) would like to thank Matthew Emerton for many conversations regarding possible integral formulations of reciprocity, and to thank Nathan Dunfield; the original calculations suggesting that a torsion Jacquet–Langlands theorem might be true were done during the process of writing our joint paper [**12**], and the data thus produced proved very useful for suggesting some of the phenomena we have studied in this paper. He would also like to thank Kevin Hutchinson for some helpful remarks concerning a theorem of Suslin. F.C. was supported during the preparation of this paper by a Sloan fellowship and by the National Science Foundation (CAREER Grant DMS-0846285).

The second author (A.V.) would like to express his gratitude to Nicolas Bergeron, with whom he wrote the paper [**6**], for many fruitful discussions and thoughts about the analytic behavior of analytic torsion; to Laurent Clozel, who suggested to him the importance of investigating torsion in cohomology; to Kartik Prasanna, for helpful discussions and references concerning his work; and to Avner Ash for encouraging words. He was supported during the preparing of this paper by a Sloan fellowship, an National Science Foundation grant, and a David and Lucile Packard Foundation fellowship; he gratefully thanks these organizations for their support.

The mathematical debt this paper owes to the work of Cheeger [**17**] and Müller [**53**] should be clear: the main result about comparison of torsion homology makes essential use of their theorem.

Some of the original ideas of this paper were conceived during the conference "Explicit Methods in Number Theory" in Oberwolfach during the summer of 2007. Various parts of the manuscript were written while the authors were resident at the following institutions: New York University, Stanford University, Northwestern University, the Institute for Advanced Study, the University of Sydney, and the Brothers K coffee shop; we thank all of these institutions for their hospitality.

We thank Romyar Sharifi and Krzysztof Klosin for helpful comments, especially on the material of Chapter 4. We thank Jean Raimbault and Gunter Harder for pointing out errors in Chapter 5. We thank Aurel Page for providing a presentation of $\mathrm{PGL}_2\left(\mathbf{Z}\left[\dfrac{1+\sqrt{-491}}{2}\right]\right)$.

CHAPTER 1

Introduction

1.1. Introduction

The main goal of this manuscript is to study the role that torsion in the homology of arithmetic groups play in the arithmetic Langlands program. In particular, we will establish a weak version of the Jacquet–Langlands correspondence in the torsion setting.

The Langlands program conjecturally relates homology of arithmetic groups to Galois representations; an emerging extension of this program predicts that this relationship exists not only for the characteristic zero homology but also for the torsion. One consequence of these conjectures is that the integral homology of arithmetic groups for different inner forms of the same group will be related in a non-obvious way, and this conjectural correspondence can be studied even without discussing Galois representations. (Indeed, since the first version of the manuscript was complete, the relationship to Galois representations has, in fact, been greatly clarified by the recent work of Scholze. We discuss this further in §2.1.5.)

An interesting class of arithmetic groups is provided by arithmetic Fuchsian subgroups $\Gamma \leqslant \mathrm{PGL}_2(\mathbf{C}) \cong \mathrm{SL}_2(\mathbf{C})/\{\pm I\}$, in particular, $\mathrm{PGL}_2(\mathscr{O}_F)$ and its congruence subgroups, where F is an imaginary quadratic field. For such groups, it has been observed (both numerically and theoretically) that $H_1(\Gamma, \mathbf{Z}) = \Gamma^{\mathrm{ab}}$ has a large torsion subgroup [**6, 64**].

The primary goal of the current paper is to study the possibility of "Jacquet–Langlands correspondences" for torsion. This amounts to relationships between $H_{1,\mathrm{tors}}(\Gamma)$ and $H_{1,\mathrm{tors}}(\Gamma')$, for certain *incommensurable* groups Γ, Γ'. We summarize some of our main theorems in the following section.

Our main tool is analytic torsion, and a significant part of the paper is handling the analysis required to apply the theorem of Cheeger and Müller to a noncompact hyperbolic manifold (Chapter 5). However, as will become clear from the theorem statements, the study of the Jacquet-Langlands correspondence also requires a careful study of the integral theory of modular forms. These issues are studied in Chapter 4. The final chapter, Chapter 6, gives the proofs.

1.1.1. Let $F/\mathbf{Q}$ be an imaginary quadratic field of odd class number[1], and $\mathfrak{n}$ an ideal of F; let $\mathfrak{p}, \mathfrak{q}$ be prime ideals of $\mathscr{O}_F$ that do not divide $\mathfrak{n}$. Let $\Gamma_0(\mathfrak{n})$ be the subgroup of $\mathrm{PGL}_2(\mathscr{O}_F)$ corresponding to matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $\mathfrak{n}|c$.

[1]The assumption on the class number is made in this section (and this section only) purely for exposition — it implies that the relevant adelic quotient has only one connected component, so that the manifold $Y_0(\mathfrak{n})$ considered below is indeed the "correct" object to consider.

Let D be the unique quaternion algebra over F ramified at $\mathfrak{p}, \mathfrak{q}$, $\mathscr{O}_D$ a maximal order in D. Let Γ'_0 be the image of $\mathscr{O}_D^\times/\mathscr{O}_F^\times$ in $\mathrm{PGL}_2(\mathbf{C})$; it is a co-compact lattice. There is a canonical conjugacy class of surjections $\Gamma'_0 \twoheadrightarrow \mathrm{PGL}_2(\mathscr{O}_F/\mathfrak{n})$, and thus one may define the analogue $\Gamma'_0(\mathfrak{n})$ of the subgroup $\Gamma_0(\mathfrak{n})$ in this context as the preimage in Γ'_0 of the upper triangular subgroup of $\mathrm{PGL}_2(\mathscr{O}_F/\mathfrak{n})$. It can be alternately described as $\mathscr{O}_{D,\mathfrak{n}}^\times/\mathscr{O}_F^\times$, where $\mathscr{O}_{D,\mathfrak{n}}$ is an Eichler order of level $\mathfrak{n}$.

Let $Y_0(\mathfrak{n})$ be the finite volume, hyperbolic 3-manifold $\mathbf{H}^3/\Gamma_0(\mathfrak{n})$. Note that $Y_0(\mathfrak{n})$ is *non-compact*. Let $Y'_0(\mathfrak{n})$ be the quotient of $\mathbf{H}^3$ by $\Gamma'_0(\mathfrak{n})$; it is a *compact* hyperbolic 3-manifold.

We refer to [**48**] for background and further details on these types of construction, especially Chapter 6 for orders in quaternion algebras.

The homology group of both $Y_0(\mathfrak{n})$ and $Y'_0(\mathfrak{n})$ is equipped with a natural action of the *Hecke algebra* (see § 3.4.3 for definitions). A theorem of Jacquet–Langlands implies that there is a *Hecke-equivariant injection*

$$\dim H_1(Y'_0(\mathfrak{n}), \mathbf{C}) \hookrightarrow \dim H_1(Y_0(\mathfrak{npq}), \mathbf{C}). \tag{1.1.1.1}$$

The proof of (1.1.1.1) is based on the fact that the length spectra of the two manifolds are closely related.

When considering *integral* homology, it will be convenient to replace the group $H_1(Y, \mathbf{Z})$ by a slightly different group $H_1^{E^*}(Y, \mathbf{Z})$ (the *dual-essential homology*, see definition 6.7.2 and § 5.4.9). The difference between the two groups is easy to understand and roughly accounted for by classes which arise from congruence quotients of the corresponding arithmetic group (cf. §3.7).

We present now some of our results:

Theorem A. *(Proved in § 6.8.) Suppose that* $\dim H_1(Y_0(\mathfrak{npq}), \mathbf{C}) = 0$. *Then the order of the finite group* $H_1^{E^*}(Y'_0(\mathfrak{n}), \mathbf{Z})$ *divides the order of the finite group* $H_1^{E^*}(Y_0(\mathfrak{npq}), \mathbf{Z})^{\mathfrak{pq}-\mathrm{new}}$*, except possibly at orbifold primes* (*which are at most* 3)*.*

The superscript $\mathfrak{pq}-$new means "new at $\mathfrak{p}$ and $\mathfrak{q}$": we take the quotient of the homology group by classes pulled-back from levels $\mathfrak{np}$ and $\mathfrak{nq}$.

The proof uses the (usual) Jacquet–Langlands correspondence, the Cheeger–Müller theorem [**17, 53**] and the congruence subgroup property for S-arithmetic SL_2. In order to apply the Cheeger–Müller theorem to *non-compact* hyperbolic manifolds, we are forced to address a certain number of technical issues, although we are able to take a shortcut especially adapted to our situation.

Remark 1.1.2. Our results apply under substantially weaker hypotheses; the strong vanishing assumption is to simplify the statement as far as possible.

However, even this rather strong vanishing hypothesis is frequently satisfied. Note, for example, that the cusps of $Y_0(\mathfrak{m})$ do not contribute to its complex cohomology (Lemma 5.4.4): their cross-section is a quotient of a torus by negation. Similarly, $\mathrm{SL}_2(\mathscr{O}_F)$ very often has base-change homology, but this often does not extend to $\mathrm{PGL}_2(\mathscr{O}_F)$, cf. Remark 4.5.2, second point.

The constraint on 2 and 3 also arises from orbifold issues: for 3 this is not a serious constraint and could presumably be removed with more careful analysis. However, we avoid the prime 2 at various other points in the text for a number of other technical reasons.

REMARK 1.1.3. There is (conditional on the congruence subgroup property for certain division algebras) a corresponding result in a situation where Y_0, Y_0' both arise from quaternion algebras. In that case, one can *very quickly* see, using the Cheeger–Müller theorem and the classical Jacquet-Langlands correspondence, that there should be *some* relationship between between the sizes of torsion homology of Y_0 and Y_0'. The reader may wish to examine this very short and simple argument (see Chapter 6, up to the first paragraph of the proof of Theorem 6.4.1).

Nonetheless, this paper focuses on the case where one of Y_0 and Y_0' is split, thus forcing us to handle, as mentioned above, a host of analytic complications. One reason is that, in order to sensibly "interpret" Theorem 6.4.1 – e.g., in order to massage it into an arithmetically suggestive form like Theorem A or (below) Theorem B, one needs the congruence subgroup property for the S-units – and, as mentioned, this is not known for a quaternion algebra. But also we find and study many interesting phenomenon, both analytic and arithmetic, peculiar to the split case (e.g., the study of Eisenstein regulator, or the relationship to K-theory provided by Theorem 4.5.1).

It is natural to ask for a more precise version of Theorem A, replacing divisibility by an equality. As we shall see, this is *false* – and false for an interesting reason related to the nontriviality of $K_2(\mathscr{O}_F)$. For example we show:

THEOREM A†. *(Proved in § 6.8; see also following sections.) Notation and assumptions as in Theorem A, suppose that* $\mathfrak{n}$ *is the trivial ideal. Then the ratio of orders*

$$\frac{|H_1^{E^*}(Y_0(\mathfrak{pq}), \mathbf{Z})^{\mathfrak{pq}-\text{new}}|}{|H_1^{E^*}(Y_0'(1), \mathbf{Z})|}$$

is divisible by the order of $K_2(\mathscr{O})$*, up to primes dividing* $6 \gcd(N\mathfrak{p} - 1, N\mathfrak{q} - 1)$.

Although our computational evidence is somewhat limited, we have no reason to suppose that the divisibility in this theorem is not actually an *equality.* We refer to Theorem 4.5.1 for a related result. To further understand the relationship of K-theory and the torsion Langlands program seems a very interesting task.

If $H_1(Y_0'(\mathfrak{n}), \mathbf{C}) \neq 0$, then we wish to compare two infinite groups. In this context, a host of new complications, including issues related to "level raising phenomena," arise. In this case, instead of controlling the ratios of the torsion subgroups between the split and non split side, we relate this ratio to a ratio of periods arising from automorphic forms π and their Jacquet–Langlands correspondent π^{JL}. This period ratio is related to work of Prasanna for $\mathrm{GL}(2)/\mathbf{Q}$ (see 6.5.1).

In this context, a sample of what we prove is:

THEOREM B. *(Proved in § 6.8.) Suppose that* l *is a prime* $\notin \{2, 3\}$ *such that:*

(1) $H_1(Y_0(\mathfrak{nq}), \mathbf{Z}_l) = 0$,
(2) $H_1(Y_0(\mathfrak{np}), \mathbf{Z}_l)$ *and* $H_1(Y_0(\mathfrak{npq}), \mathbf{Z}_l)$ *are* l*–torsion free,*
(3) $H_1(Y_0'(\mathfrak{n}), \mathbf{Q}_l) = 0$ *— equivalently:* $H_1(Y_0'(\mathfrak{n}), \mathbf{C}) = 0$.

Then l *divides* $H_1^{E^*}(Y_0'(\mathfrak{n}), \mathbf{Z})$ *if and only if* l *divides* Δ*, where* $(N(\mathfrak{p}) - 1)\Delta$ *is the determinant of* $T_{\mathfrak{q}}^2 - (N(\mathfrak{q}) + 1)^2$ *acting on* $H_1(Y_0(\mathfrak{np}), \mathbf{Q})$*, and* $T_{\mathfrak{q}}$ *denotes the* $\mathfrak{q}$*-Hecke operator.*

Thus the naive analog of Theorem A – simply asking for a divisibility between the sizes of torsion parts – is *false* here: We have extra torsion on the compact $Y_0'(\mathfrak{n})$, at primes dividing Δ. By "extra," we mean that there is no corresponding torsion for $Y_0(\mathfrak{npq})$. How should we interpret this extra torsion, i.e. how to interpret the number Δ in terms of modular forms on the split manifold Y_0?

Suppose for simplicity that ℓ does not divide $N(\mathfrak{p})-1$. Then ℓ divides Δ exactly when $T_{\mathfrak{q}}^2-(N(\mathfrak{q})+1)^2$ has a nontrivial kernel on $H_1(Y_0(\mathfrak{np}),\mathbf{F}_\ell)$. In view of this, an interpretation of Δ is given by the following "level raising theorem." We state in a somewhat imprecise form for the moment. (To compare with the previous theorem, replace $\mathfrak{m}$ by $\mathfrak{np}$).

Theorem C. *(Proved in § 5.5; see also § 4.3.) Suppose* $[c] \in H_1(Y_0(\mathfrak{m}),\mathbf{F}_l)$ *is a non-Eisenstein (see Definition 3.8.1) Hecke eigenclass, in the kernel of* $T_{\mathfrak{q}}^2 - (\mathrm{N}(\mathfrak{q})+1)^2$.

Then there exists $[\tilde{c}] \in H_1(Y_0(\mathfrak{mq}),\mathbf{F}_l)$ *with the same generalized Hecke eigenvalues as c at primes not dividing* $\mathfrak{q}$*, and not in the image of the pullback degeneracy map* $H_1(Y_0(\mathfrak{m}))^2 \to H_1(Y_0(\mathfrak{mq}))$.

To understand these Theorems (and their relationship) better, let us examine the situation over $\mathbf{Q}$ and explain analogues of Theorems B and C in that setting.

Let N be an integer and p, q primes, and denote by $\Gamma_0(N), \Gamma_0(Np)$ etc. the corresponding congruence subgroups of $\mathrm{SL}_2(\mathbf{Z})$; denote by $\Gamma_0^*(N)$ the units in the level N Eichler order inside the quaternion algebra over $\mathbf{Q}$ ramified at p, q. Define Δ analogously, replacing $H_1(Y_0(\mathfrak{np}),\mathbf{C})$ by the homology $H_1(\Gamma_0(Np),\mathbf{C})$ and $T_{\mathfrak{q}}$ by the usual Hecke operator T_q.

Then, if l divides Δ, there exists a Hecke eigenclass $f \in H_1(\Gamma_0(Np),\mathbf{F}_l)$ which is annihilated by $T_q^2-(q+1)^2$. Under this assumption there exists:

(b) a Hecke eigenform $f^* \in H_1(\Gamma_0^*(N),\mathbf{F}_l)$

(c) a "new at q" Hecke eigenform $\tilde{f} \in H_1(\Gamma_0(Npq),\mathbf{F}_l)$

both with the same (mod-l) Hecke eigenvalues as f.

Indeed, (c) follows from Ribet's level-raising theorem, and (b) then follows using (c) and Jacquet–Langlands. In terms of representation theory this phenomena can be described thus: f gives rise to a principal series representation π of $\mathrm{GL}_2(\mathbf{Q}_q)$ on an $\mathbf{F}_l$-vector space. The condition that $T_{\mathfrak{q}}^2-(N(\mathfrak{q})+1)^2$ kills f forces this representation to be *reducible*: it contains a twist of the Steinberg representation.

Our Theorems B and C can be seen as analogues of (b) and (c) respectively. In all cases one produces homology at primes l dividing Δ on a quaternion algebra or at higher level. There are significant differences: most notably, in case (b) the class g lifts to characteristic zero whereas in case (B) it is a pure torsion class. More distressing is that Theorem B, as well as Theorem A and all our related results, gives no information about Hecke eigenvalues.

1.1.4. Other work. As mentioned, the current paper is a shortened version of the manuscript [**16**]. It has been distilled, at the suggestion of a referee, to focus on the main theorems about the Jacquet-Langlands correspondence. We now briefly describe some of the contents of [**16**] that may be of interest in relation to our current results:

Chapter 5 of [**16**] is a more careful study of regulators, and, in particular, their relation to L-values. The present document retains only the most relevant part to the Jacquet–Langlands correspondence, in the present §6.5.

Chapter 7 of [**16**] gives a partial explanation of the rôle played by K_2, by interpreting K_2 in terms of certain Eisenstein Galois deformation rings.

Chapter 7 of [**16**] also studies phenomena related to restricting an even Galois representation of $\mathbf{Q}$ to an imaginary quadratic field. (For example, modularity theorems over imaginary quadratic fields gives rise to finiteness theorems for even Galois representations over $\mathbf{Q}$.)

Chapter 8 of [**16**] is an extensive list of numerical examples. Many of these examples motivated the theorems of the current paper, and give evidence for conjectures that are formulated in Chapter 6.

1.2. A guide to reading this paper

We now describe the contents of this paper (this is not, however, a complete table of contents, simply a guide to some of the important subsections).

(1) Chapter 1 is the introduction.
- § 1.1.1 Some sample theorems.
- § 1.2 A guide to the paper, which you are now reading.

(2) Chapter 2 gives a general discussion of the context of the paper – in particular, extending the Langlands program to torsion classes, and how results such as Theorem A would fit into that extension.
- § 2 Background on automorphic forms and "reciprocity over $\mathbf{Z}$."
- § 2.1.5 Discussion of the recent work of Scholze and its relationship to our work.
- § 2.2 Conjectures about reciprocity related to torsion classes for GL_2; in particular, we formulate an $R = \mathbf{T}$ conjecture that guides many of our computations and conjectures.

(3) Chapter 3 gives notation:
- § 3.1 Summary of the most important notation; the reader could read this and refer to the other parts of this Chapter only as needed.
- § 3.4 Notation concerning modular forms, and related notions such as Eisenstein classes and congruence homology.

(4) Chapter 4, *Raising the Level: newforms and oldforms* compares spaces of modular forms at different levels.
- § 4.1 Ihara's lemma over imaginary quadratic fields. Similar results have also been proven by Klosin.
- § 4.2 This section gives certain "dual" results to Ihara's lemma.
- § 4.3 We prove level raising, an analogue of a result of Ribet over $\mathbf{Q}$.
- § 4.5 We study more carefully how level raising is related to cohomology of S-arithmetic groups. In particular we are able to produce torsion classes related to K_2.

(5) Chapter 5, *The split case*, analyzes various features of the noncompact case. In particular, it gives details required to extend the Cheeger–Müller theorem in the non-compact case.
- § 5.1 Notation and basic properties of the manifolds under consideration.
- § 5.3 Definitions of analytic and Reidemeister torsion in the split case. The difficulty is that, for a non-compact Riemannian manifold M,

cohomology classes need not be representable by *square integrable* harmonic forms.

§ 5.6 A review of the theory of Eisenstein series, or that part of it which is necessary. This contains, in particular, the definition of the regulator in the non-compact case, which is used at several prior points in the paper.

§5.7 Bounds on Eisenstein torsion, and the Eisenstein part of the regulator, in terms of L-values.

§ 5.8 This contains the core part of the analysis, and contains gives the proofs. The key point is the analysis of small eigenvalues of the Laplacian on the "truncation" of a non-compact hyperbolic 3-manifold. This analysis nicely matches with the trace formula on the non-compact manifold.

(6) Chapter 6, *Jacquet–Langlands*, studies "Jacquet–Langlands pairs" of hyperbolic 3-manifolds and in particular proves Theorem A, Theorem $A^{\dagger}$, and Theorem B quoted in § 1.1.1.

§6.2 Recollections on the classical Jacquet–Langlands correspondence.

§6.3 Some background on the notion of newform.

§ 6.4 We state and prove the theorem on comparison of homology in a very crude form.

§ 6.6 We show that that certain volume factors occurring in the comparison theorem correspond exactly to congruence homology.

§ 6.7 We introduce the notion of *essential homology* and *dual-essential homology*: two variants of homology which "cut out" congruence homology in two different ways (one a submodule, one a quotient).

§ 6.8 We begin to convert the prior Theorems into actual comparison theorems between orders of newforms. We consider in this section simple cases in which we can control as many of the auxiliary phenomena (level lowering, level raising, K-theoretic classes) as possible.
In particular, we prove Theorems A , $A^{\dagger}$, and B quoted earlier in the introduction.

§ 6.9 A conjectural discussion of the general case and what can be proved.

CHAPTER 2

Some Background and Motivation

In this chapter, we assume familiarity with the basic vocabulary of the Langlands program. Our goal is to formulate precise conjectures relating homology to Galois representations — at least, in the context of inner forms of GL_2 over number fields — with emphasis on writing them in a generality that applies to torsion homology. We then explain why these conjectures suggest that a numerical Jacquet–Langlands correspondence, relating the size of torsion homology groups between two arithmetic manifolds, should hold (Lemma 2.2.10).

2.1. Reciprocity over Z

2.1.1. Arithmetic Quotients. We recall here briefly the construction of arithmetic quotients of symmetric space and their connection to automorphic representations. Let $\mathbb{G}$ be a connected linear reductive algebraic group over $\mathbf{Q}$, and write $G_\infty = \mathbb{G}(\mathbf{R})$. Let A be a maximal $\mathbf{Q}$-split torus in the center of $\mathbb{G}$, and let $A^0_\infty \subset G_\infty$ denote the connected component of the $\mathbf{R}$-points of A. Furthermore, let K_∞ denote a maximal compact of G_∞ with connected component K^0_∞.

The quotient G_∞/K_∞ carries a G_∞-invariant Riemannian metric, with respect to which it is a Riemannian symmetric space. For our purposes, it will be slightly better to work with the (possibly disconnected) quotient

$$S := G_\infty/A^0_\infty K^0_\infty,$$

because G_∞ always acts on S in an orientation-preserving fashion, and S does not have a factor of zero curvature. (Later in the document we will work almost exclusively in a case where this coincides with G_∞/K_∞, anyway).

For example:

- If $G_\infty = \mathrm{GL}_2(\mathbf{R})$, then S is the union of the upper- and lower- half plane, i.e. $S = \mathbf{C} - \mathbf{R}$ with invariant metric $\dfrac{|dz|^2}{\mathrm{Im}(z)^2}$.
- If $G_\infty = \mathrm{GL}_2(\mathbf{C})$, then $S = \mathbf{H}^3$.

Now let $\mathbb{A}$ be the adele ring of $\mathbf{Q}$, and $\mathbb{A}_f$ the finite adeles. For any compact open subgroup K of $\mathbb{G}(\mathbb{A}_f)$, we may define an "arithmetic manifold" (or rather "arithmetic orbifold") $Y(K)$ as follows:

$$Y(K) := \mathbb{G}(F)\backslash(S \times \mathbb{G}(\mathbb{A}_f))/K = \mathbb{G}(F)\backslash\mathbb{G}(\mathbb{A})/A^0_\infty K^0_\infty K.$$

The orbifold $Y(K)$ may or may not be compact, and may be disconnected; in all cases it has finite volume with respect to the measure defined by the metric structure.

Since the components of S are contractible, the connected components of $Y(K)$ will be $K(\pi,1)$-spaces, and the cohomology of each component of $Y(K)$ is canonically isomorphic to the group cohomology of the fundamental group.

2.1.2. Cohomology of arithmetic quotients and reciprocity. Assume for now that $Y(K)$ is compact. Matsushima's formula [**9**] states that the cohomology $H^i(Y(K), \mathbf{C})$ decomposes as a direct sum of the $(\mathfrak{g}, K_\infty)$-cohomology:

$$H^i(Y(K), \mathbf{C}) = \bigoplus m(\pi) H^i(\mathfrak{g}, K_\infty; \pi_\infty).$$

Here the right hand side sum is taken over automorphic representations $\pi = \pi_\infty \otimes \pi_f$ for $\mathbb{G}(\mathbb{A}) = G_\infty \times \mathbb{G}(\mathbb{A}_f)$, and $m(\pi)$ is the dimension of the K-fixed subspace in π_f. The effective content of this equation is that "cohomology can be represented by harmonic forms."

2.1.3. Reciprocity in characteristic zero. Let us suppose π is a representation that "contributes to cohomology," that is to say, $m(\pi) \neq 0$ and $H^i(\mathfrak{g}, K; \pi_\infty) \neq 0$ for suitable K. Assume moreover that $\mathbb{G}$ is *simply connected*. Let $\widehat{G}$ denote the dual group of $\mathbb{G}$, considered as a reductive algebraic group over $\overline{\mathbf{Q}}$, and let ${}^L G$ be the semi-direct product of $\widehat{G}$ with $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ being the standard one (see e.g. [**11**, §2.1]). Then one conjectures (see, for example, Conjecture 3.2.1 of [**11**] for a more throrough and precise treatment) the existence (for each p) of a continuous irreducible Galois representation

$$\rho = \mathrm{rec}(\pi) : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to {}^L G(\overline{\mathbf{Q}}_p),$$

such that

(1) For every prime $\ell \neq p$, the representation $\rho | G_\ell$ is associated to π_ℓ via the local Langlands correspondence (in the cases where it is defined, e.g. [**37**]; in particular, it is always defined for primes ℓ at which $\rho | G_\ell$ is unramified.)
(2) ρ is unramified outside finitely many primes;
(3) If c is a complex conjugation, then $\rho(c)$ is *odd* (see [**6**, Proposition 6.1] for precise definition; if $\mathbb{G}$ is split, it means that the trace of $\rho(c)$ in the adjoint representation should be minimal among all involutions).
(4) $\rho | G_{F_p}$ is de Rham for any place $v|p$, with prescribed Hodge-Tate weights: they correspond to the conjugacy class of cocharacter $\mathbb{G}_m \to \widehat{G}$ that are dual to the half-sum of positive roots for G.

For a more precise formulation the reader should consult [**11**]; see their §2.4 for a discussion of the meaning of "de Rham" for general groups.

2.1.4. Cohomological reciprocity over Z. Starting with observations at least as far back as Grunewald [**33**] in 1972 (continuing with further work of Grunewald, Helling and Mennicke [**32**], Ash [**2**], Figueiredo [**26**], and many others), it has become apparent that something much more general should be true.

Let us suppose now that $\mathbb{G}$ is split simply connected, with complex dual group $\widehat{G}(\mathbf{C})$. Recall that for almost every prime ℓ the "Hecke algebra" $\mathscr{H}_\ell$ of $\mathbf{Z}$-valued functions on $\mathbb{G}(\mathbf{Q}_\ell)$, bi-invariant by $\mathbb{G}(\mathbf{Z}_\ell)$, acts by correspondences on $Y(K)$ – in our context, this action is defined in §3.4.3. Moreover [**31**]

$$\mathscr{H}_\ell \otimes \mathbf{Z}[\ell^{-1}] \cong \mathrm{Rep}(\widehat{G}(\mathbf{C})) \otimes \mathbf{Z}[\ell^{-1}].$$

Accordingly, if σ is any representation of $\widehat{G}(\mathbf{C})$, let $T_\sigma(\ell)$ be the corresponding element of $\mathscr{H}_\ell \otimes \mathbf{Z}[\ell^{-1}]$. For example, if $\mathbb{G} = \mathrm{SL}_2$, then $\widehat{G}(\mathbf{C}) = \mathrm{PGL}_2(\mathbf{C})$; the element T_{Ad} associated to the adjoint representation of $\widehat{G}(\mathbf{C})$ is the Hecke operator usually denoted T_{ℓ^2}, of degree $\ell^2 + \ell + 1$.

We may then rephrase the reciprocity conjecture in the less precise form:

- To every *Hecke eigenclass* $\alpha \in H^i(Y(K), \overline{\mathbf{Q}}_p)$, there exists a matching Galois representation $\rho_\alpha : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \widehat{G}(\overline{\mathbf{Q}}_p)$, so conditions (2) — (4) above hold, and for every σ and almost every ℓ,

$$T_\sigma(\ell)\alpha = \mathrm{trace}(\sigma \circ \rho_\alpha(\mathrm{Frob}_\ell)).$$

  Conversely, every Galois representation satisfying conditions (2), (3), (4) arises thus (for some i).

One difficulty with this conjecture is that it only (conjecturally) explains Galois representations in characteristic zero. Thus it is of interest to replace the coefficients $\overline{\mathbf{Q}}_p$ of cohomology $H^i(Y(K), \overline{\mathbf{Q}}_p)$ with torsion coefficients. For cleanness of formulation it is simplest to replace $\overline{\mathbf{Q}}_p$ above by $\overline{\mathbf{F}}_p$, or, more generally, a finite Artinian local $W(\overline{\mathbf{F}}_p)$-algebra A. The following is the "torsion" counterpart to the conjecture above:

- Let A be a finite length local $W(\overline{\mathbf{F}}_p)$-algebra. To every *Hecke eigenclass* α in the group $H^i(Y(K), A)$, there exists a matching Galois representation ρ_α with image in $\widehat{G}(A)$ such that conditions (2), (4) above hold, and, for every σ and almost every ℓ,

$$T_\sigma(\ell)\alpha = \mathrm{trace}(\sigma \circ \rho_\alpha(\mathrm{Frob}_\ell)).$$

  Conversely, every Galois representation satisfying conditions (2), (4) arises thus (for some i).

Note that we have deemed requirement (3) to be empty. By the universal coefficient theorem, the cohomology with coefficients in A (for any A) is determined by the cohomology with **Z**-coefficients, and this motivates a study of the integral cohomology groups $H^\bullet(Y(K), \mathbf{Z})$. We therefore refer to these conjectures together as "reciprocity over **Z**." Since $H^\bullet(Y(K), \mathbf{Z})$ may possess torsion, reciprocity over **Z** is *not* a consequence of the first conjecture alone.

The majority of the numerical evidence for this conjecture — particularly in the cases where it involves mod-p torsion classes and A is a finite field — is due to A. Ash and his collaborators; see [**2**].

Since the cohomology groups of $Y(K)$ are finitely generated, any eigenclass $\alpha \in H^i(Y(K), \overline{\mathbf{Q}}_p)$ will actually come from $H^i(Y(K), L)$ for some finite extension $L/\mathbf{Q}_p$ (a similar remark applies to eigenclasses in $W(\overline{\mathbf{F}}_p)$-algebras). However, the conjectures are more naturally stated over $\overline{\mathbf{Q}}_p$ than over L, since even if $\alpha \in H^i(Y(K), L)$, and even if there is a natural choice of dual group $\widehat{G}$ over $\mathbf{Q}$, it may be the case that ρ_α may only be conjugated to lie in $\widehat{G}(L')$ for some nontrivial extension L'/L. One can already see this for representations of finite groups, for example, two dimensional representations of the quaternion group have traces in $\mathbf{Q}_p$ for all p, but can not be conjugated into $\mathrm{GL}_2(\mathbf{Q}_p)$ unless $p \equiv 1 \mod 4$.

There is some flexibility as to whether we work with integral *homology* or integral *cohomology*. It will also be useful at some points to work with $\mathbf{Q}/\mathbf{Z}$-coefficients. The universal coefficient theorem relates these groups, and there are various ways in which to phrase the conjectures. For example, if $Y = Y(K)$ happens to be a compact 3-manifold, the torsion classes that don't arise from characteristic zero classes live in $H_1(Y, \mathbf{Z})$, $H^2(Y, \mathbf{Z})$, $H^1(Y, \mathbf{Q}/\mathbf{Z})$ and $H_2(Y, \mathbf{Q}/\mathbf{Z})$. For the purposes of this paper, it will be most convenient (aesthetically) to work with $H_1(Y, \mathbf{Z})$, and hence we phrase our conjectures in terms of homology.

One may also ask which *degree* the homology classes of interest live in. This is a complicated question; the paper [**13**] suggest that the Hecke algebra completed at a maximal ideal corresponding to a representation with large image acts faithfully in degree $\frac{1}{2}(\dim(G_\infty/K_\infty) - \mathrm{rank}(G_\infty) + \mathrm{rank}(K_\infty))$.

2.1.5. The Work of Scholze. In a recent paper, Scholze [**63**] has made substantial progress on one half of the integral reciprocity conjectures, namely, the construction of Galois representations associated to torsion classes in cohomology. The natural setting of Scholze's paper is the cohomology of congruence subgroups of $\mathrm{GL}_n(\mathscr{O}_F)$ for a CM field F/F^+. Under this restriction on F, there exists a unitary group $\mathrm{U}(n,n)/F^+$ associated to a PEL Shimura variety X so that the cohomology of $\mathrm{GL}_n(\mathscr{O}_F)$ occurs in the boundary cohomology of a natural compactification of X. The point is that $\mathrm{GL}(n)/F$ is naturally a Levi for some parabolic subgroup P of $\mathrm{U}(n,n)/F^+$. Naïvely, one might hope that the Betti cohomology of X can be related to the étale cohomology of X and hence directly to the interesting Galois representations one hopes to construct. In fact, this idea does not work, since it is quite possible for an étale cohomology group to carry an interesting Hecke action but an uninteresting Galois representation (which is what happens in this case). For example, when F is an imaginary quadratic field and $n = 2$, the corresponding Galois representations in the étale cohomology of X coming from the cohomology $\mathrm{GL}_2(\mathscr{O}_F)$ are given by Grossencharacters. The reason why this is not in contradiction with the Eichler–Shimura relations is that the latter implies that the characteristic polynomial of Frobenius at v satisfies a polynomial $\Phi_v(X)$ associated to the action of the Hecke operators at v, but there is no reason to suppose (in general) that $\Phi_v(X)$ may not factor as $P_v(X)Q_v(X)$ where $Q_v(X)$ carries the interesting eigenvalues of the Hecke operators, and $P_v(X)$ is the minimal polynomial of Frobenius at v. Scholze's argument is thus by necessity somewhat more subtle, and we direct the reader to [**63**] for details (we should also point out the recent paper [**36**] which proves a related theorem in characteristic zero).

Although this paper was written before Scholze's result was announced, the results and arguments in this paper are, perhaps surprisingly, not substantially affected or simplified by their existence. There are three main reasons for this. The first is that, even in the context of 2-dimensional representations of imaginary quadratic fields, the Galois representations constructed by Scholze are not known to satisfy local–global compatibility at the primes v dividing the residual characteristic. As a concrete example, one does not yet know whether the Galois representations associated to "ordinary" torsion classes (in the sense that U_v in invertible) are "ordinary" (in the sense that the image of the decomposition group D_v has an unramified quotient). The second is that many of our results are concerned mainly with the relationships between integral cohomology classes and/or regulators which, even for $F = \mathbf{Q}$, use arguments which are not directly related to the construction of the associated Galois representations. Two examples here (which still have clear relationships to the arithmetic of the associated Galois representations) are Ribet's results on level raising and Prasanna's results on ratios of Petersson norms and level lowering. The final reason is that this entire paper was written under the supposition that the most optimistic conjectures concerning torsion classes and Galois representations are all correct!

2.1.6. Discussion. The main difference between torsion classes and characteristic zero classes occurs most noticeably when G_∞ does *not* admit discrete series (although the study of torsion in the cohomology of Shimura varieties is also very interesting!). Whenever G_∞ does *not* admit discrete series, the set of automorphic Galois representations will not be Zariski dense in the deformation space of all (odd) p-adic Galois representations into ${}^L G$. When $\mathbb{G} = \mathrm{GL}(2)$ over an imaginary quadratic field F, this follows from Theorem 7.1 of [**15**], and Ramakrishna's arguments are presumably sufficiently general to establish a similar theorem in the general setting where G_∞ does not have discrete series. Thus, if one believes that the collection of *all* Galois representations is an object of interest, the representations obtained either geometrically (via the cohomology of a variety) or automorphically (via a classical automorphic form) do not suffice to study them.

On the other hand, suppose that one is only interested in geometric Galois representations. One obstruction to proving modularity results for such representations is apparent failure of the Taylor–Wiles method when G_∞ does not have discrete series. A recent approach to circumventing these obstructions can be found in the work of the first author and David Geraghty [**14**]. One of the main theorems of [**14**] is to prove minimal modularity lifting theorems for imaginary quadratic fields *contingent* on the existence of certain Galois representations. It is *crucial* in the approach of *ibid.* that one work with the *all* the cohomology classes over $\mathbf{Z}$, not merely those which lift to characteristic zero. In light of this, it is important to study the nature of torsion classes even if the goal is ultimately only to study motives.

Having decided that the (integral) cohomology of arithmetic groups should play an important role in the study of reciprocity and Galois representations, it is natural to ask what the role of Langlands principle of *functoriality* will be. Reciprocity compels us to believe the existence of (some form of) functoriality on the level of the cohomology of arithmetic groups. One of the central goals of this paper is *to give some evidence towards arithmetic functoriality in a context in which it is not a consequence of a classical functoriality for automorphic forms.*

2.2. Inner forms of GL(2): conjectures

We now formulate more precise conjectures in the case when $\mathbb{G}$ is an inner form of PGL(2) or GL(2) over a number field F. In most of this paper we shall be working with PGL(2) rather than GL(2), but at certain points it will be convenient to have the conjectures formulated for GL(2).

In particular, we formulate conjectures concerning $R = \mathbf{T}$, multiplicity one, and Jacquet–Langlands. We will not strive for any semblance of generality: we concentrate on two cases corresponding to the trivial local system, and we shall only consider the semistable case, for ease of notation as much as anything else. The originality of these conjectures (if any) consists of conjecturally identifying the action of the entire Hecke algebra $\mathbf{T}$ of endomorphisms on cohomology in terms of Galois representations, not merely $\mathbf{T} \otimes \mathbf{Q}$ (following Langlands, Clozel, and others), nor the finite field quotients $\mathbf{T}/\mathfrak{m}$ (generazing Serre's conjecture — for the specific case of imaginary quadratic fields, this is considered in work of Figueiredo [**26**], Torrey [**68**], and also discussed in unpublished notes of Clozel [**18, 19**]). One precise formulation along these lines is also contained in [**14**], Conjecture A. The key difference in this section is to extend this conjecture to the case of inner forms.

2.2.1. Universal Deformation Rings. Let F be a number field and let Σ denote a finite set of primes in $\mathscr{O}_F$. Let k be a finite field of characteristic p, and let

$$\overline{\rho} : G_F \to \mathrm{GL}_2(k)$$

be a continuous absolutely irreducible odd Galois representation. Recall that oddness in this context means that $\overline{\rho}(c_v)$ has determinant -1 for any complex conjugation c_v corresponding to a real place v of F; no condition is imposed at the complex places.

Let $\epsilon : G_F \to \mathbf{Z}_p^\times$ denote the cyclotomic character, and for any place v, let $I_v \subset G_v$ denote inertia and decomposition subgroups of G_F, respectively.

Let us suppose for all $v|p$ the following is satisfied:

(1) $\det(\overline{\rho}) = \epsilon\phi$, where $\phi|G_v$ is trivial for $v|p$. If $\mathbb{G} = \mathrm{PGL}_2$ we require $\phi \equiv 1$.
(2) For $v|p$ and $v \notin \Sigma$, either:
  (a) $\overline{\rho}|_{G_v}$ is finite flat,
  (b) $\overline{\rho}|_{G_v}$ is ordinary, but not finite flat.
(3) For $v \in \Sigma$, $\overline{\rho}|_{G_v} \sim \begin{pmatrix} \epsilon\chi & * \\ 0 & \chi \end{pmatrix}$, where χ is an unramified character of G_v.
(4) For $v \notin \Sigma \cup \{v|p\}$, $\overline{\rho}|G_v$ is unramified.

Associated to $\overline{\rho}$, we can, in the usual way (see [**22, 49, 74**]) define a deformation problem that associates to any local Artinian ring $(A, \mathfrak{m}, k)$ deformations ρ of $\overline{\rho}$ such that:

(1) $\det(\rho) = \epsilon \cdot \widetilde{\phi}$, where ϕ is the Teichmuller lift of ϕ.
(2) For $v|p$ and $v \notin \Sigma$, either:
  (a) $\rho|G_v$ is finite flat,
  (b) $\rho|G_v$ is ordinary, and $\overline{\rho}|G_v$ is not finite flat.
(3) For $v \in \Sigma$, $\rho|G_v \sim \begin{pmatrix} \epsilon\widetilde{\chi} & * \\ 0 & \widetilde{\chi} \end{pmatrix}$, where $\widetilde{\chi}$ is the Teichmuller lift of χ.
(4) If v is a real place of F, then $\rho|G_v$ is odd. (This is only a nontrivial condition if $\mathrm{char}(k) = 2$).
(5) If $v \notin \Sigma \cup \{v|p\}$, $\rho|G_v$ is unramified.

We obtain in this way a universal deformation ring R_Σ. One may determine the naïve "expected dimension" of these algebras (more precisely, their relative dimension over $\mathbf{Z}_p$) using the global Euler characteristic formula. This expected dimension is $-r_2$, where r_2 is the number of complex places of $\mathscr{O}_F$. Equivalently, one expects that these rings are finite over $\mathbf{Z}_p$, and finite if F has at least one complex place. This computation can not be taken too seriously, however, as it is inconsistent with the existence of elliptic curves over any field F.

2.2.2. Cohomology. Let $\mathbb{G}/F$ be an inner form of GL(2) corresponding to a quaternion algebra ramified at some set of primes $S \subset \Sigma \cup \{v|\infty\}$. Let $K_\Sigma = \prod_v K_{\Sigma,v}$ denote an open compact subgroup of $\mathbb{G}(\mathbb{A}_f)$ such that:

(1) If $v \in \Sigma$ but $v \notin S$, $K_{\Sigma,v} = \left\{ g \in \mathrm{GL}_2(\mathscr{O}_v) \mid g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod \pi_v \right\}$.
(2) If $v \in S$, then $K_{\Sigma,v}$ consists of the image of $B_v^\times$ inside $\mathbb{G}(F_v)$, where B_v is a maximal order of the underlying quaternion algebra.
(3) If $v \notin \Sigma$, then $K_{\Sigma,v} = \mathbb{G}(\mathscr{O}_v)$.

Moreover, if $\mathfrak{q}$ is a finite place in Σ that is not in S (so that situation 1 above occurs), let $K_{\Sigma/\mathfrak{q}}$ be defined as K_Σ except that $K_{\Sigma/\mathfrak{q}}$ is maximal at $\mathfrak{q}$.

Let $Y(K_\Sigma)$ be the locally symmetric space associated with $\mathbb{G}$. If $\mathbb{G}$ is an inner form of PGL_2, we are identifying K_Σ with its image in $\mathrm{PGL}_2(\mathbb{A}_f)$; each component of Y is then uniformized by a product of hyperbolic 2- and 3-spaces. If $\mathbb{G}$ is an inner form of GL_2, then $Y(K_\Sigma)$ is a torus bundle over the corresponding PGL_2-arithmetic manifold.

Let q be the number of archimedean places for which $\mathbb{G} \times_F F_v$ is split.

Definition 2.2.3. *Let $\Psi^\vee : H_q(Y(K_{\Sigma/\mathfrak{q}}), \mathbf{Z})^2 \to H_q(Y(K_\Sigma), \mathbf{Z})$ be the transfer map (see § 3.4.7). Then the space of new-at-$\mathfrak{q}$ forms of level K_Σ is defined to be the group $H_q(Y(K_\Sigma), \mathbf{Z})_{\mathfrak{q}-\mathrm{new}} := \mathrm{coker}(\Psi^\vee)$.*

We note that there are other candidate definitions for the space of newforms "over $\mathbf{Z}$"; the definition above appears to work best in our applications.

There is an obvious extension of this definition to the space of forms that are new at any set of primes in $\Sigma \setminus S$. Let us refer to the space of *newforms* as the quotient $H_q(Y(K_\Sigma), \mathbf{Z})^{\mathrm{new}}$ by the image of the transfer map for *all* $\mathfrak{q}$ in $\Sigma \setminus S$.

If $Y(K_\Sigma)$ and $Y(K_{\Sigma/\mathfrak{q}})$ are the corresponding arithmetic quotients, then the integral homology and cohomology groups $H_\bullet(Y, \mathbf{Z})$ and $H^\bullet(Y, \mathbf{Z})$ admit an action by Hecke endomorphisms (see § 3.4.3 for the definition of the ring of Hecke operators). We may now define the corresponding Hecke algebras.

Definition 2.2.4. *Let $\mathbf{T}_\Sigma$ — respectively the* new Hecke algebra $\mathbf{T}^{\mathrm{new}}_\Sigma$ *— denote the subring of* $\mathrm{End}\, H_q(Y(\Sigma), \mathbf{Z})$ (*respectively* $\mathrm{End}\, H_q(Y(K_\Sigma), \mathbf{Z})^{\mathrm{new}}$) *generated by Hecke endomorphisms.*

It is easy to see that $\mathbf{T}^{\mathrm{new}}_\Sigma$ is a quotient of $\mathbf{T}_\Sigma$. Of course, if $\Sigma = S$, then $\mathbf{T}^{\mathrm{new}}_\Sigma = \mathbf{T}_\Sigma$.

We say that a maximal ideal $\mathfrak{m}$ of $\mathbf{T}_\Sigma$ is Eisenstein if the image of $\mathbf{T}_\mathfrak{q}$ in the field $\mathbf{T}_\Sigma/\mathfrak{m}$ is given by the sum of two Hecke characters evaluated at $\mathfrak{q}$, for all but finitely many $\mathfrak{q}$ (see Definition 3.8.1 for a more precise discussion of the various possible definitions of what it means to be Eisenstein).

Conjecture 2.2.5 ($R = \mathbf{T}$). *Let $\mathfrak{m}$ be a maximal ideal of $\mathbf{T}^{\mathrm{new}}_\Sigma$, with $k = \mathbf{T}_\Sigma/\mathfrak{m}$ a field of characteristic p, and suppose that $\mathfrak{m}$ not Eisenstein. Then:*

(1) *There exists a Galois representation*

$$\rho_\mathfrak{m} : G_F \to \mathrm{GL}_2(\mathbf{T}^{\mathrm{new}}_{\Sigma,\mathfrak{m}})$$

satisfying the usual compatibility between Hecke eigenvalues of the characteristic polynomial of Frobenius.

(2) *The induced map $R_\Sigma \to \mathbf{T}^{\mathrm{new}}_{\Sigma,\mathfrak{m}}$ is an isomorphism.*

Remark 2.2.6. In a recent preprint, the first author and Geraghty have proved a minimal modularity lifting theorem for imaginary quadratic fields F under the assumption that there exists a map from R_Σ to $\mathbf{T}_\mathfrak{m}$ (see [**14**]). This provides strong evidence for this conjecture, in the form that $1 \Rightarrow 2$. Moreover, in the minimal case, one also deduces (under the conjectural existence of Galois representations) that the Hecke algebra is free as an R_Σ-module, and is multiplicity one whenever it is infinite (see Conjecture 2.2.9). On the other hand, Scholze's results [**63**] mentioned previously constructs Galois representations $\rho_\mathfrak{m}$ as above without showing that they the appropriate Galois representation has the requisite local behavior at $v|p$. One way to characterize the current situation is that Scholze constructs a map $\widetilde{R}_\Sigma \to \mathbf{T}^{\mathrm{new}}_{\Sigma,\mathfrak{m}}$ from some less restrictive deformation ring $\widetilde{R}_\Sigma$; once we know that this map factors through R_Σ, then the isomorphism $R_\Sigma \simeq \mathbf{T}^{\mathrm{new}}_{\Sigma,\mathfrak{m}}$ will follow from the methods of Calegari and Geraghty.

2.2.7. Jacquet–Langlands correspondence. Let $F/\mathbf{Q}$ be a field with one complex place, let $\mathbb{G}/F$ and $\mathbb{G}'/F$ be inner forms of $\mathrm{PGL}(2)/F$ that are ramified at the set of finite primes S and S' respectively and ramified at all real infinite places of F. Let $Y(K_\Sigma)$ be the arithmetic manifold constructed as in § 2.2.2, and denote by $Y'(K_\Sigma)$ the analogous construction for $\mathbb{G}'$. We refer to a pair of such manifolds as a *Jacquet–Langlands pair.*

It follows from the Jacquet–Langlands correspondence that there is a Hecke-equivariant isomorphism

$$H_1(Y(K_\Sigma), \mathbf{C})^{\mathrm{new}} \cong H_1(Y'(K_\Sigma), \mathbf{C})^{\mathrm{new}}.$$

One is tempted to conjecture the following

$$|H_1(Y(K_\Sigma), \mathbf{Z})^{\mathrm{new}}| =^? |H_1(Y'(K_\Sigma), \mathbf{Z})^{\mathrm{new}}|,$$

and, even more ambitiously, a corresponding isomorphism of Hecke modules. We shall see, however, that this is *not* the case.

Consider, first, an example in the context of the upper half-plane: Let $N \in \mathbf{Z}$ be prime, $N \equiv 1 \mod p$, and consider the group $\Gamma_0(N) \subset \mathrm{PSL}_2(\mathbf{Z})$. Then there is a map $\Gamma_0(N) \to \Gamma_0(N)/\Gamma_1(N) \to \mathbf{Z}/p\mathbf{Z}$, giving rise to a class in $H^1(\Gamma_0(N), \mathbf{Z}/p\mathbf{Z})$. On the other hand, we see that this class capitulates when we pull back to a congruence cover (compare the annihilation of the Shimura subgroup under the map of Jacobians $J_0(N) \to J_1(N)$).

Similar situations arise in the context of the $Y(K_\Sigma)$. We denote the cohomology that arises in this way *congruence*, because it vanishes upon restriction to a congruence subgroup. See 3.7 for a precise definition. A simple argument (Lemma 3.7.5.1) shows that all congruence classes are necessarily Eisenstein. For 3-manifolds, it may be the case that congruence homology does not lift to characteristic zero.

We define the *essential* homology, denoted H_1^E, by excising congruence homology (see Definition 6.7.2) Although this definition is perhaps is ad hoc, our first impression was that the following equality should hold:

$$|H_1^E(Y(K_\Sigma), \mathbf{Z})^{\mathrm{new}}| =^? |H_1^E(Y'(K_\Sigma), \mathbf{Z})^{\mathrm{new}}|.$$

However, even this conjecture is too strong — we shall see that "corrections" to this equality may occur arising from K-theoretic classes. (In particular, see Theorem A† from the introduction, and we have numerical examples where where these two spaces are finite and have different orders. We leave open the possibility that this equality might always hold when D and D' are both non-split, equivalently, Y and Y' are both compact, although there is no reason to suppose it should be true in that case either.) In the examples in which these groups have different orders, the disparity arises from certain unusual Eisenstein classes. Thus, one may salvage a plausible conjecture by localizing the situation at a maximal ideal of the Hecke algebra.

Conjecture 2.2.8. *If $\mathfrak{m}$ is not Eisenstein, there is an equality*

$$|H_1(Y(K_\Sigma), \mathbf{Z})^{\mathrm{new}}_{\mathfrak{m}}| = |H_1(Y'(K_\Sigma), \mathbf{Z})^{\mathrm{new}}_{\mathfrak{m}}|.$$

Let us try to explain why a conjecture such as 2.2.8 might be expected. Let $\overline{\rho} : G_F \to \mathrm{GL}_2(k)$ be an absolutely irreducible Galois representation giving rise to a universal deformation ring R_Σ as in § 2.2.1. Let $Y(K_\Sigma)$ be the arithmetic manifold constructed as in § 2.2.2, and denote by $Y'(K_\Sigma)$ the analogous construction for $\mathbb{G}'$. If Conjecture 2.2.5 holds, then we obtain an identification of R_Σ with the ring of Hecke endomorphisms acting on both $H_1(Y(K_\Sigma), \mathbf{Z})^{\mathrm{new}}_{\mathfrak{m}}$ and $H_1(Y'(K_\Sigma), \mathbf{Z})^{\mathrm{new}}_{\mathfrak{m}}$.

What is the structure of these groups as Hecke modules? One might first guess that they are isomorphic. However, this hope is dashed even for $\mathrm{GL}(2)/\mathbf{Q}$, as evidenced by the failure (in certain situations) of multiplicity one for non-split quaternion algebras as shown by Ribet [**56**]. In the context of the p-adic Langlands program, this failure is not an accident, but rather a consequence of local-global compatibility, and should be governed purely by local properties of $\overline{\rho}$. A similar remarks apply to multiplicites for $p = 2$ on the split side when $\overline{\rho}$ is unramified at 2 and $\overline{\rho}(\mathrm{Frob}_2)$ is a scalar. In this spirit, we propose the following:

Conjecture 2.2.9 (multiplicity one). *Let $\overline{\rho} : G_F \to \mathrm{GL}_2(k)$ be an absolutely irreducible Galois representation, and let $\mathfrak{m}$ be the associated maximal ideal. Suppose that:*

(1) *p divides neither 2 nor d_F.*
(2) *If $\mathfrak{q} \in S$, then either:*
    (a) *$\overline{\rho}|G_{\mathfrak{q}}$ is ramified at $\mathfrak{q}$, or,*
    (b) *$\overline{\rho}|G_{\mathfrak{q}}$ is unramified at $\mathfrak{q}$, and $\overline{\rho}(\mathrm{Frob}_{\mathfrak{q}})$ is not a scalar.*

Then $H_1(Y(K_\Sigma), \mathbf{Z})^{\mathrm{new}}_{\mathfrak{m}}$ is free of rank one over $\mathbf{T}^{\mathrm{new}}_{\mathfrak{m}}$.

One might argue that condition 1 is somewhat timid, since (as for $\mathrm{GL}(2)/\mathbf{Q}$) when $p = 2$ and $2 \nmid d_F$, one should also have multiplicity one under some mild conditions on $\overline{\rho}|G_v$ for $v|2$, and when $p|d_F$ and $F/\mathbf{Q}$ is not too ramified at p, multiplicity one should also hold. (See also Remark 2.2.6.) However, our only real data along these lines is for $p = 2$ and $F = \mathbf{Q}(\sqrt{-2})$, when the failure of multiplicity one is frequent. A trivial consequence of Conjectures 2.2.9 and 2.2.5 is the following.

Lemma 2.2.10. *Assume Conjectures 2.2.9 and 2.2.5. Suppose that:*

(1) *p divides neither 2 nor d_F.*
(2) *If $\mathfrak{q} \in S$ or S', then either:*
    (a) *$\overline{\rho}|G_{\mathfrak{q}}$ is ramified at $\mathfrak{q}$, or,*
    (b) *$\overline{\rho}|G_{\mathfrak{q}}$ is unramified at $\mathfrak{q}$, and $\overline{\rho}(\mathrm{Frob}_{\mathfrak{q}})$ is not a scalar.*

Then there is an isomorphism

$$H_1(Y(K_\Sigma), \mathbf{Z})^{\mathrm{new}}_{\mathfrak{m}} \simeq H_1(Y'(K_\Sigma), \mathbf{Z})^{\mathrm{new}}_{\mathfrak{m}}.$$

Let us summarize where we are: Later in the paper, we will prove theorems relating the size of $H_1(Y(K_\Sigma), \mathbf{Z})$ and $H_1(Y'(K_\Sigma, \mathbf{Z}))$. What we have shown up to here is that, at least if one completes at non-Eisenstein maximal ideals, such a result *is to be expected* if one admits the multiplicity one conjecture Conjectures 2.2.9 and the reciprocity conjecture Conjecture 2.2.5 noted above, *as well as* a Serre conjecture implying that $\overline{\rho}_{\mathfrak{m}}$ is modular both on the inner and split forms of the group. In this way, we view the results of this paper as giving evidence for these conjectures.

It is an interesting question to formulate a conjecture on how the $\mathfrak{m}$-adic integral homology should be related when multiplicity one (i.e. Conjecture 2.2.9) fails. Asking that both sides are *isogenous* is equivalent to the rational Jacquet–Langlands (a theorem!) since any torsion can be absorbed by an isogeny. It should at least be the case that one space of newforms is zero if and only if the other space is.

2.2.11. Auxiliary level structure; other local systems. One can generalize the foregoing conjectures to add level structures at auxiliary primes. That is, we may add level structure at some set N of primes (not containing any primes in Σ or dividing p). Similarly, one can consider not simply homology with constant coefficients, but with values in a local system. We will not explicitly do so in this document; the modifications are routine and do not seem to add any new content to our theorems.

CHAPTER 3

Notation

The most important notation necessary for browsing this paper — the manifolds $Y(K)$ we consider, the notation for their homology groups, and basics on the Cheeger-Müller theorem — is contained in Sections § 3.1.

The remainder of the Chapter contains more specialized notation as well as recalling various background results.

The reader might consult these other sections as necessary when reading the text.

3.1. A summary of important notation

For A an abelian group, we denote by A_{tf} the maximal torsion-free quotient of A and by A_{div} the maximal divisible subgroup of A. We shall use the notation A_{tors} in two contexts: if A is finitely generated over $\mathbf{Z}$ or $\mathbf{Z}_p$, it will mean the torsion subgroup of A; on the other hand, if $\mathrm{Hom}(A, \mathbf{Q}/\mathbf{Z})$ or $\mathrm{Hom}(A, \mathbf{Q}_p/\mathbf{Z}_p)$ is finitely generated over $\mathbf{Z}_p$, it will mean the maximal torsion *quotient* of A. These notations are compatible when both apply at once.

Let F be a number field with one complex place and $r_1 = [F : \mathbf{Q}] - 2$ real places. We denote the ring of integers of F by $\mathscr{O}$ and set $F_\infty := F \otimes \mathbf{R} \cong \mathbf{C} \times \mathbf{R}^{r_1}$. We fix, once and for all, a complex embedding $F \hookrightarrow \mathbf{C}$ (this is unique up to complex conjugation). The adele ring of F will be denoted $\mathbb{A}$ (occasionally by $\mathbb{A}_F$ when we wish to emphasize the base field F) and the ring of *finite* adeles $\mathbb{A}_f$.

Let D be a quaternion algebra over F ramified at a set S of places that contains all real places. (By abuse of notation we occasionally use S to denote the set of finite places at which D ramifies; this abuse should be clear from context.)

Let $\mathbb{G}$ be the algebraic group over $\mathbf{Q}$ defined as

$$\mathbb{G} = \mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_1(D)/\mathbb{G}_m.$$

Let $G_\infty = \mathbb{G}(\mathbf{R})$ and let K_∞ be a maximal compact of G_∞. The associated symmetric space G_∞/K_∞ is isometric to the hyperbolic 3-space $\mathbf{H}^3$ with its standard metric; we fix such an isometric identification (see §3.5 for details).

For $\Sigma \supset S$ a subset of the finite places of F, we define the associated arithmetic quotient, a (possibly disconnected) hyperbolic three orbifold of finite volume:

$$\begin{aligned} Y(K_\Sigma) \quad & \Big(\text{alternate notation: } Y(\Sigma) \text{ or } Y_0(\mathfrak{n}), \text{ where } \mathfrak{n} = \textstyle\prod_{\mathfrak{p} \in \Sigma} \mathfrak{p}.\Big) \\ &:= \mathbb{G}(F) \backslash \mathbb{G}(\mathbb{A}) / K_\infty K_\Sigma \\ &= \mathbb{G}(F) \backslash (\mathbf{H}^3 \times \mathbb{G}(\mathbb{A}_f)) / K_\Sigma, \end{aligned}$$

where $K_\Sigma = \prod K_v$, where, fixing for every $v \notin S$ an identification of $D_v^\times$ with $\mathrm{GL}_2(F_v)$, and so also of $\mathbb{G}(F_v)$ with $\mathrm{PGL}_2(F_v)$, we have set

$$K_v = \begin{cases} \mathrm{PGL}_2(\mathscr{O}_v), & v \notin \Sigma; \\ g \in \mathrm{PGL}_2(\mathscr{O}_v) \mid g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod \pi_v, & v \in \Sigma - S;, \\ \text{the image of } B_v^\times \text{ in } \mathbb{G}(F_v) & \text{otherwise,} \end{cases}$$

where B_v is a maximal order of D_v.

Then $Y(\Sigma)$ is an analog of the modular curve $Y_0(N)$ with level N the product of all primes in Σ. Note that there is an abuse of language in all of our notation $Y(\Sigma)$, $Y(K_\Sigma)$, $Y_0(\mathfrak{n})$: we do not explicitly include in this notation the set of places S defining the quaternion algebra D. *When we write $Y(\Sigma)$, there is always an implicit choice of a fixed subset $S \subset \Sigma$.*

Almost all of our theorems carry over to general level structures. Accordingly, if $K \subset \mathbb{G}(\mathbb{A}_f)$ is an open compact subgroup, we denote by $Y(K)$ the corresponding manifold $\mathbb{G}(F)\backslash(\mathbf{H}^3 \times \mathbb{G}(\mathbb{A}_f))/K_\Sigma$; of course, when $K = K_\Sigma$, this recovers the manifold above.

It will often be convenient in this paper to use the phrase "hyperbolic three manifold" to include disconnected three-manifolds, each of whose components are hyperbolic. Similarly, in order to avoid having to add a parenthetical remark "(or possibly orbifold)" after every use of the word manifold, we shall often elide the distinction and use the word manifold. By "orbifold prime" for the orbifold M we shall mean a prime that divides the order of one of the point stabilizer groups. When we take cohomology of such an M, then, we always mean in the sense of orbifolds, that is to say: if $\tilde{M} \to M$ is a G-covering by a genuine manifold $\tilde{M}$, as will always exist in our settings, the cohomology of M is understood to be the cohomology of $M \times E/G$, where E is a space on which G acts freely.

Our primary interest in this paper is in the homology (or cohomology) of the arithmetic manifolds $Y(K_\Sigma)$ just defined. Since the expression $H_1(Y(K_\Sigma), \mathbf{Z})$ is somewhat cumbersome, we shall, hopefully without confusion, write:

$$H_1(\Sigma, \mathbf{Z}) := H_1(Y(K_\Sigma), \mathbf{Z}).$$

Since Σ is only a finite set of primes, it is hoped that no ambiguity will result, since the left hand side has no other intrinsic meaning. In the usual way (§ 3.4.3) one defines the action of a Hecke operator

$$T_\mathfrak{p} : H_1(\Sigma, \mathbf{Z}) \longrightarrow H_1(\Sigma, \mathbf{Z})$$

for every $\mathfrak{p} \notin S$. These commute for different $\mathfrak{p}$. When $\mathfrak{p} \in \Sigma - S$, this operator is what is often called the "U-operator."

If $\mathfrak{q} \in \Sigma$ is a prime that does not lie in S, then we write $\Sigma = \Sigma/\mathfrak{q} \cup \{\mathfrak{q}\}$, and hence:

$$H_1(\Sigma/\mathfrak{q}, \mathbf{Z}) := H_1(Y(K_{\Sigma/\mathfrak{q}}), \mathbf{Z}).$$

Suppose that $\mathfrak{q} \in \Sigma$ is not in S. There are two natural degeneracy maps $Y(K_\Sigma) \to Y(K_{\Sigma/\mathfrak{q}})$ (see § 3.4.3); these give rise to a pair of maps:

$$\Phi : H^1(\Sigma/\mathfrak{q}, \mathbf{Q}/\mathbf{Z})^2 \to H^1(\Sigma, \mathbf{Q}/\mathbf{Z}),$$

$$\Phi^\vee : H^1(\Sigma, \mathbf{Q}/\mathbf{Z}) \to H^1(\Sigma/\mathfrak{q}, \mathbf{Q}/\mathbf{Z})^2,$$

where the first map Φ is the pullback, and the latter map is the transfer. In the same way, we obtain maps:

$$\Psi : H_1(\Sigma, \mathbf{Z}) \to H^1(\Sigma/\mathfrak{q}, \mathbf{Z})^2,$$
$$\Psi^\vee : H_1(\Sigma/\mathfrak{q}, \mathbf{Z})^2 \to H_1(\Sigma, \mathbf{Z}).$$

The map Ψ can be obtained from Φ (and similarly $\Psi^\vee$ from $\Phi^\vee$) by applying the functor $\mathrm{Hom}(-, \mathbf{Q}/\mathbf{Z})$.

3.1.1. Additional notation in the split case. When $\mathbb{G} = \mathrm{PGL}_2$, a situation analyzed in detail in Chapter 5, we will use some further specialized notation. In this case F is imaginary quadratic. We put

$$\mathbf{B} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subset \mathrm{PGL}_2,$$

a Borel subgroup, and denote by $\mathbf{N}$ its unipotent radical $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$; we denote by $\mathbf{A}$ the diagonal torus. We denote by $\alpha : \mathbf{B} \to \mathbf{G}_m$ the positive root:

$$\alpha : \begin{pmatrix} x & * \\ 0 & 1 \end{pmatrix} \mapsto x. \tag{3.1.1.1}$$

We also define $\mathrm{PU}_2 \subset \mathrm{PGL}_2(\mathbf{C})$ to be the stabilizer of the standard Hermitian form $|x_1|^2 + |x_2|^2$. Then $\mathrm{PU}_2 \times \prod_v \mathrm{PGL}_2(\mathscr{O}_v)$, the product being taken over finite v, is a maximal compact subgroup of $\mathrm{PGL}_2(\mathbb{A})$.

Chapter 5 uses a significant amount of further notation, and we do not summarize it here since it is localized to that Chapter.

3.1.2. Regulators and the Cheeger-Müller theorem. Here is what the Cheeger–Müller theorem [**17, 54**] says about the hyperbolic manifold $Y(K)$ when it is compact (or, indeed, about any compact hyperbolic 3-manifold). We assume for the moment that $Y(K)$ has no orbifold points, the modifications in the orbifold case being discussed in § 3.10.5.

Define the *regulator of the* j*th homology* via

$$\mathrm{reg}(H_\ell(Y(K))) = \left|\det \int_{\gamma_j} \nu_k\right| \tag{3.1.2.1}$$

where γ_i give a basis for $H_\ell(Y(K), \mathbf{Z})$ modulo torsion, and ν_k an orthonormal basis of harmonic j-forms.

Another way to say this is as follows: the image of $H_\ell(Y(K), \mathbf{Z})$ inside $H_\ell(Y(K), \mathbf{R})$ is a *lattice* in the latter vector space; and $\mathrm{reg}(H_\ell)$ is the *covolume* of this lattice, with respect to the measure on $H_\ell(\mathbf{R})$ that arises with its identification as the dual to a space of harmonic forms. A more intrinsic way to write the right-hand side would be $\frac{\left|\det \int_{\gamma_j} \nu_k\right|}{|\det\langle \nu_j, \nu_k\rangle|^{1/2}}$; in this form it is independent of basis ν_j.

Now we define the *regulator of* $Y(K)$ via:

$$\mathrm{reg}(Y(K)) = \frac{\mathrm{reg}(H_1)\mathrm{reg}(H_3)}{\mathrm{reg}(H_0)\mathrm{reg}(H_2)} = \mathrm{vol} \cdot \mathrm{reg}(H_1(Y(K))^2, \tag{3.1.2.2}$$

where vol is the product of the volumes of all connected components. The second equality here follows from the easily verified facts that $\mathrm{reg}(H_i)\mathrm{reg}(H_{3-i}) = 1$ and $\mathrm{reg}(H_0) = 1/\sqrt{\mathrm{vol}}$.

Denote by $\Delta^{(j)}$ the de Rham Laplacian on the orthogonal complement of harmonic j-forms inside smooth j-forms. Let $H_{1,\mathrm{tors}}$ be the torsion part of $H_1(Y(K), \mathbf{Z})$. We then define *Reidemeister* and *analytic* torsion via the formulae:

$$\mathrm{RT}(Y(K)) = |H_{1,\mathrm{tors}}|^{-1} \cdot \mathrm{reg}(Y(K)) \tag{3.1.2.3}$$

$$\tau_{\mathrm{an}}(Y(K)) = \det(\Delta^{(1)})^{-1/2} \det(\Delta^{(0)})^{3/2}. \tag{3.1.2.4}$$

The determinants appearing in (3.1.2.4) are to be understood via zeta-regularization. Then the *equality* of Reidemeister torsion and analytic torsion, conjectured by Ray–Singer and proved by Cheeger and Müller, asserts simply

$$\mathrm{RT}(Y(K)) = \tau_{\mathrm{an}}(Y(K)). \tag{3.1.2.5}$$

Intuitively this formula expresses the fact that the the limit of a chain complex for M, as one triangulates very finely, approaches the de Rham complex.

When $Y(K)$ is noncompact, the exact analogue of (3.1.2.5) is not known; Chapter 5 is devoted to proving a partial version of it which suffices for our purposes.

3.2. Fields and adeles

3.2.1. The number field F. Let F be a number field with one complex place[1] and $r_1 = [F : \mathbf{Q}] - 2$ real places. We denote the ring of integers of F by $\mathscr{O}$ and set $F_\infty := F \otimes \mathbf{R} \cong \mathbf{C} \times \mathbf{R}^{r_1}$. We fix, once and for all, a complex embedding $F \hookrightarrow \mathbf{C}$ (this is unique up to complex conjugation).

The adele ring of F will be denoted $\mathbb{A}$, and the ring of *finite* adeles $\mathbb{A}_f$. If T is any set of places, we denote by $\mathbb{A}^{(T)}$ the adele ring omitting places in T, i.e. the restricted direct product of F_v for $v \notin T$. Similarly — if T consists only of finite places — we write $\mathbb{A}_f^{(T)}$ for the finite adele ring omitting places in T. We denote by $\mathscr{O}_{\mathbb{A}}$ or $\widehat{\mathscr{O}}$ the closure of $\mathscr{O}_F$ in $\mathbb{A}_f$; it is isomorphic to the profinite completion of the ring $\mathscr{O}_F$.

For v any place of F, and $x \in F_v$, we denote by $|x|_v$ the normalized absolute value on F_v, i.e., the effect of multiplication by x on a Haar measure. In the case of $\mathbf{C}$ this differs from the usual absolute value: $|z|_{\mathbf{C}} = |z|^2$ for $z \in \mathbf{C}$, where we will denote by $|z|$ the "usual" absolute value $|x+iy| = \sqrt{x^2+y^2}$. For $x \in \mathbb{A}$ we write, as usual, $|x| = \prod_v |x_v|_v$.

We denote by ζ_F the zeta-function of the field F, and by ζ_v its local factor at the finite place v of F. We denote by ξ_F the completed ζ_F, i.e. including Γ-factors at archimedean places.

When we deal with more general L-functions (associated to modular forms), we follow the convention that L refers to the "finite part" of an L-function, i.e., excluding all archimedean factors, and Λ refers to the completed L-function.

We denote by $\mathrm{Cl}(\mathscr{O}_F)$ (or simply $\mathrm{Cl}(F)$) the class group of $\mathscr{O}_F$, and we denote its order by h_F.

We denote by w_F the the number of roots of unity in F, i.e. (denoting by μ the set of all roots of unity in an algebraic closure, as a Galois module), the number of Galois-invariant elements of μ.

We denote by $w_F^{(2)}$ the number of Galois-invariant elements of $\mu \otimes \mu$, equivalently, the greatest common divisor of $\mathrm{N}(\mathfrak{q})^2 - 1$, where $\mathfrak{q}$ ranges through all sufficiently large prime ideals of F. One easily sees that if F is imaginary quadratic, then w_F divides 12 and $w_F^{(2)}$ divides 48. In particular it is divisible only at most by 2 and 3.

We denote by w_H the number of roots of unity in the Hilbert class field of F. If F has a real place, then $w_H = 2$; otherwise, w_H is divisible at most by 2 and 3.

3.2.2. Quaternion Algebras. Let D be a quaternion algebra over F ramified at a set S of places that contains all real places. (By abuse of notation we occasionally use S to denote the set of finite places at which D ramifies; this abuse should be clear from context.)

Recall that D is determined up to isomorphism by the set S of ramified places; moreover, the association of D to S defines a bijection between isomorphism classes of quaternion algebras, and subsets of places with even cardinality. Let ψ be the standard anti-involution[2] of D; it preserves the center F, and the reduced norm of

---

[1] Most of the notation we give makes sense without the assumption that F has one complex place.

[2] It is uniquely characterized by the property that $((t-x)(t-\psi(x))^2$ belongs to $F[t]$ and is the characteristic polynomial of multiplication by $x \in D$ on D.

D is given by $x \mapsto x \cdot \psi(x)$. The additive identity $0 \in D$ is the only element of norm zero.

Finally, let $\mathbb{G}$ be the algebraic group over F defined as

$$\mathbb{G} = \mathrm{GL}_1(D)/\mathbb{G}_m.$$

When $v \notin S$, the group $\mathbb{G} \times_F F_v$ is isomorphic to PGL_2 (over F_v). We fix once and for all an isomorphism

$$\iota_v : \mathbb{G} \times_F F_v \xrightarrow{\sim} \mathrm{PGL}_2. \tag{3.2.2.1}$$

This will be used only as a convenience to identify subgroups.

Let $B/\mathscr{O}_F$ be a maximal order in D. Let $B^\times$ denote the invertible elements of B; they are precisely the elements of B whose reduced norm lies in $\mathscr{O}_F^\times$. Let $B^1 \subset B^\times$ denote the elements of B of reduced norm one.

Later we shall compare two quaternion algebras D, D'; we attach to D' data $S', \mathbb{G}', A'$.

3.2.3. Quaternion Algebras over Local Fields. Let $F_v/\mathbf{Q}_p$ denote the completion of F with ring of integers $\mathscr{O}_{F,v}$, uniformizer π_v, and residue field $\mathscr{O}_{F,v}/\pi_v\mathscr{O}_{F,v} = k_v$.

If D_v is not isomorphic to the algebra of 2×2 matrices, it can be represented by the symbol $\left(\frac{\pi,\alpha}{F_v}\right)$, i.e.

$$F_v[i,j]/\{i^2 = \pi_v, j^2 = \alpha, ij = -ji\}.$$

where $\alpha \in \mathscr{O}_{F,v}$ is such that the image of α in k is a quadratic non-residue. Then $B_v := \mathscr{O}_{F,v}[i,j,ij]$ is a maximal order and all such are conjugate.

The maximal order B_v admits a unique maximal bi-ideal $\mathfrak{m}_v$, which is explicitly given by (i) with the presentation above. Then $\ell := B_v/\mathfrak{m}_v$ is a quadratic extension of k. The image of $B_v^\times$ in $B_v/\mathfrak{m}_v$ is $l^\times$, whereas the image of the norm one elements B_v^1 in $B_v/\mathfrak{m}_v$ is given by those elements in $l^\times$ whose norm to k is trivial. Denote this subgroup by l^1.

3.3. The hyperbolic 3-manifolds

3.3.1. Level Structure. We now fix notations for the level structures that we study. They all correspond to compact open subgroups of $G(\mathbb{A}_f)$ of "product type," that is to say, of the form $\prod_v K_v$.

We have fixed an isomorphism of $\mathbb{G}(F_v)$ with $\mathrm{PGL}_2(F_v)$ whenever $v \notin S$ (see (3.2.2.1)) We use it in what follows to identify subgroups of $\mathrm{PGL}_2(F_v)$ with subgroups of $\mathbb{G}(F_v)$.

For any finite place, set $K_{0,v} \subset \mathbb{G}(\mathscr{O}_v)$ to be the image in PGL_2 of

$$K_{0,v} = \left\{ g \in \ GL_2(\mathscr{O}_v) \mid g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod \pi_v \right\}$$

if $\mathbb{G}$ is split and the image of $B_v^\times \to \mathbb{G}(F_v)$ otherwise.

Define $K_{1,v}$ to be the image in PGL_2 of

$$K_{1,v} = \left\{ g \in \mathrm{GL}_2(\mathscr{O}_v) \mid g \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod \pi_v \right\}$$

if v is split and the image of $1 + \mathfrak{m}_v$ inside $\mathbb{G}(F_v)$ otherwise.

Thus, $K_{1,v}$ is normal in $K_{0,v}$, and the quotient is isomorphic to $k_v^\times$ or $l_v^\times/k_v^\times$, according to whether $\mathbb{G}$ is split or not; here l_v is the quadratic extension of the residue field.

Let Σ denote a finite set of primes containing every element of S. We then set

$$K_{\Sigma,0} := \prod_{v\in\Sigma} K_{0,v} \prod_{v\notin \Sigma} \mathbb{G}(\mathscr{O}_v).$$

and similarly $K_{\Sigma,1}$.

We write simply K_Σ for $K_{\Sigma,0}$.

In particular, the quotient $K_\Sigma/K_{\Sigma,1}$ is abelian, and it has size

$$\prod_{\mathfrak{q}\in S}(N(\mathfrak{q})+1)\prod_{\mathfrak{q}\in\Sigma-S}(N(\mathfrak{q})-1).$$

In all instances, let $K_v^{(1)} \subset K_v$ denote the subgroup of norm one elements, that is, the elements either of determinant one if $\mathbb{G}(\mathscr{O}_v) = \mathrm{GL}_2(\mathscr{O}_v)$ or norm one if $\mathbb{G}(\mathscr{O}_v) = B^\times$.

3.3.2. Groups and Lie algebras. Let $G_\infty = \mathbb{G}(\mathbf{R})$ and let K_∞ be a maximal compact of G_∞. We may suppose that K_∞ is carried, under the morphism $G_\infty \to \mathrm{PGL}_2(\mathbf{C})$ derived from (3.2.2.1), to the standard subgroup PU_2 corresponding to the stabilizer of the Hermitian form $|z_1|^2+|z_2|^2$.

We denote by $\mathfrak{g}$ the Lie algebra of G_∞ and by $\mathfrak{k}$ the Lie algebra of K_∞.

3.3.3. $Y(K_\Sigma)$ **and** $Y(K_{\Sigma,1})$**.** If K is a compact open subgroup of $\mathbb{G}(\mathbb{A}_f)$, we define the associated arithmetic quotient[3]

$$\begin{aligned} Y(K) &:= \mathbb{G}(F)\backslash\mathbb{G}(\mathbb{A})/K_\infty K \\ &= \mathbb{G}(F)\backslash(G_\infty/K_\infty\times\mathbb{G}(\mathbb{A}_f))/K, \end{aligned}$$

The manifold $Y(K)$ is a (possibly disconnected) hyperbolic three manifold of finite volume, since G_∞/K_∞ is isometric to $\mathbf{H}$ (see § 3.5.1 for that identification).

As before, let Σ be a finite set of finite places containing all places in S. Define the arithmetic quotients $Y(K_\Sigma)$ and $Y(K_{\Sigma,1})$ as:

$$Y(K_\Sigma) := \mathbb{G}(F)\backslash\mathbb{G}(\mathbb{A})/K_\infty K_{\Sigma,0},$$

$$Y(K_{\Sigma,1}) := \mathbb{G}(F)\backslash\mathbb{G}(\mathbb{A})/K_\infty K_{\Sigma,1}.$$

By analogy with usual notation, we sometimes write $Y_0(\mathfrak{n})$ and $Y_1(\mathfrak{n})$ for $Y(K_\Sigma)$ or $Y(K_{\Sigma,1})$, where $\mathfrak{n} = \prod_{\mathfrak{p}\in\Sigma}\mathfrak{p}$.

REMARK 3.3.4. We may express this more explicitly:

$$Y(K_\Sigma) = \coprod_A Y_0(\Sigma,\mathfrak{a}) = \coprod_A \Gamma_0(\Sigma,\mathfrak{a})\backslash\mathbf{H}^3, \tag{3.3.4.1}$$

$$Y(K_{\Sigma,1}) = \coprod_A Y_1(\Sigma,\mathfrak{a}) = \coprod_A \Gamma_1(\Sigma,\mathfrak{a})\backslash\mathbf{H}^3,$$

where:

[3]Note that, for general groups $\mathbb{G}$, it might be more suitable, depending on the application, to replace K_∞ in this definition by $K_\infty^0 A_\infty^0$ (so that $Y(K)$ has a finite invariant measure and has an orientation preserved by G_∞.

(1) The indexing set A is equal to (the finite set) $F^\times\backslash\mathbb{A}^\times/\mathscr{O}_\mathbb{A}^\times\mathbb{A}^{\times 2}$, which is the quotient of the class group of F by squares. It is important to notice that A depends only on F, and not on D or Σ.

More generally, the component set for $Y(K)$ (for general K) may be identified with a quotient of some ray class group of $\mathscr{O}_F$.

(2) Each $\Gamma_i(\Sigma,\mathfrak{a})$ is a subgroup of $\mathbb{G}(F)$, realized as a finite volume discrete arithmetic subgroup of $\mathrm{PGL}_2(\mathbf{C})$ via the unique complex embedding $F\to\mathbf{C}$ and the fixed isomorphism of $\mathbb{G}(\mathbf{C})$ with $\mathrm{PGL}_2(\mathbf{C})$. Moreover, the corresponding group will be co-compact providing that $\mathbb{G}\neq\mathrm{PGL}_2$. Explicitly, let $\alpha\in\mathbb{A}^\times$ represent the class $\mathfrak{a}\in A$; then

$$\Gamma_i(\Sigma,\mathfrak{a})\simeq\left\{\gamma\in\mathbb{G}(F)\subset\mathbb{G}(\mathbb{A})\;\middle|\;\begin{pmatrix}\alpha&0\\0&1\end{pmatrix}^{-1}\gamma\begin{pmatrix}\alpha&0\\0&1\end{pmatrix}\in K_{\Sigma,i}\right\}\tag{3.3.4.2}$$

Example 3.3.5. We specialize our notation to various classical situations in order to orient the reader:

Let $\mathbb{G}=\mathrm{GL}(2)/\mathbf{Q}$, $S=\varnothing$, $\Sigma=\{p,q\}$, and $\Gamma=\mathrm{PSL}_2(\mathbf{Z})$. Suppose instead that $\mathbb{G}'=\mathrm{GL}_1(D)/\mathbf{Q}$, where $D/\mathbf{Q}$ is ramified at p and q (and so indefinite at ∞), $S=\{p,q\}$, $\Sigma=\{p,q\}$. Denote the corresponding manifolds by Y and Y'.

Then $Y(K_\Sigma)$ can be identified with $Y_0(pq)$, and $Y(K_{\Sigma,1})$ can be identified with $Y_1(pq)$, where $Y_0(n)$ and $Y_1(n)$ are the modular curves of level n parameterizing (respectively) elliptic curves with a cyclic subgroup of order n, and elliptic curves with a point of exact order n.

On the other hand, $Y'(\Sigma)$ can be identified with a Shimura variety, namely, the variety parameterizing "fake elliptic curves" (principally polarized abelian surfaces A together with an embedding of $\mathscr{O}_D$ into $\mathrm{End}(A)$ satisfying some extra conditions). On the other hand, $Y'(K_{\Sigma,1})$ can be identified with the same moduli space together with a level structure: if $\mathfrak{p},\mathfrak{q}$ are the maximal bi-ideals of $\mathscr{O}_D$ corresponding to p,q, then the level structure is given by specifying a generator of K, where $A[\mathfrak{pq}]\simeq K\oplus K$ under the idempotents $e=\begin{pmatrix}1&0\\0&0\end{pmatrix}$ and $1-e$ of $\mathscr{O}_D/\mathfrak{pq}$, having identified the latter with a matrix algebra, cf. [**10**]).

3.3.6. Orbifold primes for $Y(K_\Sigma)$. The $Y(K_\Sigma)$ are orbifolds. As such, to each point there is canonically associated an isotropy group. It is not difficult to see that the only primes dividing the order of an orbifold isotropy group are at most the primes dividing $w_F^{(2)}$ (see § 3.2.1).

In particular, the only possible orbifold primes in the split case, or indeed if F is quadratic imaginary, are 2 and 3. At certain points in the text we will invert $w_F^{(2)}$; this is simply to avoid such complications.

We will sometimes say "$A=B$ up to orbifold primes." By this we mean that the ratio A/B is a rational number whose numerator and denominator are divisible only by orbifold primes.

3.4. Homology, cohomology, and spaces of modular foms

Recall (§ 3.1) that we denote $H_1(Y(K_\Sigma),\mathbf{Z})$ by $H_1(\Sigma,\mathbf{Z})$. These (co)homology groups may also be interpreted via group (co)homology. Suppose that $\Gamma\subset\mathrm{PGL}_2(\mathbf{C})$ is a discrete subgroup; then there are functorial isomorphisms

$$H^i(\Gamma\backslash\mathbf{H})\simeq H^i(\Gamma,A),$$

There is a similar identification on group homology. From the fact that $Y(K_\Sigma)$ is a disconnected sum of finitely many 3-manifolds which are of the form $\Gamma_0(M,\mathfrak{a})\backslash\mathbf{H}^3$, we deduce that each $\Gamma_0(M,\mathfrak{a})$ has cohomological dimension at most 3 (away from orbifold primes, i.e. on modules on which $w_F^{(2)}$ is invertible).

We make free use of compactly supported cohomology and Borel–Moore homology of a space X, denoted respectively by $H^i_c(X,\mathbf{Z})$ and $H_i^{\mathrm{BM}}(X,\mathbf{Z})$. The latter group is, by definition, homology of the complex of *locally finite* chains, and the compactly supported cohomology is the cohomology of the dual complex. If X is a manifold, then $H^i_c(X,\mathbf{C})$ is isomorphic to the cohomology of the complex of compactly supported differential forms.

3.4.1. Duality and the linking form. We discuss the compact case; for the noncompact case see § 5.1.2.

For a compact 3-manifold M, $H_0(M,\mathbf{Z}) = H_3(M,\mathbf{Z}) = \mathbf{Z}$, and Poincaré duality defines isomorphisms $H_1(M,\mathbf{Z}) \simeq H^2(M,\mathbf{Z})$ and $H_2(M,\mathbf{Z}) \simeq H^1(M.\mathbf{Z})$. From the universal coefficient theorem, we further deduce that

$$H^1(M,\mathbf{Q}/\mathbf{Z}) \simeq \mathrm{Hom}(H_1(M,\mathbf{Z}),\mathbf{Q}/\mathbf{Z})).$$

For an abelian group A, let $A^\vee = \mathrm{Hom}(A,\mathbf{Q}/\mathbf{Z})$. If A is finite, then $A^{\vee\vee} = A$. The exact sequence $0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0$ induces a long exact sequence in homology; esp.

$$H^1(\mathbf{Q}) \xrightarrow{\alpha} H^1(\mathbf{Q}/\mathbf{Z}) \xrightarrow{\delta} H_1(\mathbf{Z})^\vee \xrightarrow{\beta} H^2(\mathbf{Q}) \tag{3.4.1.1}$$

δ induces an isomorphism of $\mathrm{coker}(\alpha)$ with $\ker(\beta)$, i.e. an isomorphism of the "torsion parts" of $H^1(\mathbf{Q}/\mathbf{Z})$ and $H^1(\mathbf{Q}/\mathbf{Z})^\vee$; the pairing $(x,y) \mapsto (x,\delta y)$ or $(x,\delta^{-1}y)$ therefore induces a *perfect* pairing on these torsion groups.

This is the " linking form"; it can also be considered as a perfect pairing

$$H_1(M,\mathbf{Z})_{\mathrm{tors}} \times H_1(M,\mathbf{Z})_{\mathrm{tors}} \to \mathbf{Q}/\mathbf{Z}.$$

obtained from intersection-product together with the connecting homomorphism $H_2(\mathbf{Q}/\mathbf{Z})_{\mathrm{tors}} \xrightarrow{\sim} H_1(\mathbf{Z})_{\mathrm{tors}}$.

3.4.1.1. *Duality for orbifolds.* For compact *orbifolds*, these conclusions still hold so long as we localize away from the order of any torsion primes: If N is such that any isotropy group has order dividing N, then Poincaré duality holds with $\mathbf{Z}[\frac{1}{N}]$ coefficients, and the foregoing goes through replacing $\mathbf{Z}$ by $\mathbf{Z}[\frac{1}{N}]$ and $\mathbf{Q}/\mathbf{Z} \cong \bigoplus_p \mathbf{Q}_p/\mathbf{Z}_p$ by $\bigoplus_{(p,N)=1} \mathbf{Q}_p/\mathbf{Z}_p$.

Over $\mathbf{Z}$ the following still holds: The pairing

$$H_1(Y(K_\Sigma),\mathbf{Z})_{\mathrm{tf}} \times H_2(Y(K_\Sigma),\mathbf{Z})_{\mathrm{tf}} \to \mathbf{Z}$$

(obtained from the corresponding pairing with $\mathbf{Q}$ coefficients, rescaled so the image lies in $\mathbf{Z}$) is perfect away from orbifold primes, i.e. the discriminant of the corresponding Gram matrix is divisible only by orbifold primes.

3.4.2. Atkin-Lehner involutions. The manifold $Y(K_\Sigma)$ has a canonical action of the group $(\mathbf{Z}/2\mathbf{Z})^\Sigma$, generated by the so-called Atkin-Lehner involutions. Indeed, for every $v \in \Sigma$ for which $\mathbb{G}$ is split the element

$$w_v := \begin{pmatrix} 0 & 1 \\ \varpi_v & 0 \end{pmatrix} \in \mathbb{G}(F_v) \tag{3.4.2.1}$$

normalizes $K_{0,v}$; for $v \in \Sigma$ for which $\mathbb{G}$ is not split we take w_v to be any element of $\mathbb{G}(F_v)$ not in $K_{0,v}$.

The group generated by the w_v generates a subgroup of $N(K_\Sigma)/K_\Sigma$ isomorphic to $(\mathbf{Z}/2\mathbf{Z})^\Sigma$.

The element w_v also normalizes $K_{1,v}$ and it acts by inversion on the abelian group $K_v/K_{1,v}$ (which is isomorphic to $k_v^\times$ or $l_v^\times/k_v^\times$, according to whether $\mathbb{G}$ is split or not).

If M is a space of modular forms with an action of an (understood) Atkin-Lehner involution w, we write $M^\pm$ for the $+$ and $-$ eigenspaces of w on M. An example of particular importance is the cohomology of $Y_0(\mathfrak{q})$ for $\mathfrak{q}$ prime; thus, for example, we write

$$H_1(Y_0(\mathfrak{q}), \mathbf{Q})^- := \{z \in H_1(Y_0(\mathfrak{q}), \mathbf{Q}) : w_\mathfrak{q} z = -z\}.$$

(We do not use this notation when it is ambiguous which Atkin-Lehner involution is being referred to.)

3.4.3. Hecke Operators. For any $g \in \mathbb{G}(\mathbb{A}_f)$ we may consider the diagram

$$Y(K_\Sigma) \leftarrow Y(gK_\Sigma g^{-1} \cap K_\Sigma) \rightarrow Y(K_\Sigma) \tag{3.4.3.1}$$

that arise from, respectively, the identity on $\mathbb{G}(\mathbb{A}_f)$ and right multiplication by g on $\mathbb{G}(\mathbb{A}_f)$.

Now fix any prime $\mathfrak{p} \notin S$ and let $g_\mathfrak{p} \in \mathbb{G}(F_\mathfrak{p})$ be the preimage of $\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$ under the fixed isomorphism of (3.2.2.1), and let $g \in \mathbb{G}(\mathbb{A}_f)$ be the element with component $g_\mathfrak{p}$ at the prime $\mathfrak{p}$ and all other components trivial. In this case (3.4.3.1) produces two maps $Y(\Sigma \cup \{\mathfrak{q}\}) \to Y(\Sigma)$, often referred to as the two degeneracy maps. The corresponding operator on homology or cohomology of $Y(\Sigma)$, obtained by pullback followed by pushforward, is the $\mathfrak{p}$th Hecke operator, denoted $T_\mathfrak{p}$.

These Hecke operators preserve $H^\bullet(Y(K_\Sigma), \mathbf{Z})$, but not the cohomology of the connected components. Indeed, the action on the component group is via the determinant map on $\mathbb{G}(\mathbb{A}_f)$ and the natural action of $\mathbb{A}_f^\times$ on $A = F^\times \backslash \mathbb{A}_f^\times / U$.

We now define the Hecke algebra (cf. definition 2.2.4).

Definition 3.4.4. *Let $\mathbf{T}_\Sigma$ denote the subring of* End $H_q(Y(K_\Sigma), \mathbf{Z})$ *generated by Hecke endomorphisms $T_\mathfrak{q}$ for all primes $\mathfrak{q}$ not dividing Σ.*

Definition 3.4.5. *If f is an eigenform for the Hecke algebra, we denote by $a(f, \mathfrak{q})$ the eigenvalue of $T_\mathfrak{q}$ on f.*

3.4.6. The abstract Hecke algebra. It is sometimes convenient to use instead the *abstract* Hecke algebra.

By this we mean: For any finite set of places Ω, let $\mathscr{T}_\Omega := \mathbf{Z}[\mathcal{T}_\mathfrak{q} : \mathfrak{q} \notin \Omega]$ be the free commutative algebra on *formal* symbols $\mathcal{T}_\mathfrak{q}$. Then there is an obvious surjection $\mathscr{T}_\Omega \twoheadrightarrow \mathbf{T}_\Omega$. We will very occasionally denote $\mathscr{T}_\Omega$ simply *by* $\mathbf{T}_\Omega$; when this is so we clarify in advance.

A typical situation where this is of use is when we are working with two levels Σ, Σ' simultaneously:

Suppose given a maximal ideal $\mathfrak{m}$ of $\mathbf{T}_\Sigma$. We will on occasion refer to "the completion of the cohomology $H_q(Y(\Sigma'))$ at $\mathfrak{m}$." By convention this means – unless otherwise specified – the following: Any ideal $\mathfrak{m}$ of $\mathbf{T}_\Sigma$ gives rise to a maximal ideal $\mathfrak{m}'$ of $\mathscr{T}_{\Sigma\cup\Sigma'}$ via preimage under the maps

$$\mathscr{T}_{\Sigma\cup\Sigma'} \to \mathbf{T}_\Sigma,$$

and by $H_q(\Sigma')_\mathfrak{m}$ we mean, *by convention*, the completion of $H_q(\Sigma')$ at the maximal ideal $\mathfrak{m}'$ of $\mathscr{T}_{\Sigma\cup\Sigma'}$.

3.4.6.1. *Localization versus Completion.* The Hecke algebras $\mathbf{T}\otimes\mathbf{Z}_p$ are always semi-local rings, that is, the completions $\mathbf{T}_\mathfrak{m}$ are finite $\mathbf{Z}_p$-algebras. It follows that the *completion* $H_q(\mathbf{Z}_p)_\mathfrak{m}$ of $H_q(\mathbf{Z}_p)$ is equal to the *localization* $H_q(\mathbf{Z}_p)_{(\mathfrak{m})}$.

Similarly, whenever $H_q(\mathbf{Z})$ is a finite group, completion may be identified with localization.

When we are dealing with $\mathbf{Q}_p/\mathbf{Z}_p$-coefficients, the localization of $H_q(\mathbf{Q}_p/\mathbf{Z}_p)$ at $\mathfrak{m}$ is isomorphic not to the completion but rather to the $\mathfrak{m}^\infty$-torsion:

$$\varinjlim H_q(\mathbf{Q}_p/\mathbf{Z}_p)[\mathfrak{m}^n] = H_q(\mathbf{Q}_p/\mathbf{Z}_p)[\mathfrak{m}^\infty] \xrightarrow{\sim} H_q(\mathbf{Q}_p/\mathbf{Z}_p)_{(\mathfrak{m})}.$$

By a slight abuse of terminology, we will denote these spaces also by $H_q(\mathbf{Q}_p/\mathbf{Z}_p)_\mathfrak{m}$.

In general, the completion $\mathbf{T}_\mathfrak{m}$ is different from $\mathbf{T}_{(\mathfrak{m})}$ if $\mathbf{T}$ is infinite. We shall only ever be concerned with modules over the first algebra.

3.4.7. Degeneracy maps. Suppose that $\mathfrak{q}\in\Sigma$ is not in S. The two natural degeneracy maps $Y(K_\Sigma)\to Y(K_{\Sigma/\mathfrak{q}})$ noted in the previous section give rise to a pair of maps:

$$\Phi : H^1(\Sigma/\mathfrak{q}, \mathbf{Q}/\mathbf{Z})^2 \to H^1(\Sigma, \mathbf{Q}/\mathbf{Z}),$$
$$\Phi^\vee : H^1(\Sigma, \mathbf{Q}/\mathbf{Z}) \to H^1(\Sigma/\mathfrak{q}, \mathbf{Q}/\mathbf{Z})^2,$$

where the first map is induced from the degeneracy maps whilst the second is the transfer homomorphism induced from the first. In the same way, we obtain maps:

$$\Psi : H_1(\Sigma, \mathbf{Z}) \to H_1(\Sigma/\mathfrak{q}, \mathbf{Z})^2,$$
$$\Psi^\vee : H_1(\Sigma/\mathfrak{q}, \mathbf{Z})^2 \to H_1(\Sigma, \mathbf{Z}).$$

The map Ψ can be obtained from Φ (and similarly $\Psi^\vee$ from $\Phi^\vee$) by applying the functor $\mathrm{Hom}(-,\mathbf{Q}/\mathbf{Z})$.

By taking the difference (resp. sum) of the maps $\Phi^\vee$, we obtain

$$H_1(\Sigma,\mathbf{Z})^- \to H_1(\Sigma/\mathfrak{q},\mathbf{Z}), \quad H_1(\Sigma,\mathbf{Z})^+ \to H_1(\Sigma/\mathfrak{q},\mathbf{Z})$$

where $+$ and $-$ refer to eigenspaces for the Atkin-Lehner involution $w_\mathfrak{q}$ corresponding to the prime $\mathfrak{q}$.

When we write "degeneracy map" from either $H_1(\Sigma,\mathbf{Z})^-$ or the companion group $H_1(\Sigma,\mathbf{Z})^+$ to $H_1(\Sigma/\mathfrak{q},\mathbf{Z})$, we always have in mind these differenced or summed versions of $\Phi^\vee$. Similar remarks apply for $\Phi,\Psi,\Psi^\vee$.

LEMMA 3.4.8. *The composite maps $\Phi^\vee\circ\Phi$ and $\Psi\circ\Psi^\vee$ are equal to*

$$\begin{pmatrix} (N(\mathfrak{q})+1) & T_\mathfrak{q} \\ T_\mathfrak{q} & (N(\mathfrak{q})+1) \end{pmatrix},$$

where $T_\mathfrak{q}$ is the Hecke operator.

REMARK 3.4.9. Were we dealing with GL_2 rather than PGL_2 (or the full group of units of a quaternion algebra, and not its quotient by center) this would be replaced by

$$\begin{pmatrix} \langle\mathfrak{q}\rangle(N(\mathfrak{q})+1) & T_\mathfrak{q} \\ T_\mathfrak{q} & (N(\mathfrak{q})+1) \end{pmatrix},$$

where $\langle \mathfrak{q} \rangle$ and $T_{\mathfrak{q}}$ are the diamond and Hecke operators respectively. In many of our results where the expression $T_q^2 - (N(\mathfrak{q})+1)^2$ appears, it can be generalized to the case of GL_2 simply by replacing it by $T_q^2 - \langle \mathfrak{q} \rangle (N(\mathfrak{q})+1)^2$.

3.5. Normalization of metric and measures

3.5.1. The metric on hyperbolic 3-space. By $\mathbf{H}^3$ we denote hyperbolic 3-space , i.e. triples $(x_1, x_2, y) \in \mathbf{R}^2 \times \mathbf{R}_{>0}$. We shall sometimes identify it with $\mathbf{C} \times \mathbf{R}_{>0}$ via $(x_1, x_2, y) \mapsto (x_1 + ix_2, y)$. We equip it with the metric of constant curvature

$$g := \frac{dx_1^2 + dx_2^2 + dy^2}{y^2}.$$

There is a metric-preserving action of $\mathrm{PGL}_2(\mathbf{C})$ on $\mathbf{H}^3$, in which the subgroup PU_2 preserves the point $(0,0,1)$, the action of the element $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ is given by translation in the $x_1 + ix_2$ variable, and the action of $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ is given by $(z, y) \in \mathbf{C} \times \mathbf{R}_{>0} \mapsto (az, |a|y)$.

This normalizes a hyperbolic metric on every manifold $Y(K)$.

On the other hand, the action of $\mathrm{PGL}_2(\mathbf{C})$ on $\mathbf{H}^3$ gives rise to another canonical metric g_K, namely, that induced by the Killing form on $\mathrm{PGL}_2(\mathbf{C})$ considered as a real Lie group. (For each point $x \in \mathbf{H}^3$, the tangent space $T_x\mathbf{H}^3$ is identified with a certain quotient of the Lie algebra, on which the Killing form induces a metric.) Then[4] $g_K = 2g$, so that the Laplacian for g corresponds to four times the Casimir.

Let $\mathfrak{pgl}_2 \supset \mathfrak{pu}_2$ denote the Lie algebra of $\mathrm{PGL}_2(\mathbf{C}) \supset \mathrm{PU}_2$; the quotient $\mathfrak{p} := \mathfrak{pgl}_2/\mathfrak{pu}_2$ it is identified naturally with the tangent space to $\mathbf{H}^3$ at the PU_2-fixed point. The tangent bundle to $\mathbf{H}^3$ is then naturally identified with the quotient $\mathrm{PGL}_2(\mathbf{C}) \times \mathfrak{p}/\mathrm{PU}_2$, where the PU_2 acts via the adjoint action on $\mathfrak{p}$. This identification is $\mathrm{PGL}_2(\mathbf{C})$-equivariant, where $\mathrm{PGL}_2(\mathbf{C})$ acts on $\mathrm{PGL}_2(\mathbf{C})$ by left-translation and trivially on $\mathfrak{p}$.

3.5.2. Measures. There is a volume form on $Y(K)$ specified by the Riemannian metric. When we speak of the volume of a component of $Y(K)$, as in § 3.5.3, we shall always mean with reference to that measure, unless otherwise stated.

We fix for later reference a measure on $\mathbb{A}$: For each finite place v, equip F_v with the additive Haar measure dx that assigns mass 1 to the maximal compact subring; for each archimedean place v, we assign F_v the usual Lebesgue measure (for v complex, we mean the measure that assigns the set $\{z \in F_v : |z| \leq 1\}$ the mass π).

We equip $F_v^\times$ with the measure $\frac{dx}{|x|} \cdot \zeta_{F_v}(1)$ (this assigns measure 1 to the maximal compact subgroup).

In § 6.5.4 we will depart slightly from these normalizations for compatibility with the work of Waldspurger, to which § 6.5.4 is really a corollary. Similarly we will depart slightly in § 5.7.9.

---

[4]If $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sl}_{2,\mathbf{C}}$, then $\langle h, h \rangle = 16$. But the image of h in the tangent space at $(0,0,1)$ has hyperbolic length 2.

3.5.3. Volumes. We shall use the following convention:

> If X is — possibly disconnected — hyperbolic 3-manifold, then $\mathrm{vol}(X)$ denotes the *product* of the volumes of the connected components of X.

THEOREM 3.5.4 (Volumes; Borel [**7**]). *Let S be the set of primes that ramify in D. Let Σ be a set of primes containing all elements of S. Then*

$$\mathrm{vol}(Y_0(\Sigma,\mathfrak{a})) = \frac{\mu\cdot\zeta_F(2)\cdot|d_F|^{3/2}}{2^m(4\pi^2)^{[F:\mathbf{Q}]-1}}\cdot\prod_{\mathfrak{q}\in S}(N\mathfrak{q}-1)\prod_{\mathfrak{q}\in\Sigma\setminus S}(N\mathfrak{q}+1),$$

for some $m\in\mathbf{N}$ and some integer μ depending only on F.

According to our convention, then, the volume of $Y_0(\Sigma)$ is given by the right hand side of the above formula, *raised to the power* of the number of components, which is equal to the order of $\mathrm{Cl}(\mathscr{O}_F)/2\mathrm{Cl}(\mathscr{O}_F)$ if $\mathrm{Cl}(\mathscr{O}_F)$ is the class group of F. (See (3.3.4.2)).

3.6. S-arithmetic groups

We shall need at a few points the S-arithmetic analogue of the spaces $Y(K)$. Roughly, these bear the same relation to $\mathbb{G}(\mathscr{O}[\mathfrak{q}^{-1}])$ as the space $Y(K)$ bears to $\mathbb{G}(\mathscr{O})$.

Let $\mathfrak{q}$ be a prime that at which $\mathbb{G}$ is split. Recall that $\mathbb{G}(F_\mathfrak{q})\cong\mathrm{PGL}_2(F_\mathfrak{q})$ acts on an infinite $(\mathrm{N}(\mathfrak{q})+1)$-valent tree $\mathcal{T}_q$, the Bruhat–Tits building. For the purpose of this paper a "tree" is a topological space, not a combinatorial object: it is a CW-complex obtained by gluing intervals representing edges to vertices in the usual fashion. This can be understood as an analogue of the action of G_∞ on G_∞/K_∞. In particular $\mathcal{T}_q$ is contractible and the vertex and edge stabilizers are maximal compact subgroups.

We shall also deal with a "doubled" tree that places the role, relative to $\mathrm{PGL}_2(F_\mathfrak{q})$, of the "upper and lower half-planes" for $\mathrm{PGL}_2(\mathbf{R})$. Namely, let $\mathcal{T}_\mathfrak{q}^\pm$ be the union of two copies of $\mathcal{T}_\mathfrak{q}$, and the $\mathrm{PGL}_2(F_\mathfrak{q})$-action is given by the product of the usual action on $\mathcal{T}_\mathfrak{q}$ and the determinant-valuation action of $\mathrm{PGL}_2(F_\mathfrak{q})$ on $\mathbf{Z}/2\mathbf{Z}$ (in which g acts nontrivially if and only if $\det g$ is of odd valuation).

We set, for an arbitrary open compact subgroup $K\subset\mathbb{G}(\mathbb{A}_f)$,

$$Y(K[\frac{1}{\mathfrak{q}}]) = \mathbb{G}(F)\backslash\left(\mathbf{H}^3\times\mathcal{T}_\mathfrak{q}\times\mathbb{G}(\mathbb{A}^{(\infty,\mathfrak{q})})/K^{(\mathfrak{q})}\right).$$

$$Y(K[\frac{1}{\mathfrak{q}}])^\pm = \mathbb{G}(F)\backslash\left(\mathbf{H}^3\times\mathcal{T}_\mathfrak{q}^\pm\times\mathbb{G}(\mathbb{A}^{(\infty,\mathfrak{q})})/K^{(\mathfrak{q})}\right)$$

where $K^{(\mathfrak{q})}$ is the projection of K to $\mathbb{G}(\mathbb{A}^{(\infty,\mathfrak{q})})$.

We shall sometimes also denote this space $Y(K[\frac{1}{\mathfrak{q}}])$ simply by $Y(\frac{1}{\mathfrak{q}})$ when the choice of K is implicit. We will often denote it by $Y(\Sigma[\frac{1}{\mathfrak{q}}])$ when $K=K_\Sigma$. Evidently, $Y(\frac{1}{\mathfrak{q}})$ is the quotient of $Y^\pm(\frac{1}{\mathfrak{q}})$ by a natural involution (that switches the two components of $\mathcal{T}_\mathfrak{q}^\pm$).

We abbreviate their cohomology in a similar fashion to that previously discussed, including superscripts $\pm$ when necessary to distinguish between the above spaces, e.g.:

$$H^*(\Sigma[\frac{1}{\mathfrak{q}}],\mathbf{Z})^\pm \quad := \quad H^*(Y(K_\Sigma[\frac{1}{\mathfrak{q}}])^\pm,\mathbf{Z}). \tag{3.6.0.1}$$

Inverting the prime 2, the fixed space of $H^*(\Sigma[\frac{1}{\mathfrak{q}}], -)^{\pm}$ under the action induced by the mentioned involution is $H^*(\Sigma[\frac{1}{\mathfrak{q}}], -)$.

The two fixed points for the maximal compact $K_{\mathfrak{q}}$ on the tree $\mathcal{T}_{\mathfrak{q}}^{\pm}$ gives rise to two inclusions $Y(\Sigma/\mathfrak{q})$ into $Y(\Sigma[1/\mathfrak{q}])^{\pm}$; these two inclusions collapse to the same map in the quotient $Y(\Sigma[1/\mathfrak{q}])$. In fact, topologically,

$$Y(\Sigma[\tfrac{1}{\mathfrak{q}}])^{\pm} \text{ is } Y(\Sigma) \times [0,1] \text{ glued to two copies of } Y(\Sigma/\mathfrak{q}); \tag{3.6.0.2}$$

this structure being obtained by dividing $\mathcal{T}_{\mathfrak{q}}$ into 1-cells and 0-cells; the gluing maps are the two degeneracy maps $Y(\Sigma) \to Y(\Sigma/\mathfrak{q})$; one obtains a corresponding description of $Y(\Sigma[\frac{1}{\mathfrak{q}}])$ after quotienting by the involutions.

Example 3.6.1. *Suppose that $\mathbb{G} = \mathrm{PGL}(2)/F$, that $\mathrm{Cl}(\mathscr{O}_F)$ is odd, and that $K = \mathrm{PGL}_2(\mathscr{O}_{\mathbb{A}})$. Then the spaces $Y(K)$, $Y^+(K[\frac{1}{\mathfrak{q}}])$, and $Y(K[\frac{1}{\mathfrak{q}}])$ are all connected $K(\pi,1)$-spaces, and*

$$\begin{aligned} \pi_1(Y(K)) &\cong \mathrm{PGL}_2(\mathscr{O}); \\ \pi_1(Y(K[\frac{1}{\mathfrak{q}}])) &\cong \mathrm{PGL}_2(\mathscr{O}[\frac{1}{\mathfrak{q}}]); \\ \pi_1\left(Y(K[\frac{1}{\mathfrak{q}}]^{\pm}\right) &\cong \mathrm{PGL}_2(\mathscr{O}[\frac{1}{\mathfrak{q}}])^{(\mathrm{ev})}, \end{aligned}$$

where $\mathrm{PGL}_2^{(\mathrm{ev})} \subset \mathrm{PGL}_2(\mathscr{O}[\frac{1}{\mathfrak{q}}])$ consists of those elements whose determinant has even valuation at $\mathfrak{q}$. In particular, the homology of each of these spaces is identified with the group homology of the right-hand groups.

In general we may write

$$Y(K_{\Sigma}[\frac{1}{\mathfrak{q}}])^{\pm} = \coprod_{A/\mathfrak{q}} \Gamma_0^{(\mathfrak{q})}(\Sigma, \mathfrak{a}) \backslash \left(\mathbf{H}^3 \times \mathcal{T}_{\mathfrak{q}}^{\pm}\right)$$

with the notation of (3.3.4.1); here the group $\Gamma_0^{(\mathfrak{q})}(\Sigma, \mathfrak{a})$ is defined similarly to $\Gamma_0(\Sigma, \mathfrak{a})$, but replacing $K_{\Sigma,0}$ by $K_{\Sigma,0} \cdot \mathbb{G}(F_{\mathfrak{q}})$; and $A_{\mathfrak{q}}$ is now the quotient of A by the class of $\mathfrak{q}$.

Denote by $\Gamma_0^{(\mathfrak{q})}(\Sigma, \mathfrak{a})^{(\mathrm{ev})}$ those elements of $\Gamma_0^{(\mathfrak{q})}(\Sigma, \mathfrak{a})$ whose determinant has even valuation at $\mathfrak{q}$. Note that $\Gamma_0^{(\mathfrak{q})} \neq \Gamma_0^{(\mathfrak{q})}(\Sigma, \mathfrak{a})^{(\mathrm{ev})}$ if and only if $\Gamma_0^{(\mathfrak{q})}(\Sigma, \mathfrak{a})$ contains an element that switches the two trees in $\mathcal{T}_{\mathfrak{q}}^{\pm}$; this is so (for any $\mathfrak{a}$) if and only if $\mathfrak{q}$ is a square in the ideal class group. [5]

Therefore $Y(\Sigma[\frac{1}{\mathfrak{q}}])^{\pm}$ is homeomorphic to

$$\coprod_{A/\mathfrak{q}} \Gamma_0^{(\mathfrak{q})}(\Sigma, \mathfrak{a})^{(\mathrm{ev})} \backslash (\mathbf{H}^3 \times \mathcal{T}_{\mathfrak{q}}), \ \mathfrak{q} \text{ a square}, \tag{3.6.1.1}$$

and *two copies of the same*, if $\mathfrak{q}$ is not a square. Note that in all cases it has exactly the same number of connected components as $Y(\Sigma)$.

[5] We are asking for the existence of an element of $\mathbb{G}(F)$ that belongs to the set $\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} K_{\Sigma,v} \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}^{-1}$ for every finite v besides $\mathfrak{q}$, and whose determinant has odd valuation at $\mathfrak{q}$ itself. One now applies the strong approximation theorem.

3.7. Congruence homology

One class of elements in cohomology that should be considered "trivial" are those classes arising from congruence covers (cf. the discussion of § 2.2.7).

3.7.0.1. *Congruence homology for arithmetic groups.* Before the definition for $Y(K)$, let us give the corresponding definition for arithmetic groups:

If Γ is any arithmetic group, it admits a map to its congruence completion $\widehat{\Gamma}$: the completion of Γ for the topology defined by congruence subgroups. This yields a natural surjection

$$H_1(\Gamma, \mathbf{Z}) \longrightarrow H_1(\widehat{\Gamma}, \mathbf{Z})$$

which we call the "congruence quotient" of homology, or, by a slight abuse of notation, the "congruence homology" $H_{1,\mathrm{cong}}(\Gamma, \mathbf{Z})$.

For example, if $\Gamma = \Gamma_0(N) \subset \mathrm{SL}_2(\mathbf{Z})$, then the morphism

$$H_1(\Gamma_0(N), \mathbf{Z}) \longrightarrow (\mathbf{Z}/N\mathbf{Z})^\times$$

arising from $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a$, factors through the congruence homology.

3.7.0.2. *Congruence homology for* $Y(K)$. Recall that $Y(K) = \coprod \Gamma_i \backslash \mathbf{H}^3$ for various arithmetic subgroups Γ_i (cf. § 3.3.4). For any coefficient group A, understood to have trivial Γ_i-action, we define $H_{1,\mathrm{cong}}(Y(K), A)$ as the quotient of $H_1(Y(K), A)$ defined by the map:

$$\bigoplus H_1(\Gamma_i, A) \twoheadrightarrow \bigoplus H_1(\widehat{\Gamma}_i, A), \tag{3.7.0.2}$$

Note that, at least for H_1, this map is indeed surjective.

3.7.0.3. *Reformulation.* There is a convenient adelic formalism to work with the right-hand side of (3.7.0.2), which allows us to minimize the amount of indexing. Although it has no essential content it makes the statements more compact:

Let $\overline{\mathbb{G}(F)}$ be the closure of $\mathbb{G}(F)$ in $\mathbb{G}(\mathbb{A}_f)$. Set

$$Y(K)^\wedge := \overline{\mathbb{G}(F)} \backslash \mathbb{G}(\mathbb{A}_f)/K. \tag{3.7.0.3}$$

As explained below, we interpret the right-hand side not as a finite set, but rather as the classifying space of a certain groupoid.

As a set, $\overline{\mathbb{G}(F)} \backslash \mathbb{G}(\mathbb{A}_f)/K$ is finite; it is identified with the set of components of $Y(K)$. We consider instead the groupoid whose objects are given by $\mathbb{G}(\mathbb{A}_f)/K$ and the morphisms are given by left multiplication by $\overline{\mathbb{G}(F)}$, and we define $Y(K)^\wedge$ to be the classifying space of this groupoid.

In this groupoid, there are finitely many isomorphism classes of objects – we refer to these isomorphism classes as the "underlying set" of $Y(K)^\wedge$ – and each has a profinite isotropy group. The underlying set of $Y(K)^\wedge$ can thus be identified with components of $Y(K)$, and the isotropy group is exactly the congruence completion of the π_1 of the corresponding component of $Y(K)$.

Thus, informally speaking, one can imagine $Y(K)^\wedge$ as a "profinite orbifold": a finite set of points, to each of which is associated a profinite isotropy group. The homology of $Y(K)^\wedge$ is identified with $\bigoplus_x H_*(J_x)$, the sum being taken over a set of representatives for isomorphism classes, and J_x being the isotropy group of x.

Also there is a natural $Y(K) \to Y(K)^\wedge$ induces a map on homology, and we may reinterpret the prior definition:

For an abelian group A, $H_{1,\text{cong}}(Y(K),A)$ is the quotient of $H_1(Y(K))$ defined by

$$H_1(Y(K),A) \twoheadrightarrow H_1(Y(K)^\wedge,A),$$

and dually we define $H^1_{\text{cong}}(Y(K),A)$ as the image of

$$H^1(Y(K)^\wedge,A) \hookrightarrow H^1(Y(K),A).$$

If $[c] \in H^1(Y(K),-)$ lies in the image of $H^1_{\text{cong}}(Y(K),-)$, then we say that $[c]$ is congruence.

Now the stabilizer of the point gK in $Y(K)^\wedge$ is the subgroup $gKg^{-1} \cap \overline{\mathbb{G}(F)} = gK^+g^{-1}$, where K^+ is the subgroup of elements of K whose determinant (reduced norm, if $\mathbb{G}$ corresponds to a quaternion algebra) has the same square class (in $\mathbb{A}_f^\times/(\mathbb{A}_f^\times)^2$) as an element of $F^\times$. Any cohomology class for $Y(K)^\wedge$ thus associates to any such gK a class in $H^1(gK^+g^{-1})$, equivariantly for the action of $\overline{\mathbb{G}(F)}$.

We may thereby identify $H^1_{\text{cong}}(Y(K)^\wedge,A)$ with

$$\text{functions: } g \in \overline{\mathbb{G}(F)}\backslash\mathbb{G}(\mathbb{A}_f) \to H^1(K^+,A) \tag{3.7.0.4}$$

that are equivariant for the action of K. We discuss computing the cohomology of K^+ in § 3.7.2. Under mild conditions, the conjugation action of K on $H^1(K^+,A)$ will be trivial.

Finally, note that, by the strong approximation theorem the closure of $\mathbb{G}(F)$ contains the $\mathbb{A}_f$-points of the derived group $[\mathbb{G},\mathbb{G}]$, and therefore $\overline{\mathbb{G}(F)}$ is a *normal* subgroup of $\mathbb{G}(\mathbb{A}_f)$, with abelian quotient; *in particular,* $\mathbb{G}(\mathbb{A}_f)$ *acts on* $Y(K)^\wedge$ *by "left multiplication."* In particular, $\mathbb{G}(\mathbb{A}_f)$ acts on the homology of $Y(K)^\wedge$. This action has no counterpart for $Y(K)$. We will make use of this action later.

Remark 3.7.1. We may alternately characterize the congruence quotient of homology in the following way: It is the largest quotient of $H_1(Y(K),A)$ in which the homology of every "sufficiently deep" covering $Y(K')$ vanishes:

$$H_{1,\text{cong}}(Y(K),A) = \text{coker}\left(\varprojlim_{K'} H_1(Y(K'),A) \to H_1(Y(K),A)\right).$$

$$H^1_{\text{cong}}(Y(K),A) = \ker\left(H^1(Y(K),A) \to \varinjlim_{K'} H^1(Y(K'),A)\right).$$

3.7.1.1. *Degeneracy maps and Hecke operators.* An inclusion $K_1 \subset K_2$ induces push-forward maps $H_*(Y(K_1)^\wedge) \to H_*(Y(K_2)^\wedge)$ and transfer maps in the reverse direction. Consequently, the *degeneracy maps* (see § 3.4.7) and also Hecke operators act on the congruence quotient of homology. We will explicitly compute this Hecke action in §3.7.5.

Suppose $K_1 \subset K_2$ are level structures such that the natural map $Y(K_1)^\wedge \to Y(K_2)^\wedge$ induces bijections on the underlying sets; equivalently, $[K_2^+ : K_1^+] = [K_2 : K_1]$ or equivalently K_2^+ surjects to K_2/K_1. Then, with reference to the identification (3.7.0.4), the pushforward map $Y(K_1) \to Y(K_2)$ corresponds, at the level of such functions, simply to composing with $K_1^+ \to K_2^+$. (Note that the resulting function is really right K_2-equivariant: it is right K_2^+-invariant because K_2^+ acts trivially on its own homology, and then K_1, K_2^+ generate K_2 by assumption.) Similarly, the pullback map corresponds to composition with the transfer map associated to $K_1^+ \to K_2^+$.

3.7.1.2. *Liftable congruence homology.* We define $h_{\rm lif}(\Sigma)$ (the subscript stands for "liftable," see below for discussion) as the order of the cokernel of

$$H_1(\Sigma, \mathbf{Z})_{\rm tors} \to H_1(\Sigma, \mathbf{Z})_{\rm cong}.$$

In general, the image of $H_1(\Sigma, \mathbf{Z})_{\rm tors}$ in congruence homology need not be stable under the action of $\mathbb{G}(\mathbb{A}_f)$. Because of that fact, we will later need the following variant of the definition: For any ideal class $[\mathfrak{r}]$, we define $h_{\rm lif}(\Sigma; \mathfrak{r})$ as the quotient of $H_1(\Sigma, \mathbf{Z})_{\rm cong}$ by the span of $[\mathfrak{r}]^i H_1(\Sigma, \mathbf{Z})_{\rm tors}$ for all i, i.e. the cokernel of

$$\bigoplus_i H_1(\Sigma, \mathbf{Z})_{\rm tors} \stackrel{\bigoplus [\mathfrak{r}]^i}{\longrightarrow} H_1(\Sigma, \mathbf{Z})_{\rm cong}.$$

Note that $[\mathfrak{r}]^2$ acts trivially, so $i = 0$ and 1 suffice.

Since the map $H_1(\Sigma, \mathbf{Z}) \to H_1(\Sigma, \mathbf{Z})_{\rm cong}$ (and its analogue for $\mathbf{Z}_p$) is surjective, the cokernel of the map $H_1(\Sigma, \mathbf{Z})_{\rm tors} \to H_1(\Sigma, \mathbf{Z})_{\rm cong}$ consists of the congruence classes which can only be accounted for by characteristic zero classes, that is, they *lift* to a class of infinite order. The p-part of the order $h_{\rm lif}(\Sigma)$ of this cokernel may also be described as the order of $H^1(\Sigma, \mathbf{Q}_p/\mathbf{Z}_p)_{\rm div} \cap H^1_{\rm cong}(\Sigma, \mathbf{Q}_p/\mathbf{Z}_p)$.

3.7.2. Computation. Under mild assumptions on K, it is simple to actually explicitly compute congruence cohomology and homology. We do so in the present section (see in particular (3.7.3.1)).

Write $\mathrm{PSL}_2(F_v)$ for the image of $\mathrm{SL}_2(F_v)$ in $\mathrm{PGL}_2(F_v)$; the quotient of $\mathrm{PGL}_2(F_v)$ by $\mathrm{PSL}_2(F_v)$ is then isomorphic to the group of square classes in F_v. For the non-split quaternion algebra D_v/F_v, the image of D_v^1 in $D_v^\times/F_v^\times$ also has quotient of exponent 2. Recall the definition of $K_v^{(1)}$: it is the intersection of K_v with $\mathrm{PSL}_2(F_v)$ or the image of D_v^1, according to whether $\mathbb{G}$ splits or not at v.

3.7.2.1. *p-convenient subgroups.* The inclusion $K_v^{(1)} \to K_v$ often induces an isomorphism on $H_1(-, \mathbf{Q}_p/\mathbf{Z}_p)$. This property will be very useful:

DEFINITION 3.7.3. *K_v is* p-convenient *if the inclusion $K_v^{(1)} \to K_v$ induces an isomorphism on $H_1(-, \mathbf{Q}_p/\mathbf{Z}_p)$. A subgroup $K \subset \mathbb{G}(\mathbb{A}_f)$ of the form $\prod_v K_v$ is p-convenient if K_v is p-convenient for all v, and convenient if it is p-convenient for all odd p.*

Now suppose $K = \prod K_v$, and let K^+ be the isotropy group of any point of $Y(K)^\vee$. The natural inclusion $K^+ \to K$ fits inside

$$\prod K_v^{(1)} \hookrightarrow K^+ \hookrightarrow \prod K_v.$$

If K_v is p-convenient, then these maps induce an isomorphism

$$H^1(K^+, \mathbf{Q}_p/\mathbf{Z}_p) \stackrel{\sim}{\leftarrow} \prod_v H^1(K_v, \mathbf{Q}_p/\mathbf{Z}_p). \qquad (3.7.3.1)$$

In particular, the congruence cohomology can be computed locally from this formula and does not depend on the choice of connected component.

3.7.3.1. We now turn to showing that many of the level structures of interest are indeed p-convenient and explicitly computing their cohomologies:

LEMMA 3.7.4. *Assume that $p > 2$. We have the following:*

(i) *If either $p > 3$ or $\mathrm{N}(v) = \#\mathscr{O}_v/v > 3$, then*

$$H^1(\mathrm{PGL}_2(\mathscr{O}_v), \mathbf{Q}_p/\mathbf{Z}_p) = H^1(\mathrm{PSL}_2(\mathscr{O}_v), \mathbf{Q}_p/\mathbf{Z}_p) = 0.$$

(ii) *If $p = 3$ and $\mathscr{O}_v/v = \mathbf{F}_3$, then $H^1(\mathrm{PSL}_2(\mathscr{O}_v), \mathbf{Q}_p/\mathbf{Z}_p) = 0$, but*

$$H^1(\mathrm{PGL}_2(\mathscr{O}_v), \mathbf{Q}_p/\mathbf{Z}_p) \simeq H^1(\mathrm{PGL}_2(\mathbf{F}_3), \mathbf{Q}_3/\mathbf{Z}_3) = H^1(A_4, \mathbf{Q}_3/\mathbf{Z}_3) = \mathbf{Z}/3\mathbf{Z}.$$

(iii) *If $K_v = \mathbb{G}(\mathscr{O}_v)$ and $\mathbb{G}$ is* non-split *at v, then*

$$H^1(K_v^{(1)}, \mathbf{Q}_p/\mathbf{Z}_p) \xleftarrow{\sim} H^1(K_v, \mathbf{Q}_p/\mathbf{Z}_p) = \mathbf{Z}_p/(\mathrm{N}(v)+1)\mathbf{Z}_p.$$

(iv) *If $K_{0,v} \subset \mathrm{PGL}_2(\mathscr{O}_v)$ is of $\Gamma_0(v)$-type,*

$$H^1(K_{0,v}^{(1)}, \mathbf{Q}_p/\mathbf{Z}_p) \xleftarrow{\sim} H^1(K_{0,v}, \mathbf{Q}_p/\mathbf{Z}_p) = \mathbf{Z}_p/(\mathrm{N}(v)-1)\mathbf{Z}_p.$$

(v) *If the image of the determinant map $K_v \subset \mathrm{PGL}_2(\mathscr{O}_v) \to \mathscr{O}_v^\times/\mathscr{O}_v^{\times 2}$ is trivial, then $H^1(K_v^{(1)}, \mathbf{Q}_p/\mathbf{Z}_p) \xleftarrow{\sim} H^1(K_v, \mathbf{Q}_p/\mathbf{Z}_p)$.*

In particular, if K_v is satisfies one of (i), (iii), (iv), or (v), then it is p-convenient in the language of Definition 3.7.3.

PROOF. By inflation-restriction, there is an exact sequence:

$$H^1(\mathrm{PSL}_2(k_v), \mathbf{Q}_p/\mathbf{Z}_p) \hookrightarrow H^1(\mathrm{PSL}_2(\mathscr{O}_v), \mathbf{Q}_p/\mathbf{Z}_p) \to H^1(K(v), \mathbf{Q}_p/\mathbf{Z}_p)^{\mathrm{PSL}_2(k_v)},$$

where $K(v)$ is the kernel of $\mathrm{PSL}_2(\mathscr{O}_v) \to \mathrm{PSL}_2(k_v)$. If $\#k_v > 3$, then $\mathrm{PSL}_2(k_v)$ is simple and non-abelian, and hence the first group vanishes. If $\#k_v = 3$ and $p > 3$, then $\mathrm{PSL}_2(k_v)$ is a group of order prime to p and the cohomology still vanishes. The group $K(v)$ is pro-v, and so $H^1(K(v), \mathbf{Q}_p/\mathbf{Z}_p)$ vanishes unless $v|p$. By Nakayama's lemma, it suffices to prove that

$$H^1(K(v), \mathbf{Z}/p\mathbf{Z})^{\mathrm{PSL}_2(k_v)} = H^1(K(v), \mathbf{Q}_p/\mathbf{Z}_p)[p]^{\mathrm{PSL}_2(k_v)}$$

vanishes (the group $K(v)$ is pro-finite and so has no divisible quotients). Yet $H^1(K(v), \mathbf{F}_p)$ is the adjoint representation of $\mathrm{PSL}_2(k_v)$, which is irreducible providing that $\#k_v \geq 3$. The same argument applies to $\mathrm{PGL}_2(\mathscr{O}_v)$, with one modification: the group $\mathrm{PGL}_2(k_v)$ is the extension of a cyclic group of order 2 by a simple non-abelian group, and thus $H^1(\mathrm{PGL}_2(k_v), \mathbf{Q}_p/\mathbf{Z}_p) = 0$ providing that $\#k_v > 3$ or $p > 3$. When $N(v) = 3$ and $p = 3$, the same argument applies except that $\mathrm{PGL}_2(\mathbf{F}_3) = S_4$ and $\mathrm{PGL}_2(\mathbf{F}_3) = A_4$; the first group has abelianization $\mathbf{Z}/2\mathbf{Z}$ whereas the second has abelianization $\mathbf{Z}/3\mathbf{Z}$. This suffices to prove (i) and (ii).

Now suppose that $\mathbb{G}$ is non-split at v. Recall from 3.2.3 that we have exact sequences:

$$0 \to L(v) \to B_v^\times/\mathscr{O}_v^\times \to l^\times/k^\times \to 0,$$
$$0 \to K(v) \to B_v^1 \to l^1 \to 0,$$

where $k = \mathscr{O}_v/v$, l/k is a quadratic extension, and $l^1 \subset l$ are the elements of norm 1. (Here $K(v)$ and $L(v)$ are defined as the kernels of the corresponding reduction maps.) We certainly have

$$H^1(l^1, \mathbf{Q}_p/\mathbf{Z}_p) = H^1(l^\times/k^\times, \mathbf{Q}_p/\mathbf{Z}_p) = \mathbf{Z}_p/(N(v)+1)\mathbf{Z}_p,$$

and the natural map $l^1 \to l^\times/k^\times$ (which is multiplication by 2) induced from the map $B_v^1 \to B_v^\times/\mathscr{O}_v^\times$ induces an isomorphism on cohomology. To prove the result, it suffices to prove that $H^1(K(v), \mathbf{F}_p)$ has no $l^\times$ invariants. We may certainly assume that v has residue characteristic p. In particular, $K(v)$ and $L(v)$ are naturally isomorphic, since (by Hensel's Lemma) every element in $\mathscr{O}_q^\times$ which is $1 \mod v$ is a square. Then, given the explicit decription of B_v^1 in section 3.2.3, the Frattini quotient of $K(v)$ is given explicitly by $(1+\mathfrak{m}_v)/(1+\mathfrak{m}_v^2) \simeq (\mathbf{Z}/p\mathbf{Z})^2$, where $\mathfrak{m}_v$ is generated by i. This quotient is given explicitly by elements of the form

$$\eta = 1 + (a+bj)i = 1 + a \cdot i - b \cdot ij \mod 1 + \mathfrak{m}_v^2.$$

Yet $j\eta j^{-1} = \eta$ implies that $a = b = 0$, and thus η is trivial. This proves (iii).

Now suppose that $K = K_{v,0}$ is of type $\Gamma_0(v)$. There are sequences:

$$0 \to L(v) \to K_{v,0} \to B(\mathrm{PGL}_2(\mathscr{O}_v/v)) \to 0,$$

$$0 \to K(v) \to K_v^1 \to B(\mathrm{SL}_2(\mathscr{O}_v/v)) \to 0.$$

Here B denotes the corresponding Borel subgroup. As above, the natural map $K_v^1 \to K_{v,0}$ induces an isomorphism

$$\mathrm{Hom}(k^\times, \mathbf{Q}_p/\mathbf{Z}_p) \simeq H^1(B(\mathrm{SL}_2(\mathscr{O}_v)/v) \to H^1(B(\mathrm{PGL}_2(\mathscr{O}_v)) \simeq \mathrm{Hom}(k^\times, \mathbf{Q}_p/\mathbf{Z}_p)$$

on cohomology, which is induced explicitly by the squaring map. Once again we may assume that the residue characteristic of v is p, and hence (since $p \neq 2$) there are natural isomorphisms $K(v) \simeq L(v)$. The action of the Borel on the adjoint representation $H^1(K(v), \mathbf{F}_p)$ is a non-trivial extension of three characters of which only the middle character is trivial; in particular $H^1(K(v), \mathbf{F}_p)^B$ is trivial, proving (iv).

If the assumption of part (v) holds, then the map $K_v^{(1)} \to K_v$ is an isomorphism, and the claim is trivial. □

3.7.5. Explicit computation of Hecke action on congruence homology. Suppose that K is p-convenient in the sense of Definition 3.7.3.

For almost every prime $\mathfrak{q}$, we have an automorphism $[\mathfrak{q}]$ of $H_{1,\mathrm{cong}}(Y(K), \mathbf{Z}_p)$ or $H^1_{\mathrm{cong}}(Y(K), \mathbf{Z}_p)$, given by the "left multiplication" action (see after (3.7.0.3)) of any element of $\mathbb{G}(\mathbb{A}_f)$ whose reduced norm is an idele corresponding to $\mathfrak{q}$. (This is well-defined away from finitely many primes). With this notation, the action of $T_\mathfrak{q}$ on congruence homology or congruence cohomology is given by multiplication by

$$[\mathfrak{q}](1 + \mathrm{N}(\mathfrak{q})). \tag{3.7.5.1}$$

In a similar way, the two degeneracy maps $H_{1,\mathrm{cong}}(\Sigma, \mathbf{Z}_p) \to H_{1,\mathrm{cong}}(\Sigma/\mathfrak{q}, \mathbf{Z}_p)$ differ exactly by the action of $[\mathfrak{q}]$ on the target.

Let us prove (3.7.5.1). Let $\mathfrak{q}$ be a prime and let $g_\mathfrak{q}$ as in §3.4.3, i.e. an element of $\mathbb{G}(F_\mathfrak{q}) \subset \mathbb{G}(\mathbb{A}_f)$ supported at $\mathfrak{q}$ corresponding to the $\mathfrak{q}$-Hecke operator. The Hecke operator $T_\mathfrak{q}$ on congruence homology is described by the sequence

$$H_1(Y(K)^\wedge) \to H_1(Y(K \cap g_\mathfrak{q} K g_\mathfrak{q}^{-1})^\wedge) \to H_1(Y(K)^\wedge),$$

just as in (3.4.3.1). Write $K' = K \cap g_\mathfrak{q}^{-1} K g_\mathfrak{q}$. Using the identification of (3.7.0.4) and the identification of degeneracy maps in §3.7.1.1 we see that the Hecke operator carries a function $f : \mathbb{G}(\mathbb{A}_f) \to H_1(K^+, \mathbf{Z}_p)$, representing a class in $H_1(Y(K)^\wedge)$, to the function obtained as $\mathcal{C}f(gg_\mathfrak{q})$ where $\mathcal{C}$ is the composite

$$H_1(K^+, \mathbf{Z}_p) \xrightarrow{\text{transfer}} H_1(K'^+, \mathbf{Z}_p) \xrightarrow{\mathrm{Ad}(g_\mathfrak{q})_*} H_1(\mathrm{Ad}(g_\mathfrak{q})K'^+, \mathbf{Z}_p) \to H_1(K^+, \mathbf{Z}_p)$$

But this composite $\mathcal{C}$ is simply multiplication by $\mathrm{N}(\mathfrak{q})+1$ (this uses the assumption that K is p-convenient to do the computation "away from $\mathfrak{q}$.") Finally, the function $f(gg_{\mathfrak{q}})$ coincides with the action of $[\mathfrak{q}]$ applied to f, because we can write $f(gg_{\mathfrak{q}}) = f(ag_{\mathfrak{q}}g)$, where $a = [g_{\mathfrak{q}}, g]$ belongs to the commutator subgroup of $\mathbb{G}(\mathbb{A}_f)$, thus to $\overline{\mathbb{G}(F)}$.

3.7.6. Variations: $\mathfrak{q}$-congruence, S-arithmetic. Fix a prime $\mathfrak{q}$. In what follows we suppose that $\mathbb{G}$ is split at $\mathfrak{q}$, and $K = K_{\mathfrak{q}} \times K^{(\mathfrak{q})}$, where $K^{(\mathfrak{q})} \subset \mathbb{G}(\mathbb{A}_f^{(\mathfrak{q})})$ and the determinant of $K_{\mathfrak{q}}$ is as large as possible, that is to say, $\mathscr{O}_{\mathfrak{q}}^*/(\mathscr{O}_{\mathfrak{q}}^*)^2$.

We can construct two variants of $Y(K)^\wedge$, both of which admit maps from $Y(K)^\wedge$:

(a) ($\mathfrak{q}$-congruence homology):

Denote by $Y(K)_{\mathfrak{q}}^\wedge$ the space obtained by taking the groupoid for $Y(K)^\wedge$ but projecting each set of morphisms to to $\mathbb{G}(F_{\mathfrak{q}})$. [6]

Then for $g \in \mathbb{A}_f$, the isotropy group of gK is given by the projection of gK^+g^{-1} to $\mathbb{G}(F_{\mathfrak{q}})$, and the (co)homology of $Y(K)_{\mathfrak{q}}^\wedge$ admits a similar description to (3.7.0.4), replacing K^+ by its projection to $\mathbb{G}(F_{\mathfrak{q}})$:

$$\text{functions: } g \in \overline{\mathbb{G}(F)}\backslash\mathbb{G}(\mathbb{A}_f) \to H^1(K_{\mathfrak{q}}^+, A) \tag{3.7.6.1}$$

This arises as follows: One can imitate all the prior discussion but replacing all congruence subgroups by simply "congruence at $\mathfrak{q}$" subgroups. For example, we would replace the congruence completions in § 3.7.0.2 by the completions for the "congruence at $\mathfrak{q}$" topology, i.e., the closures inside $\mathbb{G}(F_{\mathfrak{q}})$. This leads to notions of $\mathfrak{q}$-congruence cohomology and homology e.g. the $\mathfrak{q}$-congruence homology is the quotient defined by $H_1(Y(K)) \to H_1(Y(K)_{\mathfrak{q}}^\wedge)$. We denote it by $H_{1,\mathfrak{q}-\mathrm{cong}}$.

If K is p-convenient and $p > 3$, we have an isomorphism:

$$H^1_{\mathfrak{q}-\mathrm{cong}}(\Sigma, \mathbf{Q}_p/\mathbf{Z}_p) \oplus H^1_{\mathrm{cong}}(\Sigma/\mathfrak{q}, \mathbf{Q}_p/\mathbf{Z}_p) \xrightarrow{\sim} H^1_{\mathrm{cong}}(\Sigma, \mathbf{Q}_p/\mathbf{Z}_p), \tag{3.7.6.2}$$

where we may take the second map to be either of the degeneracy maps from level $\Sigma/\mathfrak{q}$ to Σ. There is a dual statement for homology.

(b) (congruence homology away from $\mathfrak{q}$):

Denote by $Y(K)^{(\mathfrak{q}),\wedge}$ or by $Y(K[\frac{1}{\mathfrak{q}}])^{\wedge,\pm}$ – the reason for the second notation will be explained below – the space defined as

$$\overline{\mathbb{G}(F)}\backslash \left(\{\pm 1\} \times \mathbb{G}(\mathbb{A}_f^{(\mathfrak{q})})\right)/K^{(\mathfrak{q})} \simeq \overline{\mathbb{G}(F)}\backslash\mathbb{G}(\mathbb{A}_f)/\mathbb{G}(F_{\mathfrak{q}})^{(\mathrm{ev})}K$$

where the action of $\mathbb{G}(\mathbb{A}_f)$ on ± 1 is via valuation of determinant at $\mathfrak{q}$, and $G(F_{\mathfrak{q}})^{\mathrm{ev}}$ is the elements of even determinant valuation. This is understood, just as for $Y(K)^\wedge$, as the classifying space of the corresponding groupoid.

---

[6] Note that, when we regard $Y(K)^\wedge$ as a groupoid, all the morphisms sets are subsets of $\overline{\mathbb{G}(F)}$, in particular, subsets of $\mathbb{G}(\mathbb{A}_f)$. Thus we may obtain a new groupoid by projecting these morphism-sets to any quotient of $\mathbb{G}(\mathbb{A}_f)$. To say it abstractly: suppose given a groupoid in which each set $\mathrm{Hom}(x, y)$ is a subset of a single ambient group X, in such a way that the morphisms $\mathrm{Hom}(x, y) \times \mathrm{Hom}(y, z) \to \mathrm{Hom}(x, z)$ are multiplication inside X. Now, if $\bar{X}$ is a quotient group of X, we may form a new groupoid with the same objects, but whose set of morphisms from x to y is the image of $\mathrm{Hom}(x, y)$ inside $\bar{X}$. The isomorphism classes are the same; the isotropy groups are the projections of the old isotropy groups to $\bar{X}$.

The isotropy group of $\varepsilon \times gK^{(\mathfrak{q})}$, for $\varepsilon \in \{\pm 1\}$, is given by $\overline{\mathbb{G}(F)} \cap \mathbb{G}(F_{\mathfrak{q}})^{(\mathrm{ev})}(gK^{(\mathfrak{q})}g^{-1})$ where ev denotes the subgroup of elements of even determinant valuation. The projection of this to $\mathbb{G}(\mathbb{A}_f^{(\mathfrak{q})})$ has for kernel $\mathrm{SL}_2(F_{\mathfrak{q}})$, in particular, this projection induces an isomorphism on abelianizations. Moreover the image of the projection is $gK^{(+,\mathfrak{q})}g^{-1}$, where $K^{(+,\mathfrak{q})}$ denotes the projection of K^+ to $\mathbb{G}(\mathbb{A}_f^{(\mathfrak{q})})$.

Thus the (co))homology of $Y(K)^{\wedge}_{\mathfrak{q}}$ admits a similar description to (3.7.0.4), replacing K^+ by its projection to $\mathbb{G}(\mathbb{A}_f^{(\mathfrak{q})})$, i.e. with

(3.7.6.3) $$\text{functions: } g \in \overline{\mathbb{G}(F)}\backslash\mathbb{G}(\mathbb{A}_f) \to H^1(K^{+,(\mathfrak{q})}, A)$$

that are equivariant for the action of K (equivalently $K\mathbb{G}(F_{\mathfrak{q}})^{\mathrm{ev}}$).

This space $Y(K[\frac{1}{\mathfrak{q}}])^{\wedge,\pm}$ arises when describing the congruence (co)homology of S-arithmetic manifolds. The evident map $\mathbf{H}^3 \times \mathcal{T}_{\mathfrak{q}}^{\pm} \to \{\pm 1\}$ induces

(3.7.6.4) $$Y(K[\frac{1}{\mathfrak{q}}])^{\pm} \to Y(K[\frac{1}{\mathfrak{q}}])^{\wedge,\pm}$$

and the congruence cohomology of $Y(K[\frac{1}{\mathfrak{q}}]^{\pm})$ (again, defined as in § 3.7.0.2, i.e. replacing each group in (3.6.1.1) with its congruence completion) can be alternately described as the image on cohomology of the induced map of (3.7.6.4).

3.8. Eisenstein classes

DEFINITION 3.8.1 (Eisenstein classes). *A maximal ideal* $\mathfrak{m}$ *of* $\mathbf{T}_{\Sigma}$ *is* Eisenstein *if* $T_{\mathfrak{q}} \equiv \psi_1(\mathfrak{q}) + \psi_2(\mathfrak{q}) \mod \mathfrak{m}$ *for all but finitely many* $\mathfrak{q}$*, where* ψ_1 *and* ψ_2 *are characters of the adelic class group*

$$\psi_i : F^{\times}\backslash\mathbb{A}_f^{\times} \to (\mathbf{T}_{\Sigma}/\mathfrak{m})^{\times}.$$

Associated to $\mathfrak{m}$ is a natural reducible semisimple Galois representation $\overline{\rho}_{\mathfrak{m}}$, via class field theory.

A natural source of Eisenstein classes, for instance, is congruence homology, as we saw in Remark 3.7.5.1. This implies, then, that H^1_E and H^1 (and H_1^E and H_1) coincide after localization at any non-Eisenstein ideal $\mathfrak{m}$ (notation as in the prior section).

More generally, there are several variants of the Eisenstein definition, less general than the definition just presented, but useful in certain specific contexts:

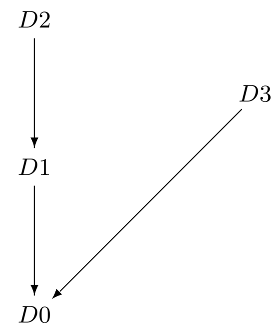
- D0: $\mathfrak{m}$ is Eisenstein if $T_{\mathfrak{q}} \mod \mathfrak{m} = \psi_1(\mathfrak{q}) + \psi_2(\mathfrak{q})$ for characters ψ_i of $F^{\times}\backslash\mathbb{A}_f^{\times}$.
- D1: $\mathfrak{m}$ is cyclotomic-Eisenstein if $T_{\mathfrak{q}} = 1 + N(\mathfrak{q}) \mod \mathfrak{m}$ for all but finitely many *principal* prime ideals $\mathfrak{q}$.
- D2: $\mathfrak{m}$ is congruence-Eisenstein in $\mathbf{T}_{\Sigma}$ if $\mathfrak{m}$ has support in $H_{1,\mathrm{cong}}(\Sigma, \mathbf{Z})$.
- D3: (only makes sense for $\mathbb{G}$ split –) $\mathfrak{m}$ is cusp-Eisenstein in $\mathbf{T}_{\Sigma}$ if it "comes from the cusps", in the sense that there does *not* exist a Hecke equivariant splitting:

$$H_1(\Sigma, \mathbf{Z})_{\mathfrak{m}} = \mathrm{im}(H_1(\partial\Sigma, \mathbf{Z})_{\mathfrak{m}}) \oplus \ker(H_1^{\mathrm{BM}}(\Sigma, \mathbf{Z})_{\mathfrak{m}} \to H_0(\partial\Sigma, \mathbf{Z})_{\mathfrak{m}}).$$

(Here we use the notation of § 5.1, which defines the right-hand groups when $\mathbb{G} = PGL_2$).

We may think of $D0$ and $D1$ as Galois-theoretic definitions, $D2$ as a homological definition, and $D3$ as an automorphic definition. §3.7.1.1 shows that $D2 \Rightarrow D1$, and it is easy to see that $D3 \Rightarrow D0$, and $D1 \Rightarrow D0$:

It is classes of the type $D3$ that are measured by special values of L-functions (see § 5.7). Since ideals of type $D3$ can only exist in the split case, there are many examples of classes that satisfy $D2$ but not $D3$. There also exist classes that are $D1$ but not $D2$. Since the order of congruence homology groups is controlled by congruence conditions on the level, and since prime divisors of L-values are not bounded in such a manner, one expects to find classes that satisfy $D3$ but not $D2$. We summarize this discussion by the following graph describing the partial ordering:



Throughout the text, when we say "Eisenstein", we shall mean Eisenstein of type $D0$ unless we specify otherwise.

3.9. Automorphic representations. Cohomological representations.

For us *automorphic representation* π is an irreducible representation of the Hecke algebra of $\mathbb{G}(\mathbb{A})$ that is equipped with an embedding $\pi \hookrightarrow C^{\infty}(\mathbb{G}(\mathbb{A})/\mathbb{G}(F))$.

Thus π factors as a restricted tensor product $\bigotimes_v \pi_v$, where π_v is an irreducible smooth representation of $\mathbb{G}(F_v)$ for v finite, and, for v archimedean, is an irreducible Harish-Chandra module for $\mathbb{G}(F_v)$. We often write

$$\pi_\infty = \otimes_{v|\infty}\pi_v, \pi_f = \otimes_{v \ finite} \ \ \pi_v.$$

Thus π_∞ is naturally a $(\mathfrak{g}, K_\infty)$-module.

We say π is *cohomological* if π_∞ has nonvanishing $(\mathfrak{g}, K_\infty)$-cohomology for every infinite place v, i.e., in the category of $(\mathfrak{g}, K_\infty)$-modules the group

$$H^i(\mathfrak{g}, K_\infty; \pi_\infty) := \mathrm{Ext}^i(\mathrm{trivial}, \pi_\infty)$$

does not vanish for some i. In the present case, this group, if nonzero, is isomorphic as a vector space to $\mathrm{Hom}_{K_\infty}(\wedge^i \mathfrak{g}/\mathfrak{k}, \pi_\infty)$.

We have Matsushima's formula:

$$H^i(Y(K), \mathbf{C}) = \bigoplus m(\pi) H^i(\mathfrak{g}, K_\infty; \pi_\infty).$$

where $m(\pi)$ is the dimension of K-invariants on π_f. (For an explicit map from the right-hand side for $i = 1$, to 1-forms on $Y(K)$, realizing the isomorphism above, see (5.7.8.1).)

Warning: Note that it is usual to define cohomological to mean that it is cohomological after twisting by some finite-dimensional representation. In that case, π contributes to the cohomology of some nontrivial local system on $Y(K)$.

For our purposes in this paper, it will be most convenient to regard cohomological as meaning with reference to the *trivial* local system.

3.10. Newforms and the level raising/level lowering complexes

3.10.1. Newforms.

DEFINITION 3.10.2 (Newforms.). *The space of newforms* $H_1(\Sigma, \mathbf{Z})^{\mathrm{new}}$ *and* $H^1(\Sigma, \mathbf{Q}/\mathbf{Z})^{\mathrm{new}}$ *are defined respectively as:*

$$\operatorname{coker}\left(\bigoplus_{\mathfrak{q}\in\Sigma\setminus S} H_1(\Sigma/\mathfrak{q}, \mathbf{Z})^2 \stackrel{\Psi^\vee}{\to} H_1(\Sigma, \mathbf{Z})\right)$$

$$\ker\left(H^1(\Sigma, \mathbf{Q}/\mathbf{Z}) \stackrel{\Phi^\vee}{\to} \bigoplus_{\Sigma\setminus S} H^1(\Sigma/\mathfrak{q}, \mathbf{Q}/\mathbf{Z})^2\right)$$

If $\mathfrak{q} \in \Sigma$, we shall sometimes use the corresponding notion of $\mathfrak{q}$-new:

$$H_1(\Sigma, \mathbf{Z})^{\mathfrak{q}-\mathrm{new}} = \operatorname{coker}\left(H_1(\Sigma/\mathfrak{q}, \mathbf{Z})^2 \longrightarrow H_1(\Sigma, \mathbf{Z})\right).$$

The corresponding notions with $\mathbf{Q}$ or $\mathbf{C}$ coefficients are obtained by tensoring. In particular, $H_1(\Sigma, \mathbf{C})^{\mathfrak{q}-\mathrm{new}} = 0$ exactly when $\dim H_1(\Sigma, \mathbf{C}) = 2 \dim H_1(\Sigma/\mathfrak{q}, \mathbf{C})$.

Similarly, given some subset $T \subset \Sigma$, we may define in an analogous way the notion of "T-new" as the cokernel of all degeneracy maps from $\Sigma/\mathfrak{q}$ to Σ, where $\mathfrak{q} \in T$.

3.10.3. Essential newforms and dual essential newforms. Later in the document we will introduce variants of this: "essential" new forms and "dual essential" newforms, denoted $H_1^E(\Sigma, \mathbf{Z})^{\mathrm{new}}$ and $H_1^{E*}(\Sigma, \mathbf{Z})^{\mathrm{new}}$. These notions are defined just as in Definition 3.10.2, but replacing H_1 by the essential homology H_1^E or the dual-essential homology H_1^{E*} defined in Definition 6.7.2 (see Remark 6.7.5 for a discussion of related points.)

3.10.4. Level-raising and level lowering complexes. The maps used in Definition 3.10.2 extends in a natural way to a complex, obtained by composing the maps $\Psi^\vee$ and $\Phi^\vee$ in the obvious way, alternating signs appropriately:

$$0 \to H_1(S, \mathbf{Z})^{2^d} \to \ldots \to \bigoplus_{\Sigma\setminus S} H_1(\Sigma/\mathfrak{q}, \mathbf{Z})^2 \to H_1(\Sigma, \mathbf{Z}),$$

$$H^1(\Sigma, \mathbf{Q}/\mathbf{Z}) \to \bigoplus_{\Sigma\setminus S} H^1(\Sigma/\mathfrak{q}, \mathbf{Q}/\mathbf{Z})^2 \to \ldots \to H^1(S, \mathbf{Q}/\mathbf{Z})^{2^d} \to 0,$$

where $d = |\Sigma \setminus S|$. These are referred to as — respectively — the level raising and level lowering complexes. The level raising complex is exact everywhere but the last term after tensoring with $\mathbf{Q}$; the level lowering complex is similarly exact everywhere but the first term after tensoring with $\mathbf{Q}$.

These complexes arise naturally in our paper because, in our analysis of analytic torsion, we are led naturally to an alternating ratio of orders of torsion groups that exactly correspond to the complexes above. *But they also arise naturally in the analysis of cohomology of S-arithmetic groups*, as we explain in Chapter 4. We are not able to use this coincidence in as definitive way as we should like, owing to our lack of fine knowledge about the cohomology of S-arithmetic groups.

We will see Chapter 4, using the relationship to the cohomology of S-arithmetic groups, that these complexes are *not* in general exact over $\mathbf{Z}$. But has one at least the following result:

> For every $\mathfrak{q} \in \Sigma\backslash S$, there exists a natural self-map of degree -1 $H_{\mathfrak{p}}$ of either complex so that $dH_{\mathfrak{q}} + H_{\mathfrak{q}}d = O_{\mathfrak{q}}$, where $O_{\mathfrak{q}}$ is defined as $\Phi_{\mathfrak{q}}^{\vee} \circ \Phi_{\mathfrak{q}}, \Phi_{\mathfrak{q}} \circ \Phi_{\mathfrak{q}}^{\vee}, \Psi_{\mathfrak{q}} \circ \Psi_{\mathfrak{q}}^{\vee}, \Psi_{\mathfrak{q}}^{\vee} \circ \Psi_{\mathfrak{q}}$, according to what makes sense on each direct summand.

For example, if we consider the level lowering complex on the torsion-free quotients $H_{1,\mathrm{tf}}$, the only homology arises at primes ℓ for which there exists $f \in H_1(S, \mathbf{Z}/\ell)$ annihilated by *all* $T_{\mathfrak{q}}^2 - (\mathrm{N}(\mathfrak{q}) + 1)^2$, for $\mathfrak{q} \in \Sigma - S$.

3.10.5. Generalizing the Cheeger-Müller theorem to the equivariant case/ orbifold case. The theorem of Cheeger–Müller is for Riemannian manifolds; but we are naturally dealing with *orbifolds*. In short, it continues to hold, but possibly with errors divisible by prime numbers that divide the order of isotropy groups, i.e.

$$\mathrm{RT}(Y(K)) = \tau_{an}(Y(K)).u, \quad u \in \mathbf{Q}^{\times}, \tag{3.10.5.1}$$

where the numerator and denominator of u are *supported at orbifold primes.* By this we mean that $u = a/b$, where a and b are integers divisible only by primes dividing the order of the isotropy group of some point on $Y(K)$. The definitions of RT, τ_{an} are as previous. (Note that, in the orbifold case, infinitely many homology groups $H_i(Y(K), \mathbf{Z})$ can be nonzero, but this does not affect our definition, which used only $H_1(Y(K), \mathbf{Z})_{\mathrm{tors}}$.)

The relation (3.10.5.1) is presumably valid quite generally, but we explain the proof only in the case of interest, where the orbifolds can be expressed as global quotients of manifolds: In our situation, where $M = Y(K)$, we may choose a sufficiently small subgroup K', normal in K, such that $Y(K')$ is a genuine manifold — that is to say, every conjugate of K' intersects $\mathbb{G}(F)$ in a torsion-free group. Let $\Delta = K/K'$. Then $Y(K)$ is the quotient $Y(K')/\Delta$ in the sense of orbifolds.

Suppose, more generally, that $\tilde{M}$ is an odd-dimensional compact Riemannian manifold with isometric Δ-action, and let $M = \tilde{M}/\Delta$ (with the induced metric, so that $\tilde{M} \to M$ is an isometry). For any $x \in M$, let Δ_x be the (conjugacy class of the) isotropy group of a preimage $\tilde{x} \in \tilde{M}$. The orbifold primes for M are those which divides the order of some Δ_x.

Fix a cell decomposition of $\tilde{M}$ and let $C^*(\tilde{M})$ be the corresponding integral chain complex. Then, if ℓ is not an orbifold prime, the cohomology groups of

$$C^*(\tilde{M} \otimes \mathbf{Z}_\ell)^G$$

agree with the orbifold cohomology groups of M.[7] On the other hand, it is shown by Lott and Rothenberg [**46**] that the Whitehead torsion of the complex

$$C^*(\tilde{M}, \mathbf{Z})^G$$

coincides with the analytic torsion of M, and our claim (3.10.5.1) follows easily.

---

[7] Indeed, the orbifold cohomology can be defined as the hypercohomology of $C^*(\tilde{M} \otimes \mathbf{Z}_\ell)$ (in the category of complexes of $\mathbf{Z}_\ell[G]$-modules) and there is a corresponding spectral sequence, converging to the orbifold cohomology of M, whose E_1 term is $H^i(G, C^j \otimes \mathbf{Z}_\ell)$. But for $i > 0$, the group $H^i(G, C^j(\tilde{M}) \otimes \mathbf{Z}_\ell)$ vanishes if ℓ is not a orbifold prime, by Shapiro's lemma.

For later applications we will also discuss later (see § 5.8.5) the case when M (thus $\tilde{M}$) has a boundary, which has been treated by Lück [**47**].

CHAPTER 4

Raising the Level: newforms and oldforms

In this chapter, we give a treatment of several matters related to comparing spaces of modular forms at different levels.

Among the main results are Theorem 4.3.1 (level-raising) and Theorem 4.5.1 (relationship between K_2 and spaces of modular forms).

Our results in this chapter are relevant both in that they show that torsion homology behaves as Langlands program predicts (e.g. level-raising from § 4.3, or the results of § 4.5), and in that they are important in our attempt to understand better the Jacquet-Langlands correspondence for torsion (see Chapter 6).

4.1. Ihara's lemma

See also [**44**]. We give a self-contained treatment.

4.1.1. Remarks on the congruence subgroup property. Let T be a finite set of places of F.

Recall that, for $\Gamma \leqslant \mathbb{G}(F)$ a T-arithmetic group, the *congruence kernel* is the kernel of the map $\Gamma^* \to \widehat{\Gamma}$, where Γ^* and $\widehat{\Gamma}$ are the completions of Γ for the topologies defined by all finite index subgroups and congruence subgroups respectively. This congruence kernel depends only on $\mathbb{G}$, the field F, and the set T; these being fixed, it is independent of choice of Γ. Indeed, it coincides with the kernel of the map $\mathbb{G}(F)^* \to \widehat{G(F)}$, where these are the completions of $G(F)$ for the topologies defined by finite index or congruence subgroups of Γ, respectively; these topologies on $\mathbb{G}(F)$ are independent of Γ.

We refer to this as *the congruence kernel for* $\mathbb{G}$ *over* $\mathscr{O}_F[T^{-1}]$.

To say that the congruence kernel has prime-to-p order implies, in particular, the following statement: Any normal subgroup $\Gamma_1 \leqslant \Gamma$ of p-power index is, in fact, congruence. (Indeed, a homomorphism from Γ to the finite p-group $P := \Gamma/\Gamma_1$ extends to $\Gamma^* \to P$, and then must factor through $\widehat{\Gamma}$.)

In particular this implies that the natural map $\Gamma \to \widehat{\Gamma}$ from Γ to its congruence completion induces an isomorphism $H^1(\Gamma, M) \to H^1(\widehat{\Gamma}, M)$, whenever M is a p-torsion module with trivial action.

4.1.2. Statement of Ihara's lemma.

Lemma 4.1.3 (Ihara's Lemma). *Suppose that* $\mathfrak{q}$ *is a finite prime in* Σ*, and*

(4.1.3.1) *The congruence kernel for* $\mathbb{G}$ *over* $\mathscr{O}[\frac{1}{\mathfrak{q}}]$ *is finite of prime-to-p order*

Suppose that one of the following conditions is satisfied:

(1) $p > 3$,
(2) $N(\mathfrak{q}) > 3$,

Then:

(1) *The kernel of the level raising map* $\Phi : H^1(\Sigma/\mathfrak{q}, \mathbf{Q}_p/\mathbf{Z}_p)^2 \to H^1(\Sigma, \mathbf{Q}_p/\mathbf{Z}_p)$ *is isomorphic to the congruence cohomology* $H^1_{\mathrm{cong}}(\Sigma/\mathfrak{q}, \mathbf{Q}_p/\mathbf{Z}_p)$*, embedded in* $H^1(\Sigma/\mathfrak{q})^2$ *via the twisted-diagonal embedding:*

$$H^1_{\mathrm{cong}}(\Sigma/\mathfrak{q}, \mathbf{Q}_p/\mathbf{Z}_p) \overset{x\mapsto(-x,[\mathfrak{q}]x)}{\hookrightarrow} H^1_{\mathrm{cong}}(\Sigma/\mathfrak{q}) \oplus H^1_{\mathrm{cong}}(\Sigma/\mathfrak{q}) \hookrightarrow H^1(\Sigma/\mathfrak{q})^2,$$

where $[\mathfrak{q}]$ *is as defined in* (3.7.5.1)*.*

(2) *The cokernel of the level lowering map* Ψ *on* $H_1(\Sigma, \mathbf{Z}_p) \to H_1(\Sigma/\mathfrak{q}, \mathbf{Z}_p)^2$ *is* $H_{1,\mathrm{cong}}(\Sigma/\mathfrak{q}, \mathbf{Z}_p)$ *considered as a quotient of* $H_1(\Sigma/\mathfrak{q}, \mathbf{Z}_p)^2$ *via the dual of the twisted-diagonal embedding.*

By a theorem of Serre [**66**, p 499, Corollaire 2] and an easy argument to pass between PGL_2 and SL_2, the condition above (concerning the congruence kernel) holds for $\mathbb{G} = \mathrm{PGL}_2$, when p does not divide the number of roots of unity $w_F = |\mu_F|$ in F.

The proof uses the cohomology of S-arithmetic groups. Speaking roughly, the proof goes as follows: $\mathbb{G}(\mathscr{O}[\mathfrak{q}^{-1}])$ is (almost) an amalgam of two copies of $\mathbb{G}(\mathscr{O})$ along a congruence subgroup $\Gamma_0(\mathfrak{q})$; the homology exact sequence associated to an amalgam will yield the result. Practically, it is more useful to phrase the results in terms of the S-arithmetic spaces introduced in § 3.6.

LEMMA 4.1.4. *There are short exact sequences*

$$0 \longrightarrow H^1(\Sigma[\frac{1}{\mathfrak{q}}]), \mathbf{Q}/\mathbf{Z})^{\pm} \longrightarrow H^1(\Sigma/\mathfrak{q}, \mathbf{Q}/\mathbf{Z})^2 \xrightarrow{\Phi} H^1(\Sigma, \mathbf{Q}/\mathbf{Z}).$$

and dually

(4.1.4.1)

$$H_2(\Sigma[\frac{1}{\mathfrak{q}}], \mathbf{Z})^{\pm} \longrightarrow H_1(\Sigma, \mathbf{Z}) \xrightarrow{\Psi} H_1(\Sigma/\mathfrak{q}, \mathbf{Z})^2 \longrightarrow H_1(\Sigma[\frac{1}{\mathfrak{q}}], \mathbf{Z})^{\pm} \longrightarrow 0.$$

PROOF. Apply a Mayer-Vietoris sequence to the description of $Y(K_\Sigma[\frac{1}{\mathfrak{q}}])$ – given in (3.6.0.2), where we cover by the open set $U = Y(\Sigma) \times (0,1)$ and $V =$ a neighbourhood of the two copies of $Y(\Sigma/\mathfrak{q})$. (See § 4.1.8 for a group-theoretic argument.) □

The action of the natural involution on the spaces in (3.6.0.2) allows us to split the sequence into "positive" and "negative" parts. Warning: The involution on $Y(\Sigma[\frac{1}{\mathfrak{q}}])$ induces an involution on each group of the sequence, but the induced map of commutative diagrams does not commute; there are sign factors, because the sign of the connecting homomorphism in Mayer–Vietoris depends implicitly on the choice of *order* of the covering sets.

For example, the part of the sequence that computes the positive eigenspace on $H_*(\Sigma[\frac{1}{\mathfrak{q}}], \mathbf{Z}[\frac{1}{2}])$ is:

(4.1.4.2)

$$H_2(\Sigma[\frac{1}{\mathfrak{q}}], \mathbf{Z}[\frac{1}{2}]) \longrightarrow H_1(\Sigma, \mathbf{Z}[\frac{1}{2}])^- \xrightarrow{\bar{\Psi}} H_1(\Sigma/\mathfrak{q}, \mathbf{Z}[\frac{1}{2}]) \longrightarrow H_1(\Sigma[\frac{1}{\mathfrak{q}}], \mathbf{Z}[\frac{1}{2}]) \to 0,$$

where $H_1(\Sigma, \mathbf{Z}[\frac{1}{2}])^-$ is the $-$ eigenspace for the Atkin-Lehner involution $w_\mathfrak{q}$, and $\bar{\Psi}$ here denotes the map $H_1(\Sigma) \to H_1(\Sigma/\mathfrak{q})$ that is obtained as the difference of the two push-forward maps, or, equivalently, the composite $H_1(\Sigma) \longrightarrow H_1(\Sigma)^2 \xrightarrow{\Psi} H_1(\Sigma/\mathfrak{q})$, the first map being $x \mapsto (x, -x)$.

PROOF. (of Ihara's lemma 4.1.3). We describe the argument in cohomology, the argument in homology being dual.

Return to the sequence

$$0 \longrightarrow H^1(\Sigma[\frac{1}{\mathfrak{q}}], \mathbf{Q}_p/\mathbf{Z}_p)^{\pm} \longrightarrow H^1(\Sigma/\mathfrak{q}, \mathbf{Q}_p/\mathbf{Z}_p)^2 \xrightarrow{\Phi} H^1(\Sigma, \mathbf{Q}_p/\mathbf{Z}_p)..$$

Now

$$H^1(\Sigma[\frac{1}{\mathfrak{q}}], \mathbf{Q}_p/\mathbf{Z}_p)^{\pm} = H^1_{\mathrm{cong}}(\Sigma[\frac{1}{\mathfrak{q}}], \mathbf{Q}_p/\mathbf{Z}_p)^{\pm},$$

this follows from the analogue of (3.7.0.2), and the (assumed) fact that, for each of the arithmetic groups $X = \Gamma_0^{(\mathfrak{q})}(\Sigma, \mathfrak{a})^{(\mathrm{ev})}$ the map $X \to \hat{X}$ to its congruence completion induces an isomorphism on H^1 with p-torsion coefficients. Thus we have a diagram

$$\begin{array}{ccccc}
H^1_{\mathrm{cong}}(\Sigma[\frac{1}{\mathfrak{q}}], \mathbf{Q}_p/\mathbf{Z}_p)^{\pm} & \longrightarrow & H^1_{\mathrm{cong}}(\Sigma/\mathfrak{q}, \mathbf{Q}_p/\mathbf{Z}_p)^2 & & \\
\downarrow \sim & & \downarrow & & \\
H^1(\Sigma[\frac{1}{\mathfrak{q}}], \mathbf{Q}_p/\mathbf{Z}_p)^{\pm} & \longrightarrow & H^1(\Sigma/\mathfrak{q}, \mathbf{Q}_p/\mathbf{Z}_p)^2 & \xrightarrow{\Phi} & H^1(\Sigma, \mathbf{Q}_p/\mathbf{Z}_p)
\end{array}$$

We claim that the upper horizontal arrow has image the twisted-diagonally embedded congruence cohomology. Indeed, the map $Y(K)^{\wedge} \to Y(K[\frac{1}{\mathfrak{q}}])^{\wedge,\pm}$ induces a bijection on the underlying sets, as discussed in § 3.7.6, and the computations of (3.7.2) give the desired result. □

REMARK 4.1.5. If $p = 3$ and $N(\mathfrak{q}) = 3$, the argument above also applies as long as

(1) The square class of -1 belongs to $\det(K_v)$ for every finite v;
(2) If $\lambda \in F^{\times}$ has even valuation at all finite primes, then the class of $\lambda \in F_v^{\times}/F_3^{\times 2}$ coincides with the class of $+1$ or -1 for every $v|3$.

Explicitly, under the conditions of the theorem above, one can appeal to Lemma 3.7.4, part (ii).

4.1.6. Amalgams. We shall later (see § 4.5.7.4) need to compute explicitly certain connecting maps, and for this a discussion of the homology sequence associated to an amalgamated free product will be useful. This will also allow us to give a purely group-theoretic proof of Ihara's lemma.

Let G be a group, and let A and B be two groups (not necessarily distinct) together with fixed embeddings $G \to A$ and $G \to B$ respectively. Recall that the *amalgamated free product* $A *_G B$ of A and B along G is the quotient of the free product $A * B$ obtained by identifying the images of G with each other. For short we say simply that $A *_G B$ is the "free amalgam" of A and B over G.

Suppose a group X acts on a regular tree $\mathcal{T}$, transitively on edges but preserving the natural bipartition of the vertices. Let e be an edge, with stabilizer G, and let A, B be the stabilizers of the two end vertices of e. Then there are clearly embeddings $G \hookrightarrow A, G \hookrightarrow B$, and moreover

(4.1.6.1) The canonical map $A *_G B \to X$ is an isomorphism.

This fact (4.1.6.1) is a consequence of the Seifert–Van Kampen theorem [**39**, §1.2] applied to the space $\mathcal{T} \times E/G$; here E is a contractible space on which G acts properly discontinuously.

Here is an example of particular interest. Let $\mathfrak{q}$ be a place of F, so that $\mathrm{PSL}_2(F_\mathfrak{q})$ acts on the tree $\mathcal{T}_\mathfrak{q}$. The remarks above give

$$A *_G B \xrightarrow{\sim} \mathrm{PSL}_2(F_\mathfrak{q})$$

where A is the image of $\mathrm{SL}_2(\mathscr{O}_\mathfrak{q})$, $B = a_\pi A a_\pi^{-1}$ where $a_\pi = \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}$ and G is equal to $A \cap B$. There is no corresponding decomposition of $\mathrm{PGL}_2(F_\mathfrak{q})$ as it does not preserve a bipartition of the vertices of the tree.

We return to the general case. Associated to A, B, G, and $A *_G B$ are the following long exact sequences, due to Lyndon (see Serre [**66**], p.169)):

$$\to H_{i+1}(A *_G B, M) \to H_i(G, M) \to H_i(A, M) \oplus H_i(B, M) \to H_i(A *_G B, M) \to \dots$$

(4.1.6.2)

$$\to H^i(A *_G B, M) \to H^i(A, M) \oplus H^i(B, M) \to H^i(G, M) \to H^{i+1}(A *_G B, M) \to \dots$$

which can also be derived by applying a Mayer–Vietoris sequence to $\mathcal{T} \times E/G$.

For later reference we detail how to construct the connecting homomorphism $H_2(X, \mathbf{Z}) \to H_1(G, \mathbf{Z})$, or rather its dual

$$H^1(G, \mathbf{Q}/\mathbf{Z}) \longrightarrow H^2(X, \mathbf{Q}/\mathbf{Z}) \tag{4.1.6.3}$$

Lemma 4.1.7. *Take $\kappa \in H^1(G, \mathbf{Q}/\mathbf{Z})$; its image in $H^2(X, \mathbf{Q}/\mathbf{Z})$ is represented by the central extension of X*

$$\tilde{X}_\kappa = \frac{(A * B \times \mathbf{Q}/\mathbf{Z})}{N},$$

where N is the normal closure of G in $A * B$, embedded in $A * B \times \mathbf{Q}/\mathbf{Z}$ via the graph of the unique extension $\tilde{\kappa} : N \to \mathbf{Q}/\mathbf{Z}$.

Note that $\tilde{X}_\kappa$ is indeed, via the natural map, a central extension of X, which is isomorphic to the quotient of $A * B$ by the *normal subgroup* N generated by the image of $G \hookrightarrow A * B$. Note also that there are canonical splittings $A \to \tilde{X}_\kappa$ and similarly for B, descending from the natural inclusion $A \hookrightarrow A * B$ and $B \hookrightarrow A * B$.

We leave the proof to the reader, but we explain why κ extends uniquely to N: Compare the above exact sequence (4.1.6) in low degree to the exact sequence for the cohomology of a group quotient:

$$\begin{array}{ccccc}
H^1(X, \mathbf{Q}/\mathbf{Z}) & \longrightarrow & H^1(A * B, \mathbf{Q}/\mathbf{Z}) & \longrightarrow & H^1(N, \mathbf{Q}/\mathbf{Z})^X \\
\downarrow \sim & & \downarrow \sim & & \downarrow \\
H^1(X, \mathbf{Q}/\mathbf{Z}) & \longrightarrow & H^1(A, \mathbf{Q}/\mathbf{Z}) \oplus H^1(B, \mathbf{Q}/\mathbf{Z}) & \longrightarrow & H^1(G, \mathbf{Q}/\mathbf{Z})
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & H^2(X, \mathbf{Q}/\mathbf{Z}) & \longrightarrow & H^2(A * B, \mathbf{Q}/\mathbf{Z}) \\
& \downarrow = & & \downarrow \sim \\
\longrightarrow & H^2(X, \mathbf{Q}/\mathbf{Z}) & \longrightarrow & H^2(A, \mathbf{Q}/\mathbf{Z}) \oplus H^2(B, \mathbf{Q}/\mathbf{Z})
\end{array}$$

This is *commutative* (the only nontrivial point to be checked is the third square) from which we conclude that the restriction map

$$H^1(N, \mathbf{Q}/\mathbf{Z})^X \to H^1(G, \mathbf{Q}/\mathbf{Z})$$

is an isomorphism, i.e., every homomorphism $\kappa : G \to \mathbf{Q}/\mathbf{Z}$ extends uniquely to a homomorphism $\tilde{\kappa} : N \to \mathbf{Q}/\mathbf{Z}$ which is also X-invariant.

4.1.8. Group-theoretic proof of Lemma 4.1.4. We give, as mentioned after the proof of this Lemma, a group-theoretic proof. This method of proof follows work of Ribet [**57**]. It will result from exact sequence (4.1.6) applied to the fundamental groups of $Y(K_\Sigma[\frac{1}{\mathfrak{q}}])$, $Y(K_{\Sigma/\mathfrak{q}})$, and $Y(K_\Sigma)$ (playing the roles of $A *_G B$, $A = B$, and G respectively); to be precise, we need to take into account the disconnectedness of these spaces.

We follow the notation after Example 3.6.1; in particular,

$$Y(K_\Sigma[\frac{1}{\mathfrak{q}}])^{\pm} = \coprod_{A_\mathfrak{q}} \Gamma_0^{(\mathfrak{q})}(\Sigma, \mathfrak{a}) \backslash (G_\infty/K_\infty \times \mathcal{T}_\mathfrak{q}^{\pm}).$$

The action of $\Gamma_0^{(\mathfrak{q})}(\Sigma, \mathfrak{a})$ on $\mathcal{T}_\mathfrak{q}^{\pm}$ induces an isomorphism:

$$\Gamma_0^{(\mathfrak{q})}(\Sigma, \mathfrak{a})^{(\mathrm{ev})} \cong \Gamma_0(\Sigma/\mathfrak{q}, \mathfrak{a}) *_{\Gamma_0(\Sigma,\mathfrak{a})} \Gamma_0(\Sigma/\mathfrak{q}, \mathfrak{a}\mathfrak{q}), \tag{4.1.8.1}$$

We take the direct sum of the Lyndon homology sequence (4.1.6) over $\mathfrak{a} \in A_\mathfrak{q}$, i.e.

(4.1.8.2)

$$H_i\Gamma_0(\Sigma, \mathfrak{a}) \to \bigoplus_{\mathfrak{a}\in A} (H_i\Gamma_0(\Sigma/\mathfrak{q}, \mathfrak{a}) \oplus H_i\Gamma_0(\Sigma/\mathfrak{q}, \mathfrak{a}\mathfrak{q})) \to \bigoplus_{\mathfrak{a}\in A} H_i(\Gamma_0^{(\mathfrak{q})}(\Sigma, \mathfrak{a})^{(\mathrm{ev})}) \to \dots$$

For brevity, we have omitted the coefficient group.

As we have seen in (3.6.1.1), the final group is identified with $H_i(\Sigma[\frac{1}{\mathfrak{q}}])^{\pm}$, whereas both $\bigoplus_{\mathfrak{a}\in A} H_i\Gamma_0(\Sigma/\mathfrak{q}, \mathfrak{a})$ and $\bigoplus_A H_i\Gamma_0(\Sigma/\mathfrak{q}, \mathfrak{a}\mathfrak{q})$ are identified with $H_i(\Sigma/\mathfrak{q})$.

This gives an exact sequence as in Lemma 4.1.4; we omit the verification that the maps appearing are indeed Φ and Ψ.

4.2. No newforms in characteristic zero.

Suppose as before that $\mathfrak{q}$ is a finite prime in Σ. In this section, we study consequences of Ihara's lemma for *dual* maps $\Phi^\vee, \Psi^\vee$ under the following assumption:

There are no newforms in characteristic zero,

that is to say $H_1(\Sigma, \mathbf{C})^{\mathfrak{q}-\mathrm{new}} = 0$, so that $H_1(\Sigma/\mathfrak{q}, \mathbf{C})^2 \cong H_1(\Sigma, \mathbf{C})$ via the natural maps in either direction.

The advantage of this condition is as follows: If $H_1(\Sigma, \mathbf{C})$ were literally zero, and $Y(\Sigma)$ is compact, then the torsion in H_1 is self-dual, and so we may "dualize" the statement of Ihara's lemma if one has only torsion classes to worry about. It turns out that the $\mathfrak{q}$-new part being zero is enough.

We make our task more complicated by not localizing away from Eisenstein primes. This is because we wish to contol certain numerical factors in this case, although it is irrelevant to the applications of § 4.3.

LEMMA 4.2.1. *Suppose* $H_1(\Sigma, \mathbf{C})^{\mathfrak{q}-\mathrm{new}} = 0$. *Suppose that* $p > 2$ *satisfies the conditions of Ihara's Lemma 4.1.3. Then cokernel of the map* $\Psi_{\mathrm{tf}}^{\vee} : H_1(\Sigma/\mathfrak{q}, \mathbf{Z}_p)^2_{\mathrm{tf}} \to H_1(\Sigma, \mathbf{Z}_p)_{\mathrm{tf}}$ *has order, up a p-adic unit, given by*

$$\frac{1}{h_{\mathrm{lif}}(\Sigma/\mathfrak{q};\mathfrak{q})} \det\left(T_{\mathfrak{q}}^2 - (1+N(\mathfrak{q}))^2 \middle| H_1(\Sigma/\mathfrak{q}, \mathbf{C})\right),$$

Recall that the notation h_{lif} has been defined in § 3.7.1.2: it roughly speaking measures congruence classes that lift to characteristic zero.

REMARK 4.2.2. (1) If one assumes that the congruence subgroup property holds for $\mathbb{G}$ over $\mathscr{O}[1/\mathfrak{q}]$ (as one expects), then an analogous statement holds if $\mathbf{Z}_p$ is replaced by $\mathbf{Z}$, at least up to powers of 2 and 3.
(2) The determinant appearing above is none other than (the p-power part of) $\prod_f (a(f,\mathfrak{q})^2 - (1+N(\mathfrak{q}))^2)$ the product ranging over a basis of Hecke eigenforms for $H_1(\Sigma/\mathfrak{q}, \mathbf{C})$, and $a(f,\mathfrak{q})$ denotes the eigenvalue of $T_{\mathfrak{q}}$ on f.

PROOF. By Ihara's lemma, the cokernel of $\Psi : H_1(\Sigma, \mathbf{Z}_p) \to H_1(\Sigma/\mathfrak{q}, \mathbf{Z}_p)^2$. is congruence homology. Consider the following diagram, with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & (\ker\Psi)_{\mathrm{tors}} & \longrightarrow & \ker\Psi & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_1(\Sigma,\mathbf{Z}_p)_{\mathrm{tors}} & \xrightarrow[\Psi]{} & H_1(\Sigma,\mathbf{Z}_p) & \longrightarrow & H_1(\Sigma,\mathbf{Z}_p)_{\mathrm{tf}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Psi_{\mathrm{tors}}} & & \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \Psi_{\mathrm{tf}}} & & \\
0 & \longrightarrow & H_1(\Sigma/\mathfrak{q},\mathbf{Z}_p)^2_{\mathrm{tors}} & \xrightarrow[\Psi]{} & H_1(\Sigma/\mathfrak{q},\mathbf{Z}_p)^2 & \longrightarrow & H_1(\Sigma/\mathfrak{q},\mathbf{Z}_p)^2_{\mathrm{tf}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{coker}(\Psi_{\mathrm{tors}}) & \longrightarrow & \mathrm{coker}(\Psi) & \longrightarrow & \mathrm{coker}(\Psi_{\mathrm{tf}}) & \longrightarrow & 0
\end{array}$$

We see that the order of $\mathrm{coker}(\Psi_{\mathrm{tf}})$ is equal to the order of the cokernel of the map from $\mathrm{coker}(\Psi_{\mathrm{tors}})$ to $\mathrm{coker}(\Psi)$, or equivalently, the order of the cokernel of $H_1(\Sigma/\mathfrak{q}, \mathbf{Z}_p)^2_{\mathrm{tors}} \to H_1(\Sigma/\mathfrak{q}, \mathbf{Z}_p)_{\mathrm{cong}}$. This map factors as

$$H_1(\Sigma/\mathfrak{q}, \mathbf{Z}_p)^2_{\mathrm{tors}} \to H_1(\Sigma/\mathfrak{q}, \mathbf{Z}_p)^2_{\mathrm{cong}} \to H_1(\Sigma/\mathfrak{q}, \mathbf{Z}_p)_{\mathrm{cong}},$$

where the last map is given by $(x, y) \mapsto (x + [\mathfrak{q}]y)$. We deduce that the cokernel has order $h_{\mathrm{lif}}(\Sigma/\mathfrak{q};\mathfrak{q})$.

Consider the sequence:

$$H_1(\Sigma/\mathfrak{q}, \mathbf{Z})^2_{\mathrm{tf}} \stackrel{\Psi_{\mathrm{tf}}^{\vee}}{\to} H_1(\Sigma, \mathbf{Z})_{\mathrm{tf}} \stackrel{\Psi_{\mathrm{tf}}}{\to} H_1(\Sigma/\mathfrak{q}, \mathbf{Z})^2_{\mathrm{tf}}.$$

Since these groups are torsion free, there is an equality $|\mathrm{coker}(\Psi_{\mathrm{tf}}^{\vee} \circ \Psi_{\mathrm{tf}})| = |\mathrm{coker}(\Psi_{\mathrm{tf}})| \cdot |\mathrm{coker}(\Psi_{\mathrm{tf}}^{\vee})|$. On the other hand, by Lemma 3.4.8, we therefore deduce that

$$|\mathrm{coker}(\Psi_{\mathrm{tf}} \circ \Psi_{\mathrm{tf}}^{\vee})| = \det\left(T_{\mathfrak{q}}^2 - (1+N(\mathfrak{q}))^2 \middle| H_1(\Sigma/\mathfrak{q}, \mathbf{C})\right).$$

Since we have already computed the order of $\mathrm{coker}(\Psi_{\mathrm{tf}})$, this allows us to determine $\mathrm{coker}(\Psi_{\mathrm{tf}}^{\vee})$. □

THEOREM 4.2.3. *Suppose $H_1(\Sigma, \mathbf{C})^{\mathfrak{q}-\mathrm{new}} = 0$. Let $\mathfrak{m}$ be an non-Eisenstein maximal*[1] *ideal of $\mathbf{T}_{\Sigma}$, whose residue characteristic p satisfies the conditions of Ihara's lemma. Suppose that $\ker(\Phi_{\mathfrak{m}}^{\vee})$ is finite.*

Then

$$\Phi_{\mathfrak{m}}^{\vee} : H^1(\Sigma, \mathbf{Q}_p/\mathbf{Z}_p)_{\mathfrak{m}} \rightarrow H^1(\Sigma/\mathfrak{q}, \mathbf{Q}_p/\mathbf{Z}_p)_{\mathfrak{m}}^2$$

is surjective. Equivalently, by duality, the map

$$\Psi^{\vee} : H_1(\Sigma/\mathfrak{q}, \mathbf{Z}_p)_{\mathfrak{m}}^2 \rightarrow H_1(\Sigma, \mathbf{Z}_p)_{\mathfrak{m}}$$

is injective.

It should be noted that the assumption is rare over $\mathbf{Q}$, but rather common over imaginary quadratic fields, especially after localizing at a maximal ideal $\mathfrak{m}$.

REMARK 4.2.4. It is too much to ask that the map $H^1(\Sigma, \mathbf{Q}_p/\mathbf{Z}_p) \rightarrow H^1(\Sigma/\mathfrak{q}, \mathbf{Q}_p/\mathbf{Z}_p)^2$ be surjective; this will not be so simply because of the behavior of congruence homology at p. The theorem shows that this if we localize away from Eisenstein ideals (thus killing the congruence homology) there is no remaining obstruction. In § 6.7, we introduce refined versions of cohomology and cohomology in which this congruence homology is excised. For an appropriate version of this cohomology, the corresponding map of Theorem 4.2.3 *will* be surjective; see Theorem 6.7.6.

PROOF. We give the proof for $\mathbb{G}$ nonsplit; see § 5.5.1 for the split case.

The argument will be the same whether we first tensor with $\mathbf{T}_{\Sigma,\mathfrak{m}}$ or not, hence, we omit $\mathfrak{m}$ from the notation. Consider the following *commutative* diagram:

$$\begin{array}{ccccccc}
0 & & \ker(\Phi_{\mathfrak{m}}^{\vee}) & & \ker(\Psi_{\mathfrak{m}}) & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^1(\Sigma, \mathbf{Q}_p)_{\mathfrak{m}} & \longrightarrow & H^1(\Sigma, \mathbf{Q}_p/\mathbf{Z}_p)_{\mathfrak{m}} & \xrightarrow[\delta_{\Sigma}]{} & H_1(\Sigma, \mathbf{Z}_p)_{\mathfrak{m}} & \longrightarrow & H_1(\Sigma, \mathbf{Q}_p)_{\mathfrak{m}} \\
\downarrow \Phi_{\mathbf{Q},\mathfrak{m}}^{\vee} & & \downarrow \Phi_{\mathfrak{m}}^{\vee} & & \downarrow \Psi_{\mathfrak{m}} & & \downarrow \\
H^1(\Sigma/\mathfrak{q}, \mathbf{Q}_p)_{\mathfrak{m}}^2 & \longrightarrow & H^1(\Sigma/\mathfrak{q}, \mathbf{Q}_p/\mathbf{Z}_p)_{\mathfrak{m}}^2 & \xrightarrow[\delta_{\Sigma/\mathfrak{q}}]{} & H_1(\Sigma/\mathfrak{q}, \mathbf{Z}_p)_{\mathfrak{m}}^2 & \longrightarrow & H_1(\Sigma/\mathfrak{q}, \mathbf{Q}_p)_{\mathfrak{m}}^2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & \mathrm{coker}(\Phi_{\mathfrak{m}}^{\vee}) & & 0 & & 0
\end{array}$$

Here every vertical column is exact (that $\Psi_{\mathfrak{m}}$ is surjective follows from Ihara's lemma) and the middle rows are exact.

Since $\ker(\Phi_{\mathfrak{m}}^{\vee})$ is finite, so is $\ker(\Psi_{\mathfrak{m}})$.

Consider an element $[c]$ in $H^1(\Sigma/\mathfrak{q}, \mathbf{Q}_p/\mathbf{Z}_p)_{\mathfrak{m}}^2$. All elements in this group are torsion, and hence $m[c] = 0$ for some positive integer m. Lift $\delta_{\Sigma/\mathfrak{q}}([c])$ to a class

[1] As the proof will make clear, it is even enough that $\mathfrak{m}$ not be cyclotomic-Eisenstein, or in the nonsplit case, that $\mathfrak{m}$ not be congruence-Eisenstein.

$[b] \in H_1(\Sigma, \mathbf{Z}_p)$. Since $\Psi(m[b]) = m\delta_{\Sigma/\mathfrak{q}}([c]) = \delta_{\Sigma/\mathfrak{q}}(m[c]) = 0$, it follows that $m[b] \in \ker(\Psi)$. Since $\ker(\Psi)$ is finite it follows that $m[b]$ has finite order, and hence $[b]$ has finite order. Since $H_1(\Sigma, \mathbf{Q}_p)_{\mathfrak{m}}$ is torsion free, δ_Σ surjects onto the torsion classes in $H_1(Y(K_\Sigma), \mathbf{Z}_p)$, and $[b]$ lifts to a class $[a] \in H^1(\Sigma, \mathbf{Q}_p/\mathbf{Z}_p)$:

$$\delta_\Sigma([a]) = [b].$$

Now $\Phi^\vee_{\mathfrak{m}}([a]) - [c]$ lies in the kernel of $\delta_{\Sigma/\mathfrak{q}}$, i.e. to the image of $H^1(\Sigma/\mathfrak{q}, \mathbf{Q}_p)^2_{\mathfrak{m}}$; since $\Phi^\vee_{\mathbf{Q},\mathfrak{m}}$ is an isomorphism we conclude that we may modify $[a]$ so that its image is exactly $[c]$. □

Using the prior computations, it is possible to understand precisely the change of regulator when one changes the level and there are no new forms. (Recall that the regulator has been defined in §3.1.2, and will play a key role in the final Chapter.)

Theorem 4.2.5. *(For $\mathbb{G}$ nonsplit; see § 5.5.5 for $\mathbb{G}$ split). Suppose $\mathfrak{q} \in \Sigma$ and $H_1^{\mathfrak{q}-\mathrm{new}}(\Sigma, \mathbf{C}) = \{0\}$. Write*

$$D := \det\left(T_{\mathfrak{q}}^2 - (1 + N(\mathfrak{q}))^2 \middle| H_1(\Sigma/\mathfrak{q}, \mathbf{C})\right).$$

Then, up to orbifold primes,

$$\frac{\mathrm{reg}(H_1(\Sigma/\mathfrak{q}))^2}{\mathrm{reg}(H_1(\Sigma))} = \frac{\sqrt{D}}{h_{\mathrm{lif}}(\Sigma/\mathfrak{q};\mathfrak{q})}$$

Recall the definition of h_{lif} from Lemma 4.2.1: It measures the amount of congruence homology which lifts to characteristic zero.

Proof. The proof will be an easy consequence of Lemma 4.2.1 We write the proof assuming there are no orbifold primes, for simplicity.

Consider the map $\Psi^\vee : H_1(\Sigma/\mathfrak{q}, \mathbf{Z})^2_{\mathrm{tf}} \to H_1(\Sigma, \mathbf{Z})^2_{\mathrm{tf}}$ and its dual over $\mathbf{R}$: $\Phi^\vee_{\mathbf{R}} : H^1(\Sigma, \mathbf{R}) \to H^1(\Sigma/\mathfrak{q}, \mathbf{R})$. Choose an orthonormal basis $\omega_1, \ldots, \omega_{2k}$ for $H^1(\Sigma, \mathbf{R})$, and a basis $\gamma_1, \ldots, \gamma_{2k}$ for $H_1(\Sigma/\mathfrak{q}, \mathbf{Z})^2_{\mathrm{tf}}$. Then

$$\det\langle \gamma_i, \Phi^\vee_{\mathbf{R}}\omega_j\rangle = \det\langle \Psi^\vee(\gamma_i), \omega_j\rangle = |\mathrm{coker}(\Psi)| \cdot \mathrm{reg}(H_1(\Sigma)).$$

Now (by Lemma 4.2.1) $|\mathrm{coker}(\Psi^\vee)| = \frac{D}{h_{\mathrm{lif}}(\Sigma/\mathfrak{q};\mathfrak{q})}$. The $\Psi^*_{\mathbf{R}}\omega_j$ do not form an orthonormal basis; indeed, the map $\Psi^*_{\mathbf{R}}$ multiplies volume by $\sqrt{D}$, i.e.

$$\|\Phi^\vee_{\mathbf{R}}\omega_1 \wedge \Phi^\vee_{\mathbf{R}}\omega_2 \wedge \ldots \Phi^\vee_{\mathbf{R}}\omega_{2k}\| = \sqrt{D}.$$

Consequently,

$$\mathrm{reg}(H_1(\Sigma/\mathfrak{q})) = \frac{\det\langle \gamma_i, \Phi^\vee_{\mathbf{R}}\omega_j\rangle}{\|\Phi^\vee_{\mathbf{R}}\omega_1 \wedge \Phi^\vee_{\mathbf{R}}\omega_2 \wedge \ldots \Phi^\vee_{\mathbf{R}}\omega_{2k}\|} = \mathrm{reg}(H_1(\Sigma)) \cdot \frac{\sqrt{D}}{h_{\mathrm{lif}}(\Sigma/\mathfrak{q};\mathfrak{q})},$$

which implies the desired conclusion. □

4.3. Level raising

Suppose that $\mathfrak{q} \notin \Sigma$, and fix a pair of levels Σ and $\Sigma' = \Sigma \cup \{\mathfrak{q}\}$. Associated to these levels are various Hecke algebras which we now compare:

The Hecke algebra $\mathbf{T}_\Sigma$ at level Σ contains $T_{\mathfrak{q}}$, which is not in $\mathbf{T}_{\Sigma'}$. We also have the Hecke algebra $\mathbf{T}^{\mathrm{new}}_{\Sigma'}$ which is the image of $T_{\Sigma'}$ inside the endomorphism ring of the the space of $\mathfrak{q}$-new forms of level Σ'.

We have the following diagrams of rings:

$$\begin{array}{ccc} \mathbf{T}_{\Sigma}^{\mathrm{an}} & \twoheadleftarrow & \mathbf{T}_{\Sigma'} \\ \big\downarrow & & \big\downarrow \\ \mathbf{T}_{\Sigma} & & \mathbf{T}_{\Sigma'}^{\mathrm{new}} \end{array}$$

Since $\mathbf{T}_\Sigma$ is finite over $\mathbf{Z}$, any maximal ideal $\mathfrak{m}_\Sigma$ of $\mathbf{T}_\Sigma$ has finite residue field. It follows that any such maximal ideal gives rise to a maximal ideal in $\mathbf{T}_\Sigma^{\mathrm{an}}$, and hence a maximal ideal of $\mathbf{T}_{\Sigma'}$. Let us call these ideals $\mathfrak{m}_\Sigma$, $\mathfrak{m}_\Sigma^{\mathrm{an}}$, and $\mathfrak{m}_{\Sigma'}$ respectively. In these terms, the problem of *level raising* can be phrased:

When does $\mathfrak{m}_{\Sigma'}$ give rise to a maximal ideal of $\mathbf{T}_{\Sigma'}^{\mathrm{new}}$?

i.e., when is $\mathfrak{m}_{\Sigma'}$ obtained by pulling back a maximal ideal of $\mathbf{T}_{\Sigma'}^{\mathrm{new}}$?

We reformulate this in slightly more down-to-earth terms. The Hecke algebra $\mathbf{T}_{\Sigma'}$ acts faithfully (by definition) on $H^1(\Sigma', \mathbf{Q}_p/\mathbf{Z}_p)$, and the quotient $\mathbf{T}_{\Sigma'}^{\mathrm{new}}$ acts faithfully on the subspace $H^1(\Sigma', \mathbf{Q}_p/\mathbf{Z}_p)^{\mathrm{new}}$. The ideal $\mathfrak{m}_{\Sigma'}$ of $\mathbf{T}_{\Sigma'}$ gives rise to a corresponding ideal of $\mathbf{T}_{\Sigma'}^{\mathrm{new}}$ if and only if the module $H^1(\Sigma', \mathbf{Q}_p/\mathbf{Z}_p)^{\mathfrak{q}-\mathrm{new}}$ has support at $\mathfrak{m}_{\Sigma'}$ – that is to say, if $H^1(\Sigma', \mathbf{Q}_p/\mathbf{Z}_p)^{\mathfrak{q}-\mathrm{new}}[\mathfrak{m}_{\Sigma'}] \neq 0$. More explicitly, given a mod-p eigenform of level Σ, when is there a mod-p newform which has the same Hecke eigenvalues for $T_\mathfrak{p}$ for all $\mathfrak{p}$ (not dividing Σ') as the original eigenform?

We give an answer to this question below, which is the imaginary quadratic analogue of a theorem of Ribet [**57, 58**].

THEOREM 4.3.1. *Assume the congruence kernel for $\mathbb{G}$ over $\mathscr{O}[\frac{1}{\mathfrak{q}}]$ has prime-to-p order. Let $\mathfrak{m}_\Sigma \subset \mathbf{T}_\Sigma$ be a non-Eisenstein maximal ideal; as above, $\mathfrak{m}_\Sigma$ gives rise to a maximal ideal $\mathfrak{m} := \mathfrak{m}_{\Sigma'}$ of $\mathbf{T}_{\Sigma'}$.*

If $T_\mathfrak{q}^2 - (1 + N(\mathfrak{q}))^2 \in \mathfrak{m}_\Sigma$ then $\mathfrak{m}$ gives rise to a maximal ideal of $\mathbf{T}_{\Sigma'}^{\mathrm{new}}$, in the sense described above. In particular, there exist non-zero eigenclasses $[c] \in H^1(\Sigma', \mathbf{Q}_p/\mathbf{Z}_p)^{\mathrm{new}}$ which are annihilated by $\mathfrak{m}$. If, furthermore, $H^1(\Sigma, \mathbf{Q}_p/\mathbf{Z}_p)_\mathfrak{m}$ is finite, then the order of $H^1(\Sigma', \mathbf{Q}_p/\mathbf{Z}_p)_\mathfrak{m}$ is strictly larger than $H^1(\Sigma, \mathbf{Q}_p/\mathbf{Z}_p)_\mathfrak{m}^2$.

REMARK 4.3.2. In the classical setting, this amounts to the assertion that, if the eigenvalue a_p of a level $\Gamma_0(N)$ modular form satisfies $a_p^2 \equiv (p+1)^2$ modulo ℓ, then there exists a congruent newform of level $\Gamma_0(N\ell)$. In our setting, the same result is true — although the newform (or the oldform, for that matter) *may not lift to characteristic* 0. In our numerical experiments, we found that this provides a surprising efficient way of *finding* lifts of torsion classes to characteristic zero by adding $\Gamma_0(\mathfrak{q})$-structure.

REMARK 4.3.3. On the level of Galois representations, the condition on Hecke eigenvalues above exactly predicts that the residual representation $\overline{\rho}_\mathfrak{m}$ at the place v corresponding to $\mathfrak{q}$ satisfies $\overline{\rho}|G_v \sim \begin{pmatrix} \epsilon\chi & * \\ 0 & \chi \end{pmatrix}$. Indeed, for $\mathfrak{q}$ prime to p, this follows from Scholze's construction [**63**]. This, in turn, is exactly the condition that is required to *define* the universal deformation ring $R_{\Sigma'}^{\mathrm{new}}$ recording deformations of $\overline{\rho}$ that are "new of level $\Gamma_0(\mathfrak{q})$" at $\mathfrak{q}$. Thus this theorem could be thought of in the following way: $\mathbf{T}_{\Sigma',\mathfrak{m}}^{\mathrm{new}}$ is non-zero exactly when $R_{\Sigma'}^{\mathrm{new}}$ is non-zero. This is, of course, a weaker version of the claim that $R_{\Sigma'}^{\mathrm{new}} = \mathbf{T}_{\Sigma',\mathfrak{m}}^{\mathrm{new}}$.

Proof. (of Theorem 4.3.1). The Hecke algebra $\mathbf{T}_{\Sigma'}$ acts on all the relevant modules, and thus it makes sense to consider the $\mathfrak{m}$-torsion of any such module where $\mathfrak{m} = \mathfrak{m}_{\Sigma'}$. Notation as in the statement, we must show that the level lowering map:

$$\Phi^{\vee}[\mathfrak{m}] : H^1(\Sigma', \mathbf{Q}_p/\mathbf{Z}_p)[\mathfrak{m}] \rightarrow H^1(\Sigma, \mathbf{Q}_p/\mathbf{Z}_p)[\mathfrak{m}]^2$$

has a nontrivial kernel. It suffices to show the same for $(\Phi^{\vee} \circ \Phi)[\mathfrak{m}]$, because by Ihara's lemma, the map $\Phi[\mathfrak{m}]$ is injective (since we have completed at a non-Eisenstein ideal, we need not worry about congruence cohomology and essential cohomology). But

$$\Phi^{\vee} \circ \Phi = \begin{pmatrix} (N(\mathfrak{q})+1) & T_{\mathfrak{q}} \\ T_{\mathfrak{q}} & (N(\mathfrak{q})+1) \end{pmatrix}.$$

Since the "determinant" $T_{\mathfrak{q}}^2 - (1+N(\mathfrak{q}))^2$ acts trivially on $H^1(\Sigma, \mathbf{Q}_p/\mathbf{Z}_p)[\mathfrak{m}_{\Sigma}]$, it has nontrivial kernel on the (possibly larger) space $H^1(\Sigma, \mathbf{Q}_p/\mathbf{Z}_p)^2[\mathfrak{m}]$ and we have proved the first assertion.

In order to see that $H^1(\Sigma', \mathbf{Q}_p/\mathbf{Z}_p)_{\mathfrak{m}}$ is strictly bigger than $H^1(\Sigma, \mathbf{Q}_p/\mathbf{Z}_p)^2_{\mathfrak{m}}$, we may assume that $\ker(\Phi^{\vee}_{\mathfrak{m}})$ is finite. By Theorem 4.2.3, it follows that $\Phi^{\vee}_{\mathfrak{m}}$ is surjective, and hence

$$|H^1(\Sigma', \mathbf{Q}_p/\mathbf{Z}_p)_{\mathfrak{m}}| = |\ker(\Phi^{\vee}_{\mathfrak{m}})| \cdot |H^1(\Sigma, \mathbf{Q}_p/\mathbf{Z}_p)^2|.$$

Since we just proved that $\ker(\Phi^{\vee}_{\mathfrak{m}})$ is non-trivial, the result follows. (For an example of what happens when $H^1(\Sigma, \mathbf{Q}_p/\mathbf{Z}_p)_{\mathfrak{m}}$ is infinite, see the remark at the end of the proof of this theorem.)

□

Remark 4.3.4. There is a slightly larger Hecke algebra $\widetilde{\mathbf{T}}_{\Sigma'}$ which contains $\mathbf{T}_{\Sigma'}$ together with the Hecke operator $U_{\mathfrak{q}}$. Although $U_{\mathfrak{q}}$ does not act on $H^1(\Sigma, \mathbf{Q}_p/\mathbf{Z}_p)$, it *does* act naturally on the image of $H^1(\Sigma', \mathbf{Q}_p/\mathbf{Z}_p)$ in $H^1(\Sigma, \mathbf{Q}_p/\mathbf{Z}_p)^2$ (this is surjective away from Eisenstein primes). Moreover, on this space, the action of $U_{\mathfrak{q}}$ satisfies the equation

$$U_{\mathfrak{q}}^2 - T_{\mathfrak{q}} U_{\mathfrak{q}} + N(\mathfrak{q}) = 0.$$

Under the level raising hypothesis, it follows that an $\mathfrak{m}$ in $\mathbf{T}_{\Sigma}$ also gives rise to an $\mathfrak{m}$ in $\widetilde{\mathbf{T}}^{\mathrm{new}}_{\Sigma'}$. Conversely, suppose that $[c] \in H^1(\Sigma', \mathbf{F}_p) \subset H^1(\Sigma', \mathbf{Q}_p/\mathbf{Z}_p)$ generates a Steinberg representation at $\mathfrak{q}$; then, by a standard calculation, $U_{\mathfrak{q}}$ acts on $[c]$ by ± 1, and thus (from the above relation) the corresponding ideal $\mathfrak{m}$ of $\widetilde{\mathbf{T}}_{\Sigma'}$ may arise via level raising only when $1 \pm T_{\mathfrak{q}} + N(\mathfrak{q}) \in \mathfrak{m}$, or equivalently when $T_{\mathfrak{q}}^2 - (1+N(\mathfrak{q}))^2 \in \mathfrak{m}$.

Remark 4.3.5. Statement (ii) implies that $H^1(\Sigma', \mathbf{Q}_p/\mathbf{Z}_p)_{\mathfrak{m}}$ is not isomorphic to $H^1(\Sigma, \mathbf{Q}_p/\mathbf{Z}_p)^2_{\mathfrak{m}}$; equivalently, that $H_1(\Sigma', \mathbf{Z}_p)_{\mathfrak{m}}$ is not isomorphic to $H_1(\Sigma, \mathbf{Z}_p)^2_{\mathfrak{m}}$. This is *false* without the finiteness assumption, as we now explain:

Fix a level Σ and put $Y = Y(K_{\Sigma})$. Let $\mathfrak{m}$ be a non-Eisenstein maximal ideal such that $\mathbf{T}/\mathfrak{m} = \mathbf{F}_{\ell}$, the finite field of size ℓ; denote by $\mathbf{Z}_{\ell}$ the Witt vectors of this field. Suppose that $H_1(Y, \mathbf{Z})_{\mathfrak{m}} \simeq \mathbf{Z}_{\ell}$. Suppose that $\mathfrak{p}$ is a prime such that

$$T_{\mathfrak{p}} \equiv 1 + N(\mathfrak{p}) \mod \pi, \quad N(\mathfrak{p}) \not\equiv -1 \mod \pi$$

At level $\Sigma \cup \{\mathfrak{p}\}$, let $\mathfrak{m}$ denote the maximal ideal of the corresponding ring $\mathbf{T}$ on which $U_{\mathfrak{p}}$ and $U_{\mathfrak{q}}$ act by $+1$. By level raising, we know that $H_1(\Sigma \cup \mathfrak{p}, \mathbf{Z})^{\mathrm{new}}_{\mathfrak{m}} \neq 0$; nonetheless, is perfectly consistent that there is an isomorphism

$$H_1(\Sigma \cup \mathfrak{p}, \mathbf{Z})^{-}_{\mathfrak{m}} \simeq \mathbf{Z}_{\ell}.$$

In fact, all that the the prior proof shows is that, in this case, the cokernel of the *transfer* homomorphism

$$H_1(\Sigma, \mathbf{Z})_{\mathfrak{m}} \to H_1(\Sigma \cup \{\mathfrak{p}\}, \mathbf{Z})^-_{\mathfrak{m}}$$

is $\mathbf{Z}_\ell/\eta_\mathfrak{p}$, where $\eta_\mathfrak{p} = T_\mathfrak{p} - 1 - N(\mathfrak{p})$. (And this situation does indeed occur in numerical experiments.)

However, it seems likely that this situation will not happen with level raising at *multiple* primes. We analyze this later in Theorem 4.4.6.

4.4. The spectral sequence computing the cohomology of S-arithmetic groups

In this section § 4.4 we allow F to be a general number field, i.e., we relax the assumption that F has a unique complex place. Most of the notation established in Chapter 3 still applies; we comment on adjustments when necessary. Put $R = \mathbf{Z}[\frac{1}{w_F^{(2)}}]$, where w_F is the number of roots of unity in F, and $w_F^{(2)}$ is as in § 3.2.1.

We shall analyze the cohomology of analogues of $Y(K_\Sigma[\frac{1}{\mathfrak{q}}])$ where one inverts not merely $\mathfrak{q}$ but an arbitrary (possibly infinite) set T of finite primes. Most of our applications of this are conditional on conjectures about cohomology of S-arithmetic groups, but for an example of an unconditional statement see Theorem 4.5.1.

Since we work over a general field F let us briefly recall our notation: $\mathbb{G}$ is an inner form of PGL_2, ramified at a set of places S. By a slight abuse of notation, use the same letter for the set of *finite* places in S. We fix a set of finite places Σ containing S – the "level" and a further set T of finite places disjoint from Σ – the "primes to be inverted." For K an open compact subgroup of $\mathbb{G}(\mathbb{A}_f)$, we $K^{(T)}$ be the projection of K to $\mathbb{G}(\mathbb{A}_f^{(T)})$. Finally put $G_\infty = \mathbb{G}(F \otimes_\mathbf{Q} \mathbf{R})$ and K_∞ a maximal compact subgroup of G_∞.

Let $\mathscr{B}_T$ be the product of the Bruhat–Tits buildings of $\mathrm{PGL}_2(F_v)$, for $v \in T$; we regard each building as a contractible simplicial complex, and so $\mathscr{B}_T$ is a contractible square complex. In particular, $\mathscr{B}_T$ has a natural filtration:

$$\mathscr{B}_T^0 \subset \mathscr{B}_T^1 \subset \mathscr{B}_T^2 \subset \dots$$

where $\mathscr{B}_T^{(j)}$ comprises the union of cells of dimension $\leq j$.

Consider the quotient

$$Y(K_\Sigma[\frac{1}{T}]) := \mathbb{G}(F)\backslash \left(G_\infty/K_\infty \times \mathscr{B}_T \times \mathbb{G}(\mathbb{A}_f^{(T)})/K_\Sigma^{(T)}\right).$$

This is compatible with the notation of § 3.6, in the case when F is imaginary quadratic. As in that case, we use the abbreviations $Y(\Sigma[1/T])$ and similar notation for its cohomology.

This has a natural filtration by spaces Y_T^j defined by replacing $\mathscr{B}_T$ with $\mathscr{B}_T^j$. The space $Y_T^j - Y_T^{j-1}$ is seen to be a smooth manifold of dimension $\dim(Y_{\{\infty\}}) + j$.

Let $\epsilon : T \to \{\pm 1\}$ be a choice of sign for every place $v \in T$. Associated to ϵ there is a natural character $\chi_\epsilon : \mathbb{G}(F) \to \{\pm 1\}$, namely $\prod_{v \in T: \epsilon(v) = -1} \chi_v$; here χ_v is the "parity of the valuation of determinant", obtained via the natural maps

$$\mathbb{G}(F) \xrightarrow{\det} F^\times/(F^\times)^2 \to \prod_v F_v^\times/(F_v^\times)^2 \xrightarrow{v} \pm 1,$$

where the final map is the parity of the valuation.

Correspondingly, we obtain a *sheaf of R-modules*, denoted $\mathcal{F}_\epsilon$, on the space $Y(K_\Sigma[\frac{1}{T}])$. Namely, the total space of the local system $\mathcal{F}_\epsilon$ corresponds to the quotient of

$$\left(G_\infty/K_\infty \times \mathscr{B}_T \times \mathbb{G}(\mathbb{A}_f^{(T)})/K^T\right) \times R$$

by the action of $\mathbb{G}(F)$: the natural action on the first factor, and the action via χ_ϵ on the second factor.

Write $\Sigma = S \cup T$, and $R = \mathbf{Z}[\frac{1}{w_F^{(2)}}]$.

THEOREM 4.4.1. *There exists an E_1 homology spectral sequence abutting to the homology groups $H_*(Y(K_\Sigma[\frac{1}{T}]), \mathcal{F}_\epsilon)$, where*

$$E^1_{p,q} = \bigoplus_{V\subset T, |V|=p} H_q(S\cup V, R)^{\overline{\epsilon}},$$

the superscript $\overline{\epsilon}$ denotes the eigenspace where each Atkin-Lehner involution $w_\mathfrak{p}$ (see § 3.4.2 for definition) acts by $-\epsilon_\mathfrak{p}$, for $\mathfrak{p}\in V$. Up to signs, the differential

$$d_1 : H_q(S\cup V\cup\{\mathfrak{q}\})^{\overline{\epsilon}} \to H_q(S\cup V)^{\overline{\epsilon}},$$

given by the difference (resp. sum) of the two degeneracy maps, according to whether $\epsilon(\mathfrak{q}) = 1$ (resp. -1).

Here, the "difference of the two degeneracy maps" means

$$H_q(S\cup V\cup\{\mathfrak{q}\}, \mathbf{Z}) \xrightarrow{(\mathrm{id},-\mathrm{id})} H_q(S\cup V\cup\{\mathfrak{q}\}, \mathbf{Z})^2 \xrightarrow{\Psi} H_q(S\cup V, \mathbf{Z}).$$

For example, when $\epsilon_\mathfrak{p} = 1$ for every $\mathfrak{p}$, the E_1 term of this sequence looks like:

$$\begin{array}{ccccc} H_2(S,R) & \longleftarrow & \bigoplus H_2(S\cup\{\mathfrak{p}\},R)^- & \longleftarrow & \bigoplus H_2(S\cup\{\mathfrak{p},\mathfrak{q}\},R)^{--} \\ H_1(S,R) & \longleftarrow & \bigoplus H_1(S\cup\mathfrak{p},R)^- & \longleftarrow & \bigoplus H_1(S\cup\{\mathfrak{p},\mathfrak{q}\},R)^{--} \\ R & \longleftarrow & 0 & \longleftarrow & 0 \end{array} \tag{4.4.1.1}$$

where, for instance, $H_1(S\cup\{\mathfrak{p},\mathfrak{q}\},R)^{--}$ is the subspace of $H_1((S\cup\{\mathfrak{p},\mathfrak{q}\})$ where $w_\mathfrak{p}, w_\mathfrak{q}$ *both* act by -1. This sequence is converging to the cohomology of $Y(K_\Sigma[\frac{1}{T}])$; if the class number of F is odd, this is simply the group cohomology of $\mathrm{PGL}_2(\mathscr{O}_F[\frac{1}{T}])$.

One could likely prove this theorem simply by iteratively applying the argument of Lemma 4.1.4, thus successively passing to larger and larger S-arithmetic groups, but we prefer to give the more general approach. See also the paper [**52**].

PROOF. The filtration Y_T^i gives rise to a homology spectral sequence (see, for example, the first chaper of [**38**]):

$$E^1_{pq} = H_q(Y_T^p, Y_T^{p-1}; \mathcal{F}_\epsilon) \implies H_{p+q}(Y(K_\Sigma[\frac{1}{T}]); \mathcal{F}_\epsilon).$$

The space $\mathring{Y}_T^p = Y_T^p - Y_T^{p-1}$ is a smooth manifold diffeomorphic to

$$\bigcup_{V\subset T,|V|=p} (0,1)^V \times Y(S\cup V)/\langle w_\mathfrak{p}\rangle,$$

the quotient of $Y(S \cup V) \times (0,1)^p$ by the group $W_V = \langle w_{\mathfrak{p}} \rangle_{\mathfrak{p} \in V}$ of Atkin-Lehner involutions $w_{\mathfrak{p}}$ for $\mathfrak{p} \in V$; here, each $w_{\mathfrak{p}}$ $(\mathfrak{p} \in V)$ acts on $(0,1)$ via $x \mapsto 1-x$; the restriction of the sheaf $\mathcal{F}_\epsilon$ is the local system corresponding to the map $w_{\mathfrak{p}} \to -\epsilon_{\mathfrak{p}}$ for each $\mathfrak{p} \in V$.

On the other hand, Y_T^p may be identified with

$$\left(\bigcup_{V \subset T, |V| \leq p} [0,1]^V \times Y(S \cup V)/W_V \right) / \sim,$$

where the equivalence relation $\sim$ is generated by the rule

$$(4.4.1.2) \qquad (t \in [0,1]^{V \cup \mathfrak{q}} \times y) \sim \begin{cases} (t|_V \in [0,1]^V \times \bar{y}), & t(\mathfrak{q}) = 0 \\ (t|_V \in [0,1]^V \times \bar{y}'), & t(\mathfrak{q}) = 1; \end{cases}$$

and $y \mapsto \bar{y}, \bar{y}'$ are the two degeneracy maps $Y(S \cup V \cup \mathfrak{q}) \to Y(S \cup V)$.

We have a canonical (Thom) isomorphism

$$H_{j-s}(M) \xrightarrow{\sim} H_j([0,1]^s \times M, \partial[0,1]^s \times M)$$

for any topological space M; here we have relative homology on the right, and $\partial[0,1]^s$ denotes the boundary of $[0,1]^s$. This isomorphism sends a chain c on M to the relative chain $[0,1]^s \times c$ on the right-hand side. In our case, this yields an isomorphism

$$\bigoplus_{|V|=p} H_{q-p}(S \cup V, R)^{\bar{\epsilon}} \xrightarrow{\sim} H_q(Y_T^p, Y_T^{p-1}; \mathcal{F}_\epsilon)$$

which leads to the conclusion after interpreting the connecting maps as degeneracy maps. □

It is also useful to take the direct sum over ϵ in Theorem 4.4.1.

COROLLARY 4.4.2. *Let $\mathcal{F} = \bigoplus_\epsilon \mathcal{F}_\epsilon$, denote the direct sum over all ϵ (see page 53) and write $\Sigma = S \cup T$. As before set $R = \mathbf{Z}[\frac{1}{w_F^{(2)}}]$.*

There exists an spectral sequence abutting to $H_*(Y(K_\Sigma[1/T], \mathcal{F})$, where

$$E^1_{p,q} = \bigoplus_{V \subset T, |V|=p} H_q(S \cup V, R)^{2^{d-p}}.$$

Up to signs, the differential

$$d_1 : H_q(S \cup V \cup \{\mathfrak{q}\}, R) \to H_q(S \cup V, R)^2,$$

is given by the two degeneracy maps.

4.4.3. Relationship to level lowering/raising. It is now clear that the cohomology of S-arithmetic groups is closely tied to the level raising/lowering complex, and to make significant progress we will need to assume:

CONJECTURE 4.4.4. *Suppose that $|T| = d$. Then $H_i(Y(K[\frac{1}{T}]), \mathbf{Z})$ is Eisenstein for $i \leq d$. Equivalently, $H^i(Y(K[\frac{1}{T}]), \mathbf{Q}/\mathbf{Z})$ is Eisenstein for $i \leq d$.*

If $i = 1$, this conjecture is a consequence of the congruence subgroup property (for the associated S-arithmetic group Γ) whenever CSP is known to hold. On the other hand, the conjecture for $i = 1$ is strictly weaker than asking that Γ satisfy the CSP: one (slightly strengthened) version of the conjecture for $i = 1$ says that any finite index normal subgroup of a congruence subgroup $\Gamma(N)$ of Γ with *solvable* quotient is congruence. It is an interesting question whether there are be natural groups which satisfy this property but *not* the congruence subgroup property, for example, various compact lattices arising from inner forms of $U(2,1)$.

Conjecture 4.4.4 implies that

> If $\mathfrak{m}$ is an non-Eisenstein maximal ideal of $\mathbf{T}$, and if $H^1(M, \mathbf{Q}_p/\mathbf{Z}_p)_\mathfrak{m}$ is finite, the level raising homology complex (respectively, the level lowering cohomology complex) is exact.

For example, suppose that $H_1(\Sigma, \mathbf{Z}_p)_\mathfrak{m}$ is finite, p is odd, and that $\mathfrak{m}$ is non-Eisenstein. Then the homology spectral sequence completed at $\mathfrak{m}$ is empty away from the second row where it consists of the following complex:

$$0 \leftarrow H_1(S, \mathbf{Z}_p)^{2^d}_\mathfrak{m} \leftarrow \ldots \bigoplus_{\Sigma \setminus S} H_1(\Sigma/\mathfrak{q}, \mathbf{Z}_p)^2_\mathfrak{m} \leftarrow H_1(\Sigma, \mathbf{Z}_p)_\mathfrak{m}.$$

Thus, if $H_i(Y(K_\Sigma[1/T], \mathcal{F})_\mathfrak{m}$ is zero in the appropriate range, this is exact up until the final term. Since all the groups are finite, we may take the Pontryagin dual of this sequence to deduce that the level raising complex is exact.

4.4.5. Level raising at two primes produces "genuinely" new cohomology. As a sample of what is implied by Conjecture 4.4.4, we revisit the issue raised after Theorem 4.3.1, namely, that level raising does not necessarily produce "more" cohomology. We show that Conjecture 4.4.4 predicts that there will *always* be "more" cohomology when there are at least two level raising primes.

Let $\mathfrak{m}$ be a non-Eisenstein maximal ideal of $\mathbf{T}_\Sigma$ such that $\mathbf{T}/\mathfrak{m} = \mathbf{F}_\ell$, a field of size ℓ not divisible by any orbifold prime; suppose that $\mathbf{T}/\mathfrak{m}$ it is generated by Hecke operators away from $\Sigma, \mathfrak{p}, \mathfrak{q}$. Let $\mathbf{Z}_\ell$ be the Witt vectors of $\mathbf{F}_\ell$ and let π be a uniformizer in $\mathbf{Z}_\ell$.

Suppose that $H_1(\Sigma, \mathbf{Z})_\mathfrak{m} \simeq \mathbf{Z}_\ell$. Suppose that $\mathfrak{p}$ and $\mathfrak{q}$ are primes such that

$$T_\mathfrak{p} \equiv 1 + N(\mathfrak{p}) \mod \pi, \qquad T_\mathfrak{q} \equiv 1 + N(\mathfrak{q}) \mod \pi,$$

and suppose in addition that $N(\mathfrak{p}), N(\mathfrak{q}) \not\equiv -1 \mod \pi$.

THEOREM 4.4.6. *Assuming Conjecture 4.4.4,* $H_1(\Sigma \cup \{\mathfrak{p}, \mathfrak{q}\}, \mathbf{Z})^{--}_\mathfrak{m} \neq \mathbf{Z}_\ell$.

In other words, level raising at *two* primes produces "more" cohomology. Contrast the situation discussed in Remark 4.3.5.

PROOF. Suppose not. Since $\mathfrak{m}$ is assumed not to be Eisenstein, $H_3(\mathbf{Z})_\mathfrak{m} = H_0(\mathbf{Z})_\mathfrak{m} = 0$. Theorem 4.4.1 gives a spectral sequence abutting to $H_*(\Sigma[\frac{1}{\mathfrak{p}\mathfrak{q}}], \mathbf{Z})$. The E_1 term of this sequence, after localizing at $\mathfrak{m}$, is

$$H_2(\Sigma, \mathbf{Z})_\mathfrak{m} \longleftarrow H_2(\Sigma \cup \{\mathfrak{p}\}, \mathbf{Z})^-_\mathfrak{m} \oplus H_2(\Sigma \cup \{\mathfrak{q}\}, \mathbf{Z})^-_\mathfrak{m} \longleftarrow H_2(\Sigma \cup \{\mathfrak{p}, \mathfrak{q}\}), \mathbf{Z})^{--}_\mathfrak{m}$$

$$H_1(\Sigma, \mathbf{Z})_\mathfrak{m} \longleftarrow H_1(\Sigma \cup \{\mathfrak{p}\}, \mathbf{Z})^-_\mathfrak{m} \oplus H_1(\Sigma \cup \{\mathfrak{q}\}, \mathbf{Z})^-_\mathfrak{m} \longleftarrow H_1(\Sigma \cup \{\mathfrak{p}, \mathfrak{q}\}, \mathbf{Z})^{--}_\mathfrak{m}$$

$$0 \longleftarrow 0 \longleftarrow 0$$

By Ihara's Lemma, projected to the negative eigenspace of Atkin-Lehner involutions, the difference-of-degeneracy maps $H_1(\Sigma \cup \mathfrak{p} \cup \mathfrak{q}, \mathbf{Z}_\ell)^{--}_\mathfrak{m} \to H_1(\Sigma \cup \mathfrak{p}, \mathbf{Z}_\ell)^-_\mathfrak{m}$ and $H_1(\Sigma \cup \mathfrak{p}, \mathbf{Z}_\ell)^-_\mathfrak{m} \to H_1(\Sigma, \mathbf{Z}_\ell)_\mathfrak{m}$ are surjective, and so must in fact be *isomorphisms.*

Put $\eta_\mathfrak{p} = T_\mathfrak{p} - 1 - N(\mathfrak{p})$ (considered as an element of $\mathbf{Z}_\ell$, since it acts via $\mathbf{Z}_\ell$-endomorphisms of $H_1(\Sigma \cup \mathfrak{p} \cup \mathfrak{q})$) and define $\eta_\mathfrak{q}$ similarly.

By our remarks about Ihara's lemma, the second row of the above sequence is exact. However the top row is now easily verified to be isomorphic, as a sequence of $\mathbf{Z}_\ell$-modules, to the sequence

$$\mathbf{Z}_\ell \leftarrow \mathbf{Z}_\ell \oplus \mathbf{Z}_\ell \leftarrow \mathbf{Z}_\ell,$$

where the first map is $(x, y) \mapsto (\eta_\mathfrak{p} x + \eta_\mathfrak{q} y)$, and the second map is $x \mapsto (\eta_\mathfrak{q} x, -\eta_\mathfrak{p} x)$.

Write n for the highest power of π dividing both $\eta_\mathfrak{p}$ and $\eta_\mathfrak{q}$ (in $\mathbf{Z}_\ell$). Then the $E^2_{1,*}$ and $E^2_{2,*}$ rows of the spectral sequence have the following shape:

$$\begin{array}{ccc} \mathbf{Z}_\ell/\pi^n & \mathbf{Z}_\ell/\pi^n & 0 \\ & \nwarrow^{d} & \\ 0 & 0 & 0 \end{array}$$

We have a contradiction: our conjecture posits that $H_2(Y[1/\mathfrak{pq}], \mathbf{Z})_\mathfrak{m}$ is zero, but the differential d cannot possibly be surjective. □

It is interesting in this case to consider what the space of new at $\mathfrak{p}$ and $\mathfrak{q}$ forms will be. By definition, it will be the kernel of the map:

$$\mathbf{Z}_\ell \oplus \mathbf{Z}_\ell = H_1(Y_0(\mathfrak{p}), \mathbf{Z})_\mathfrak{m} \oplus H_1(Y_0(\mathfrak{q}), \mathbf{Z})_\mathfrak{m} \longrightarrow H_1(Y_0(\mathfrak{pq}), \mathbf{Z})_\mathfrak{m} = \mathbf{Z}_\ell \oplus A.$$

We know that the maps send $(x, y) \in \mathbf{Z}_\ell^2$ to $\eta_\mathfrak{q} x - \eta_\mathfrak{p} y \in \mathbf{Z}_\ell$, but the image in A is inaccessible. If $A = k$, for example, then the cokernel could either be $\mathbf{Z}_\ell/\pi^n$ or $\mathbf{Z}_\ell/\pi^n \oplus k$.

4.5. $\zeta(-1)$ and the homology of PGL_2

In this section § 4.5, as in the prior Section, we continue to allow F to be a general number field, i.e., we relax the assumption that F has a unique complex place. However, we specialize to the case of $\mathbb{G} = \mathrm{PGL}_2$, rather than an inner form thereof. Most of the notation established in Chapter 3 still applies; we comment on adjustments when necessary.

We have tried to write this section to be as self-contained as possible, because the main Theorem 4.5.1 is of independent interest.

Our main theorem asserts, roughly speaking, that the second K-group $K_2(\mathscr{O}_F)$ of the ring of integers "shows up" in the Hecke-trivial part of automorphic homology at level 1 (if F has more than one archimedean place) or at every prime level $\mathfrak{p}$ (if F is quadratic imaginary). This implies[2] for example, that if F is a totally real field of class number 1 with $[F : \mathbf{Q}] > 2$, then the odd part of the numerator of $\zeta_F(-1)$ divides the order of (the finite group) $H_2(\mathrm{PGL}_2(\mathscr{O}_F), \mathbf{Z})$ — thus the title of this subsection.

The interest of the theorem is twofold – the methods of the proof shed light on our definition of newform, and moreover it can be shown that the results match a certain computation in Galois deformation theory (this second point is discussed in Chapter 8 of [**16**]).

[2]The implication is via the "Birch–Tate" conjecture for the size of K_2; this is a theorem, thanks to Tate's theorem relating K_2 and Galois cohomology, and the validity of the Main Conjecture of Iwasawa theory.

As commented, we take $\mathbb{G} = \mathrm{PGL}_2$ but allow F now to be an arbitrary field. Let $\mathfrak{p}$ be a prime ideal. Set, as in Chapter 3,

$$Y_0(1) = \mathrm{PGL}_2(F)\backslash (G_\infty/K_\infty \times \mathrm{PGL}_2(\mathbb{A}_f)/\mathrm{PGL}_2(\mathscr{O}_{\mathbb{A}}))) \tag{4.5.0.1}$$

$$Y_0(\mathfrak{p}) = \mathrm{PGL}_2(F)\backslash (G_\infty/K_\infty \times \mathrm{PGL}_2(\mathbb{A}_f)/K_{\mathfrak{p}}) . \tag{4.5.0.2}$$

As discussed in Chapter 3, if F has odd class number, we have a canonical identification

$$H_*(Y_0(1), \mathbf{Z}) \cong H_*(\mathrm{PGL}_2(\mathscr{O}_F), \mathbf{Z}).$$

Let $\mathfrak{I}$ be the ideal of the Hecke algebra generated by all $T - \deg(T)$, for T relatively prime to the level. The ideal $\mathfrak{I}$ is (some version of) the *Eisenstein ideal.*

Theorem 4.5.1. *Let w be the number of roots of unity in F and set $R = \mathbf{Z}[\frac{1}{6}]$, and other notation as above. Then:*

(i) *If F has at least two archimedean places, there is a surjection*

$$H_2(Y_0(1), R)/\mathfrak{I} \twoheadrightarrow K_2(\mathscr{O}_F) \otimes R.$$

(ii) *Suppose F is totally imaginary and $\mathfrak{q}$ is any prime. Suppose moreover that $H_1(Y_0(1), \mathbf{C}) = 0$. Set*

$$\mathcal{K}_{\mathfrak{q}} = \ker\left(H_1(Y_0(\mathfrak{q}), R)^- \to H_{1,\mathrm{cong}}(Y_0(\mathfrak{q}), R) \times H_1(Y_0(1), R)\right).$$

Here the $-$ superscript is the negative eigenspace of the Atkin-Lehner involution, and the map $H_1(Y_0(\mathfrak{q}))^- \to H_1(Y_0(1))$ is the difference of the two degeneracy maps.[3] *Then there is a surjection*

$$\mathcal{K}_{\mathfrak{q}}/\mathfrak{I} \twoheadrightarrow K_2(\mathscr{O}_F) \otimes R, \tag{4.5.1.1}$$

The same conclusion holds localized at p (i.e. replacing R by $\mathbf{Z}_p$) so long as no cohomological cusp form $f \in H_1(Y_0(1), \mathbf{C})$ has the property that its Hecke eigenvalues $\lambda_{\mathfrak{p}}(f)$ satisfy $\lambda_{\mathfrak{p}}(f) \equiv \mathrm{N}(\mathfrak{p}) + 1$ modulo p, for almost all prime ideals $\mathfrak{p}$.

The proof of this Theorem takes up the remainder of this section § 4.5. The basic idea is to begin with the known relationship between $K_2(F)$ and the homology of $\mathrm{PGL}_2(F)$, and descend it to the ring of integers, using the congruence subgroup property many times to effect this.

Remark 4.5.2. *Some remarks on this theorem and its proof:*

- Part (ii) of this theorem actually arose out our attempt to understand some of our numerical observations; these observations are documented elsewhere [**16**, Chapter 9]. Speaking very roughly, such a class shows up at every multiple of a given level $\mathfrak{n}$, but not at level $\mathfrak{n}$ itself (the theorem corresponds to $\mathfrak{n}$ the trivial ideal). We note that the proof of (ii) gives a general result, but for convenience we have presented the simplest form here.
- Note that the assumption in part (ii) that there are no cusp forms of level 1 is not necessarily a rarity — the non-vanishing theorems proved of Rohlfs and Zimmert (see [**59, 75**]) apply only to $\mathrm{SL}_2(\mathscr{O}_F)$, not $\mathrm{PGL}_2(\mathscr{O}_F)$, and indeed one might expect the latter group to often have little characteristic zero H_1.

[3] The group $\mathcal{K}_{\mathfrak{q}}$ is a dual notion to the new at $\mathfrak{q}$ essential homology; see Theorem 4.5.4 below.

More precisely, there are several classes of representations over $\mathbf{Q}$ which base-change to cohomological representations for $\mathrm{SL}_2(\mathscr{O}_F)$ (see [**27**] for a description); but for PGL_2 such classes exist only when there is a real quadratic field F' such that FF' is unramified over F; then the base change of a weight 2 holomorphic form of level $\mathrm{disc}(F')$ and Nebentypus the quadratic character attached to F' gives rise to cohomology classes for $\mathrm{PGL}_2(\mathscr{O}_F)$. See for example [**27**, Question 1.8] for discussion of the possibility that "usually" such base-change classes give all the characteristic zero homology.

- It is would be interesting to investigate whether the surjections of the Theorems are isomorphisms. Indeed, more generally, it appears likely that "Hecke-trivial" classes in homology are related to K-theory. It seems very interesting to investigate this further.

REMARK 4.5.3. The theorem implies the existence of torsion in certain Hilbert modular varieties:

Suppose, for instance, that $F \neq \mathbf{Q}$ is a totally real field of class number 1. Now let Σ be the set of embeddings $F \hookrightarrow \mathbf{R}$ and $\mathscr{Y} := \mathrm{SL}_2(\mathscr{O}_F)\backslash \left(\mathbf{H}^2\right)^{\Sigma}$ the associated Hilbert modular variety. Write Z_F for the 2-group $\mathrm{PGL}_2(\mathscr{O}_F)/\mathrm{SL}_2(\mathscr{O}_F) \cong \mathscr{O}_F^{\times}/(\mathscr{O}_F^{\times})^2$. It acts on Y, and the quotient $\mathscr{Y}/Z_F$ is isomorphic to what we have called $Y_0(1)$.

We have a decomposition

$$H_2(\mathscr{Y}, \mathbf{Z}[1/2]) = \bigoplus_{\chi: Z_F \to \{\pm 1\}} H_2(\mathscr{Y}, \mathbf{Z}[1/2])^{\chi},$$

and moreover — denoting by χ_0 the trivial character of Z_F — the space of invariants $H_2(\mathscr{Y}, \mathbf{Z}[1/2])^{\chi_0}$ is naturally isomorphic to $H_2(Y_0(1), \mathbf{Z}[1/2])$, with notations as above.

Now $H^2(\mathscr{Y}, \mathbf{C})$ is generated freely by the image of differential forms $\omega_i := dz_v \wedge \overline{dz_v}$, where z_v is the standard coordinate on the vth copy ($v \in \Sigma$) of $\mathbf{H}^2$ (see [**29**]). For $\epsilon \in \mathscr{O}_F$, the action of the corresponding element of Z on ω_i is simply $[\epsilon]^*\omega_i = \mathrm{sign}(v(\epsilon))\omega_i$. In particular, $H_2(\mathscr{Y}, \mathbf{C})^{\chi_0} = 0$, since the invariants under $-1 \in \mathscr{O}_F$ are already trivial.

This implies that $H_2(\mathscr{Y}, \mathbf{Z}[1/2])^{\chi_0} = H_2(Y_0(1), \mathbf{Z}[1/2])$ is pure torsion; thus $H_2(\mathscr{Y}, \mathbf{Z}[1/2])$ has order divisible by (the odd part of the numerator of) $\zeta_F(-1)$.

The quotient $\mathbf{T}/\mathfrak{I}$ is a quotient of $\mathbf{Z}$. In particular, if p is any prime, then $\mathfrak{m} = (\mathfrak{I}, p)$ is a maximal ideal of $\mathbf{T}$ which is Eisenstein (of cyclotomic type, see Definition 3.8.1). Part (ii) of Theorem 4.5.1 implies the following:

THEOREM 4.5.4. *Suppose that F is imaginary quadratic. Then, for every prime $\mathfrak{q}$ with $\dim H_1(Y_0(\mathfrak{q}), \mathbf{C}) = 0$, there is a divisibility*

$$\# K_2(\mathscr{O}_F) \mid \# H_1^E(Y_0(\mathfrak{q}), \mathbf{Z})$$

away from the primes 2 and 3. Indeed $H_1^E(Y_0(\mathfrak{q}), \mathbf{Z}_p)$ has support at $\mathfrak{m} = (p, \mathfrak{I})$ for all primes $\mathfrak{q}$.

This result uses, of course, the definition of essential homology: it is simply the kernel of the map from H_1 to $H_{1,\mathrm{cong}}$, and we refer forward to §6.7 for discussion. This result is an immediate consequence of the prior Theorem.

4.5.5. Background on K-theory. If F is a field, the second K-group $K_2(F)$ is defined to be the universal symbol group

$$F^\times \wedge F^\times / \langle x \wedge (1-x) : x \in F - \{0,1\}\rangle.$$

If $\mathfrak{p}$ is a finite prime of F, there is a *tame symbol* $K_2(F) \to k_{\mathfrak{p}}^\times$, defined, as usual, by the rule

$$x \wedge y \mapsto \frac{x^{v(y)}}{y^{v(x)}}(-1)^{v(x)v(y)} \bmod \mathfrak{p}.$$

A nice reference is [**20**]. The K_2 of the *ring of integers* can be defined as

$$K_2(\mathscr{O}_F) := \ker(K_2(F) \longrightarrow \bigoplus_{\mathfrak{p}} k_{\mathfrak{p}}^\times).$$

This group is known to be finite. For example, $K_2(\mathbf{Z}) \cong \{\pm 1\}$; indeed, the map that sends $x \wedge y$ to 1 unless x, y are both negative gives an explicit isomorphism. On the other hand, if $F = \mathbf{Q}(\sqrt{-303})$, then, according to [**3**], $K_2(\mathscr{O}_F)$ has order 22, and is generated by $5 \cdot (-(3\alpha + 17) \wedge (\alpha - 37))$, where $\alpha = \frac{1+\sqrt{-303}}{2}$.

In the present section we shall use:

Theorem 4.5.6 (Suslin). *Let F be an infinite field. The map $F^\times \to B(F)$, given by $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ induces an isomorphism on $H_2(-, \mathbf{Z}[\frac{1}{2}])$. Moreover, the induced map*

$$F^\times \otimes F^\times \to H_2(B(F), \mathbf{Z}[\frac{1}{2}]) \to H_2(\mathrm{PGL}_2(F), \mathbf{Z}[\frac{1}{2}]),$$

where the first map is the cup-product, is a universal symbol for $\mathbf{Z}[\frac{1}{2}]$-modules. In particular, for any $\mathbf{Z}[1/2]$-algebra R,

$$H_2(\mathrm{PGL}_2(F), R) \cong K_2(F) \otimes R.$$

Here $B(F)$ denotes the Borel subgroup.

sketch. We indicate a computational proof that follows along very similar lines to that of Hutchinson [**40**] and Mazzoleni [**50**] for $\mathrm{GL}_2(F)$ and $\mathrm{SL}_2(F)$ respectively (cf. also Dupont and Sah [**23**].)

Firstly, note that the inclusion $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ induces an isomorphism $H_i(F^\times) \simeq H_i(B(F))$ for $i \in \{1, 2\}$ for F infinite; that is a consequence of the Hochschild–Serre spectral sequence applied to $0 \to F \to B(F) \to F^\times \to 0$ (cf. the proof of Lemma 4 of [**50**]).

Denote by $C_k(\mathcal{S})$ denote the free abelian group of distinct $k+1$ tuples of elements of $\mathcal{S}$ (for a set $\mathcal{S}$); Consider the complex

$$C_0(\mathbf{P}^1(F)) \leftarrow C_1(\mathbf{P}^1(F)) \leftarrow C_2(\mathbf{P}^1(F)) \dots$$

of $\mathrm{GL}_2(F)$-modules. Computing its G-hypercohomology (i.e. taking G-invariants on a resolution by injective G-modules) gives a spectral sequence converging to the group homology of $\mathrm{GL}_2(F)$; see p.187 of [**40**].

The action of $\mathrm{GL}_2(F)$ on $\mathbf{P}^1(F)$ factors through $\mathrm{PGL}_2(F)$ — and we are interested in the corresponding spectral sequence for $\mathrm{PGL}_2(F)$. The calculations are very similar to those in [**40**], p.185. As $\mathrm{PGL}_2(F)$-modules, $C_0(\mathbf{P}^1(F))$ is the induction of the trivial module $\mathbf{Z}$ from the Borel subgroup $B(F)$, $C_1(\mathbf{P}^1(F))$ the induction of the trivial module from the torus $F^\times \subset B(F)$, and $C_2(\mathbf{P}^1(F))$ the induction of the trivial module from the trivial subgroup (i.e. the regular representation). The first page of the spectral sequence corresponding to the complex is given as follows (cf. [**40**]. p.185):

$$\begin{array}{ccccccccc}
H_2(B) & \longleftarrow & H_2(F^\times) & \longleftarrow & 0 & & & & \\
H_1(B) & \longleftarrow & H_1(F^\times) & \longleftarrow & 0 & & & & \\
H_0(B) & \longleftarrow & H_0(F^\times) & \longleftarrow & \mathbf{Z} & \longleftarrow & \bigoplus_{x\notin\{\infty,0,1\}}[x]\mathbf{Z} & \longleftarrow & \bigoplus_{x\neq y}[x,y]\mathbf{Z}
\end{array}$$

The maps $H_i(F^\times) \to H_i(B)$ are given by the composition $w-1$ with the natural mapping $H_i(F^\times) \to H_i(B)$. Here w is an element of the normalizer of $F^\times$ not belonging to $F^\times$. Yet the action of w on $H_1(F^\times)$ is given by

$$x = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} = x^{-1} = (-1)\cdot x,$$

where the latter indicates the action of -1 on $F^\times = H_1(F^\times)$ as a $\mathbf{Z}$-module. In particular, the map $H_1(F^\times) \to H_1(B)$ has kernel $\mu_2(F)$, and is an isomorphism after tensoring with any $\mathbf{Z}[1/2]$-algebra R. In particular, the transgression $d_2 : E^2_{3,0} \to E^2_{1,1}$ is trivial after tensoring with such an R. Similarly, the map $H_2(F^\times) \to H_2(B)$ is the zero map, since w acts trivially on $H_2(F^\times)$. The PGL_2-spectral sequence thus yields the following exact sequence after tensoring with R:

$$P(F) = E^2_{0,3} \to H_2(B(F)) \to H_2(\mathrm{PGL}_2(F)) \to E^2_{1,1} \simeq \mu_2(F) \to 0.$$

In the diagram above, the map from $\bigoplus_{x\notin\{\infty,0,1\}}[x]\mathbf{Z}$ to $\mathbf{Z}$ is trivial; thus $P(F)$ is generated by classes $[x]$ for $x \in \mathbf{P}^1(F) - \{0,1,\infty\}$ and the image of $[x]$ under

$$P(F) \to H_2(B(F)) \xrightarrow{\sim} H_2(F^\times) = \wedge^2 F^\times$$

is is $2(x \wedge (1-x))$. This is, in effect, the computation carried out in the Appendix of [**23**]; the group $P(F)$ corresponds to the group described in (A28) of that reference, and the map is their φ.

Hence, tensoring with any $\mathbf{Z}[1/2]$-algebra R, we obtain the desired result. □

4.5.7. Proof of theorem 4.5.1. We handle first the case when F has more than one archimedean place. In particular, by a result of Serre ([**65**], Theorem 2, p.498), $\mathrm{SL}_2(\mathscr{O}_F)$ has the congruence subgroup property "tensored with R", i.e. the congruence kernel completed at R is trivial. (The congruence kernel is a finite group of order $|\mu_F| = w_F$ which is invertible in $R = \mathbf{Z}[1/6]$.)

We shall apply the spectral sequence of Theorem 4.4.1, with Σ to be the empty set, T the set of all finite places, and $\epsilon = +1$ for all primes. Thus one obtains an E_1 sequence converging to $H_*(\mathrm{PGL}_2(F))$:

$$H_2(1,R) \longleftarrow \bigoplus H_2(\mathfrak{p},R)^- \longleftarrow \bigoplus H_2(\{\mathfrak{p},\mathfrak{q}\},R)^{--}$$

$$H_1(1,R) \longleftarrow \bigoplus H_1(\mathfrak{p},R)^- \longleftarrow \bigoplus H_1(\{\mathfrak{p},\mathfrak{q}\},R)^{--}$$

$$H_0(1,R) \longleftarrow \bigoplus H_0(\mathfrak{p},R)^- \longleftarrow \bigoplus H_0(\{\mathfrak{p},\mathfrak{q}\},R)^{--}$$

Note that the set of connected components of $Y_0(1), Y_0(\mathfrak{p}), Y_0(\mathfrak{pq})$, etc. are all identified, via the determinant map, with the quotient C_F/C_F^2 of the class group of F by squares. In what follows we denote this group by C, i.e.

$$C := C_F/C_F^2.$$

Consequently, we may identify $H_0(Y_0(\mathfrak{p}),R)$ with the set of functions $C \to R$; now, the action of $w_\mathfrak{p}$ is then given by multiplication by the class of $\mathfrak{p}$, and so we may identify

$$H_0(Y_0(\mathfrak{p}),R)^- \cong \{f : C \to R : \quad f(I\mathfrak{p}) = -f(I)\},$$

and so on.

4.5.7.1. *The H_0-row.* We claim that the first row of the spectral sequence is exact.

Indeed if M is any $R[C]$-module, we may decompose M according to characters $\chi : C \to \pm 1$ of the elementary abelian 2-group C, i.e.

$$M = \bigoplus_\chi M_\chi, \quad M_\chi = \{m \in M : c \cdot m = \chi(c)m\}.$$

The first row of the spectral sequence splits accordingly; if we write $S(\chi)$ for the set of primes $\mathfrak{p}$ for which $\chi(\mathfrak{p}) = -1$, the χ-component is isomorphic to the sequence

$$\mathcal{S}_\chi^\bullet : R \leftarrow \bigoplus_{\mathfrak{p}\in S(\chi)} R \leftarrow \bigoplus_{\mathfrak{p},\mathfrak{q}\in S(\chi)} R \leftarrow \dots \tag{4.5.7.1}$$

This is verified to be acyclic as long as $S(\chi)$ is nonempty; by Chebotarev's density, this is so as long as χ is nontrivial.

In particular, the first row of E_2 looks like $R,\ 0,\ 0,\ 0\dots.$

4.5.7.2. *The H_1-row.* We now analyze similarly the second row of the spectral sequence,

$$H_1(1,R) \leftarrow \bigoplus H_1(\mathfrak{p},R)^- \leftarrow \bigoplus H_1(\{\mathfrak{p},\mathfrak{q}\},R)^{--} \leftarrow \cdots$$

We claim it is exact away from the left-most term.

The covering $Y_1(\mathfrak{p}) \to Y_0(\mathfrak{p})$ with Galois group $k_\mathfrak{p}^\times$ (see before (3.3.4.1) for the definition of the space Y_1) induces a map

$$\theta : \ H_1(Y_0(\mathfrak{p}),\mathbf{Z}) \to k_\mathfrak{p}^\times. \tag{4.5.7.2}$$

If X is any connected component of $Y_0(\mathfrak{p})$, the congruence subgroup property implies that the induced map $H_1(X,R) \to k_\mathfrak{p}^\times \otimes R$ is an *isomorphism.* (Recall that we are supposing that F has more than one archimedean place at the moment.)

Therefore, we may canonically identify

$$H_1(Y_0(\mathfrak{p}),R) \cong \left(k_\mathfrak{p}^\times \otimes R\right)^C = \{ \text{ functions } f : C \to k_\mathfrak{p}^\times \otimes R\}.$$

Explicitly, given a homology class $h \in H_1$, we associate to it the function given by $c \in C \mapsto \theta(h|_c)$, where $h|_c$ is the homology class that agrees with h on the connected component corresponding to c, and is zero on all other components.

Let us compute the action of $w_{\mathfrak{p}}$. Firstly, $w_{\mathfrak{p}}$ negates θ: the automorphism $w_{\mathfrak{p}}$ of $Y_0(\mathfrak{p})$ extends to an automorphism W of the covering $Y_1(\mathfrak{p})$ such that$WaW = a^{-1}$ for $a \in k_{\mathfrak{p}}^{\times} = \mathrm{Aut}(Y_1(\mathfrak{p})/Y_0(\mathfrak{p}))$. Consequently, the action of $w_{\mathfrak{p}}$ is given by translation by $\mathfrak{p}$ followed by negation.[4]

Similarly,

$$H_1(Y_0(\{\mathfrak{p},\mathfrak{q}\}), R) \cong \left(\left(k_{\mathfrak{p}}^{\times} \oplus k_{\mathfrak{q}}^{\times}\right) \otimes R\right)^C = \{\text{ functions } f : C \to \left(k_{\mathfrak{p}}^{\times} \oplus k_{\mathfrak{q}}^{\times}\right) \otimes R\}.$$

Here the action of $w_{\mathfrak{p}}$ translates by $\mathfrak{p}$ on C and by $(-1,1)$ on $k_{\mathfrak{p}}^{\times} \oplus k_{\mathfrak{q}}^{\times}$; the action of $w_{\mathfrak{q}}$ translates by $\mathfrak{q}$ on C and acts by $(1,-1)$ on $k_{\mathfrak{p}}^{\times} \oplus k_{\mathfrak{q}}^{\times}$.

Write $V(\mathfrak{pq})$ for the $--$ part of this space, which we may write as a sum of a $\mathfrak{p}$-piece and a $\mathfrak{q}$-piece:

$$\begin{aligned} V(\mathfrak{pq})_{\mathfrak{p}} &= \{f : C \to k_{\mathfrak{p}}^{\times} \otimes R : f(\mathfrak{p}I) = f(I),\ f(\mathfrak{q}I) = -f(I).\} \\ V(\mathfrak{pq})_{\mathfrak{q}} &= \{f : C \to k_{\mathfrak{q}}^{\times} \otimes R : f(\mathfrak{p}I) = -f(I),\ f(\mathfrak{q}I) = f(I).\} \end{aligned}$$

The H_1-row of the above spectral sequence is thereby identified with

$$R \otimes \left(\bigoplus_{\mathfrak{p}} V(\mathfrak{p}) \leftarrow \bigoplus_{\mathfrak{p},\mathfrak{q}} V(\mathfrak{pq})_{\mathfrak{p}} \oplus V(\mathfrak{pq})_{\mathfrak{q}} \leftarrow \dots \right). \tag{4.5.7.3}$$

For example, the morphism $V(\mathfrak{pq})_{\mathfrak{p}} \to V(\mathfrak{p})$ sends a function f to $f(\mathfrak{q}I) - f(I) = 2f$; this follows from the existence of a commutative diagram

$$\begin{array}{ccc} Y_1(\{\mathfrak{p},\mathfrak{q}\}) & \longrightarrow & Y_1(\mathfrak{p}) \\ \downarrow & & \downarrow \\ Y_0(\{\mathfrak{p},\mathfrak{q}\}) & \longrightarrow & Y_0(\mathfrak{p}) \end{array}$$

that respects the map $k_{\mathfrak{p}}^{\times} \oplus k_{\mathfrak{q}}^{\times} \to k_{\mathfrak{p}}^{\times}$ on automorphism groups.

This sequence (4.5.7.3) decomposes in a natural way indexed by prime ideals $\mathfrak{p}$ (the $\mathfrak{p}$ piece is "all those terms involving $k_{\mathfrak{p}}^{\times}$; for instance $V(\mathfrak{pq})_{\mathfrak{p}}$ is the $\mathfrak{p}$-part of $V(\mathfrak{pq})$). We further split (4.5.7.3) up according to characters $\chi : C \to \pm 1$. The $\chi, \mathfrak{p}$-component of the above sequence is only nonzero if $\chi(\mathfrak{p}) = 1$. In that case, it is identified with

$$k_{\mathfrak{p}}^{\times} \otimes \mathcal{S}_{\chi}^{\bullet},$$

at least up to multiplying the morphisms by powers of 2; here $\mathcal{S}_{\chi}^{\bullet}$ is as in (4.5.7.1). As before, this is acyclic unless $\chi = 1$. In that case, its homology is concentrated in one degree and is isomorphic to $R \otimes k_{\mathfrak{p}}^{\times}$.

We conclude that the second row of E_2 looks like

[4]Here is the proof: given a homology class $h \in H_1$, with associated function $f_h : C \to k_{\mathfrak{p}}^{\times}$, we have

$$f_{w_{\mathfrak{p}}h}(c) = \theta((w_{\mathfrak{p}}h)|c) = \theta(w_{\mathfrak{p}}h|_{w_{\mathfrak{p}}c}) = -h|_{w_{\mathfrak{p}}c} = -f_h(\mathfrak{p}c).$$

$$\begin{array}{ccc} H_2(Y_0(1),R)/\bigoplus H_2(Y_0(\mathfrak{p}),R)^- & ? & ? \\ 0 & \bigoplus_{\mathfrak{p}} k_{\mathfrak{p}}^{\times} \otimes R & 0 \\ R & 0 & 0 \end{array} \tag{4.5.7.4}$$

4.5.7.3. *The edge exact sequence.* From (4.5.7.4) we now obtain an edge exact sequence

$$H_2(Y_0(1),R)/\bigoplus_{\mathfrak{p}} H_2(Y_0(\mathfrak{p}),R)^- \longrightarrow H_2(\mathrm{PGL}_2(F),R) \xrightarrow{\theta} \bigoplus_{\mathfrak{p}}(k_{\mathfrak{p}}^{\times} \otimes R),$$

where θ is the morphism arising from the spectral sequence. We claim that the composite

$$K_2F \otimes R \xleftarrow{\sim} H_2(\mathrm{PGL}_2(F),R) \xrightarrow{\theta} \bigoplus(k_{\mathfrak{p}}^{\times} \otimes R)$$

is *none other than the tame symbol map.*

This will complete the proof of the theorem in the case when F has more than two archimedean places: we will have exhibited an isomorphism

$$H_2(Y_0(1),R)/\bigoplus H_2(Y_0(\mathfrak{p}),R)^- \cong K_2(\mathscr{O}) \otimes R, \tag{4.5.7.5}$$

and it remains only to observe that for any $h \in H_2(Y_0(1),R)$ and any prime $\mathfrak{q}$, if $\alpha, \beta : Y_0(\mathfrak{q}) \to Y_0(1)$ are the two degeneracy maps, that

$$\begin{aligned} T_{\mathfrak{q}}h = \alpha_*\beta^*h &= \beta_*\beta^*h + (\alpha_* - \beta_*)\beta^*h \\ &= (\mathrm{N}(\mathfrak{q})+1)h + (\alpha_* - \beta_*)\beta^*h \\ &\in (\mathrm{N}(\mathfrak{q})+1)h + \mathrm{image}(H_2(Y_0(\mathfrak{q}),R)^-). \end{aligned} \tag{4.5.7.6}$$

In other words, each Hecke operator $T_{\mathfrak{q}} - \mathrm{N}(\mathfrak{q}) - 1$ acts trivially on the left-hand of (4.5.7.5); therefore $\mathfrak{I}$ annihilates this quotient, completing the proof.

4.5.7.4. *The edge map is (an invertible multiple of) the tame symbol.* It remains to check that the map $K_2 \to k_{\mathfrak{p}}^{\times}$ is the tame symbol map. To do, compare the above sequence with a corresponding sequence for $F_{\mathfrak{p}}$. This can be done via the natural maps $\mathrm{PGL}_2(F) \to \mathrm{PGL}_2(F_{\mathfrak{p}})$ and $\bar{S} \times \mathscr{B}_T \times \mathbb{G}(\mathbb{A}^T)/K^T \to \mathscr{B}_{\mathfrak{p}}$. These morphisms preserve the filtrations on the spaces (the filtration on $\mathscr{B}_{\mathfrak{p}}$ being two-step: the vertices and the whole space). One obtains then a morphism of the corresponding spectral sequences and thereby a commuting sequence:

$$\begin{array}{ccccc} H_2(Y_0(1),R)/\bigoplus H_2(Y_0(\mathfrak{p}),R)^- & \longrightarrow & H_2(\mathrm{PGL}_2(F),R) & \longrightarrow & \bigoplus(k_{\mathfrak{p}}^{\times} \otimes R) \\ \downarrow & & \downarrow & & \downarrow \\ H_2(\mathrm{PGL}_2(\mathscr{O}_{\mathfrak{p}}),R)/H_2(K_{0,\mathfrak{p}},R)^- & \longrightarrow & H_2(\mathrm{PGL}_2(F_v),R) & \longrightarrow & k_{\mathfrak{p}}^{\times} \otimes R \end{array}$$

This reduces us to computing the morphism $H_2(\mathrm{PGL}_2(F_{\mathfrak{p}}),R) \to k_{\mathfrak{p}}^{\times} \otimes R$. This morphism arises from the action of $\mathrm{PGL}_2(F_{\mathfrak{p}})$ on the tree $\mathscr{B}_p$; considering the action of $\mathrm{PSL}_2(F_{\mathfrak{p}})$ (the image of SL_2 in PGL_2) on the same gives a similar map $H_2(\mathrm{PSL}_2(F_{\mathfrak{p}}),R) \to k_{\mathfrak{p}}^{\times} \otimes R$. This yields a commutative diagram

$$\begin{array}{ccccc} (F_p^\times \wedge F_\mathfrak{p}^\times)\otimes R & \longrightarrow & H_2(\mathrm{PSL}_2(F_\mathfrak{p}), R) & \longrightarrow & (k_\mathfrak{p}^\times \otimes R) \\ \downarrow{\scriptstyle \times 4} & & \downarrow & & \downarrow \\ (F_p^\times \wedge F_\mathfrak{p}^\times)\otimes R & \longrightarrow & H_2(\mathrm{PGL}_2(F_\mathfrak{p}), R) & \longrightarrow & k_\mathfrak{p}^\times \otimes R \end{array} \tag{4.5.7.7}$$

Here the first top (resp. bottom) horizontal map is induced by $x \mapsto \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$ (resp. $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$) together with the natural identification $H_2(F_\mathfrak{p}^\times, R) \cong \left(F_\mathfrak{p}^\times \wedge F_\mathfrak{p}^\times\right) \otimes R$. So it is enough to check that the top row is a multiple of the tame symbol.

We check that by passing to group homology, interpreting the connecting map as a connecting map in the Lyndon sequence, and using the explicit description already given. Let π be a uniformizer in $F_\mathfrak{q}$. Take $X = \mathrm{SL}_2(F_\mathfrak{q}), A = \mathrm{SL}_2(\mathscr{O}_\mathfrak{q})$ and $B = \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} A \begin{pmatrix} \pi^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ and $G = A \cap B$; then the natural map $A *_G B \to X$ is an isomorphism. Moreover, the connecting homomorphism in group homology

$$H_2(X, \mathbf{Z}) \to H_1(G, \mathbf{Z}) \to k_\mathfrak{p}^\times$$

the latter map induced by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a$, is identified with the lower composite of (4.5.7.7). In these terms, we need to evaluate

$$\wedge^2 F_\mathfrak{q}^\times = H_2(F_\mathfrak{q}, \mathbf{Z}) \to H_2(X, \mathbf{Z}) \to H_1(G, \mathbf{Z}) \to k_\mathfrak{p}^\times, \tag{4.5.7.8}$$

where the first map is again induced by $x \mapsto \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$. We will show this is twice the tame symbol.

Let $\kappa : k_\mathfrak{p}^\times \to \mathbf{Q}/\mathbf{Z}$ be a homomorphism; we will prove that the composite

$$\wedge^2 F_\mathfrak{q}^\times = H_2(F_\mathfrak{q}, \mathbf{Z}) \to H_2(X, \mathbf{Z}) \to H_1(G, \mathbf{Z}) \to k_\mathfrak{p}^\times \xrightarrow{\kappa} \mathbf{Q}/\mathbf{Z} \tag{4.5.7.9}$$

takes $u \wedge \pi \mapsto \kappa(\bar{u})^2$ for any unit $u \in \mathscr{O}_q$ and any uniformizer π. This shows that (4.5.7.8) is twice the tame symbol.

By pullback we regard κ as a homomorphism $\kappa : G \to \mathbf{Q}/\mathbf{Z}$. By the discussion following (4.1.6.3) we associate to it a central extension $\tilde{X}_\kappa$ of X by $\mathbf{Q}/\mathbf{Z}$, equipped with splittings over A and B. As a shorthand, we denote these splittings as $g \in A \mapsto g^{\tilde{A}} \in \tilde{X}_\kappa$, and similarly for B. Note that, for $g \in G$,

$$g^{\tilde{A}} = g^{\tilde{B}} \kappa(g).$$

Set $a_x = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$. Let $\widetilde{a_x}$ denote any lift of a_x to $\tilde{X}_\kappa$. The composite (4.5.7.9) is given by

$$x \wedge y \mapsto \widetilde{a_x}\widetilde{a_y}\widetilde{a_x}^{-1}\widetilde{a_y}^{-1} \in \ker(\tilde{X}_\kappa \to X) \cong \mathbf{Q}/\mathbf{Z}. \tag{4.5.7.10}$$

Note this is independent of choice of lift.

Only an ugly computation remains. For $u \in \mathscr{O}_{\mathfrak{q}}^{\times}$ a unit, we may suppose that $\widetilde{a_u} = a_u^{\tilde{A}}$. On the other hand, we may write $a_\pi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\pi^{-1} \\ \pi & 0 \end{pmatrix}$, and thus may suppose that $\widetilde{a_\pi} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\tilde{A}} \begin{pmatrix} 0 & -\pi^{-1} \\ \pi & 0 \end{pmatrix}^{\tilde{B}}$.

Then $\widetilde{a_\pi}\widetilde{a_u}\widetilde{a_\pi}^{-1}\widetilde{a_u}^{-1}$ equals

$$\begin{aligned}
&\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\tilde{A}} \begin{pmatrix} 0 & -\pi^{-1} \\ \pi & 0 \end{pmatrix}^{\tilde{B}} \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}^{\tilde{A}} \begin{pmatrix} 0 & \pi^{-1} \\ -\pi & 0 \end{pmatrix}^{\tilde{B}} \\
&\times \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\tilde{A}} \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix}^{\tilde{A}} \\
&= \kappa(a_u) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\tilde{A}} \begin{pmatrix} 0 & -\pi^{-1} \\ \pi & 0 \end{pmatrix}^{\tilde{B}} \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}^{\tilde{B}} \begin{pmatrix} 0 & \pi^{-1} \\ -\pi & 0 \end{pmatrix}^{\tilde{B}} \\
&\times \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\tilde{A}} \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix}^{\tilde{A}} \\
&= \kappa(a_u) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\tilde{A}} \left(\begin{pmatrix} 0 & -\pi^{-1} \\ \pi & 0 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \begin{pmatrix} 0 & \pi^{-1} \\ -\pi & 0 \end{pmatrix} \right)^{\tilde{B}} \\
&\times \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\tilde{A}} \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix}^{\tilde{A}}
\end{aligned}$$

which becomes

$$\begin{aligned}
&= \kappa(a_u) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\tilde{A}} \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix}^{\tilde{B}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\tilde{A}} \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix}^{\tilde{A}} \\
&= \kappa(a_u)^2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\tilde{A}} \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix}^{\tilde{A}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\tilde{A}} \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix}^{\tilde{A}} = \kappa(a_u)^2.
\end{aligned}$$

We have now proven that the composite map (4.5.7.8), which is also the PSL_2-row of (4.5.7.7) is the square of the tame symbol. By the commutativity of (4.5.7.7), we are done.

4.5.7.5. *The case when F has one archimedean place; proof of (ii) in Theorem 4.5.1.* In this case, we proceed exactly as for (4.5.7.5) (with the ring of integers replaced by $\mathscr{O}_F[\frac{1}{\mathfrak{q}}]$ – i.e., when constructing the spectral sequence as in §4.4, one uses just the filtration on $\mathscr{B}_{T-\{\mathfrak{q}\}}$) to construct an isomorphism

$$H_2(Y[\frac{1}{\mathfrak{q}}], R) / \bigoplus_{\mathfrak{p} \neq \mathfrak{q}} H_2(\{\mathfrak{p}\}[\frac{1}{\mathfrak{q}}], R)^- \xrightarrow{\sim} K_2(\mathscr{O}_F[\frac{1}{\mathfrak{q}}]) \otimes R. \tag{4.5.7.11}$$

Here $Y(\frac{1}{\mathfrak{q}})$ is defined as in § 3.6 with $\Sigma = \emptyset$ and $\mathbb{G} = \mathrm{PGL}_2$.

Note, for later use, that the abstract prime-to-$\mathfrak{q}$ Hecke algebra $\mathscr{T}_{\mathfrak{q}}$ (see § 3.4.6) acts on $H_2(Y[\frac{1}{\mathfrak{q}}], R)$ and – as discussed around (4.5.7.6) – this action descends to the left-hand quotient above; the quotient action is simply that every $T \in \mathscr{T}_{\mathfrak{q}}$ acts by its degree $\deg(T)$.

Now we have constructed an exact sequence (the $-$ eigenspace of Lemma 4.1.4, extended a little to the left)

(4.5.7.12)
$$H_2(Y_0(\mathfrak{q}),R)^- \to H_2(Y_0(1),R) \to H_2(Y[\frac{1}{\mathfrak{q}}],R) \twoheadrightarrow \ker(H_1(Y_0(\mathfrak{q}),R)^- \to H_1(Y_0(1),R)).$$

Compare (4.5.7.12) and its analogue replacing level 1 by level $\mathfrak{p}$:

(4.5.7.13)
$$\begin{array}{ccccccc} \bigoplus_{\mathfrak{p}} H_2(Y_0(\mathfrak{pq}),R)^- & \longrightarrow & \bigoplus_{\mathfrak{p}} H_2(Y_0(\mathfrak{p}),R) & \longrightarrow & \bigoplus_{\mathfrak{p}} H_2(Y\{\mathfrak{p}\}[\frac{1}{\mathfrak{q}}],R) & \longrightarrow \\ \downarrow & & \downarrow & & \downarrow \delta & \\ H_2(Y_0(\mathfrak{q}),R)^- & \longrightarrow & H_2(Y_0(1),R) & \longrightarrow & H_2(Y[\frac{1}{\mathfrak{q}}],R) & \longrightarrow \end{array}$$

(4.5.7.14)
$$\begin{array}{cl} \twoheadrightarrow & \bigoplus_{\mathfrak{p}} \ker(H_1(Y_0(\mathfrak{pq}),R)^- \to H_1(Y_0(\mathfrak{p}),R)) \\ & \downarrow \\ \twoheadrightarrow & \bigoplus_{\mathfrak{p}} \ker(H_1(Y_0(\mathfrak{q}),R)^- \to H_1(Y_0(1),R)). \end{array}$$

On the top line, we take summations over all primes $\mathfrak{p}$ not equal to $\mathfrak{q}$. Also, on the top line, the $-$ subscripts refer to the $\mathfrak{q}$-Atkin Lehner involution. We take the vertical maps to be differences between the two degeneracy maps. (It would therefore be possible to replace the top vertical row by the corresponding $-$ eigenspaces for the $\mathfrak{p}$-Atkin Lehner involution.)

The cokernel of the map δ, as we have seen, is isomorphic to $K_2(\mathscr{O}_F[\frac{1}{\mathfrak{q}}]) \otimes R$. Chasing the above diagram, we obtain a sequence, exact at the middle and final term:

(4.5.7.15)
$$\frac{H_2(Y_0(1),R)}{\sum_{\mathfrak{p}} \text{ image of } H_2(Y_0(\mathfrak{p}),R)} \to K_2(\mathscr{O}_F[\frac{1}{\mathfrak{q}}]) \otimes R \to \mathcal{K}^*.$$

where we set

$$\mathcal{K}^* = \frac{\ker(H_1(Y_0(\mathfrak{q}),R)^- \to H_1(Y_0(1),R))}{\sum_{\mathfrak{p}} \text{ image of } \ker\left(H_1(Y_0(\mathfrak{pq}),R)^- \twoheadrightarrow H_1(Y_0(\mathfrak{p}),R)\right)}.$$

The last two groups of (4.5.7.15) admit compatible maps to $k_{\mathfrak{q}}^{\times}$, i.e., there's a commutative square

(4.5.7.16)
$$\begin{array}{ccc} K_2(\mathscr{O}_F[\frac{1}{\mathfrak{q}}]) \otimes R & \longrightarrow & \mathcal{K}^* \\ \downarrow & & \downarrow \overline{\theta} \\ k_{\mathfrak{q}}^{\times} & = & k_{\mathfrak{q}}^{\times}. \end{array}$$

where $\overline{\theta}$ is the map induced by the map θ of (4.5.7.2). In order to see that θ descends to $\mathcal{K}^*$, we need to see that θ is trivial on the image, in $H_1(Y_0(\mathfrak{q}),R)$, of $H_1(Y_0(\mathfrak{pq}),R)$ (via the difference of the two degeneracy maps). But θ pulls back via both degeneracy maps to the *same* map $H_1(Y_0(\mathfrak{pq}), R) \to k_{\mathfrak{q}}^{\times}$; thus θ descends to $\mathcal{K}^*$, as required.

That means we get in fact a sequence, exact at middle and final terms:

$$\frac{H_2(Y_0(1), R)}{\sum_{\mathfrak{p}} \text{ image of } H_2(Y_0(\mathfrak{p}), R)} \longrightarrow K_2(\mathscr{O}_F) \otimes R \twoheadrightarrow \text{a quotient of } \mathcal{K}_{\mathfrak{q}}, \tag{4.5.7.17}$$

writing, as in the statement of the theorem,

$$\mathcal{K}_{\mathfrak{q}} = \text{ the kernel of } H_1(Y_0(\mathfrak{q}), R)^- \longrightarrow H_1(Y_0(1), R) \times \left(k_{\mathfrak{q}}^{\times} \otimes R\right).$$

The assertion (4.5.1.1) of the theorem may now be deduced: If there are no cohomological cusp forms of level 1, the first group of (4.5.7.17) vanishes. We need to refer ahead to Chapter 6 for the proof of that: In the notations of that Chapter, in the case at hand, the maps $H_2(\partial Y_0(1), R) \to H_2(Y_0(1), R)$ are surjective (as in (5.4.2.1)) but, as we check in § 5.4.8.1, the quotient of $H_2(\partial Y_0(1), R)$ by the image of all $H_2(\partial Y_0(\mathfrak{p}), R)$ is zero. (For this it is important that w_F is invertible in R).

Now, as in the very final assertion of Theorem 4.5.1 , let us localize at p. Put $R = \mathbf{Z}_p$. As before let $\mathscr{T}_{\mathfrak{q}}$ be the abstract prime-to-$\mathfrak{q}$ Hecke algebra. Then in fact $\mathscr{T}_{\mathfrak{q}}$ acts on all three groups of (4.5.7.15) where, according to our discussion after (4.5.7.11), each $T \in \mathscr{T}_{\mathfrak{q}}$ acts on the middle group by its degree.

Write $\mathfrak{m} = (\mathfrak{I}, p) \subset \mathscr{T}_{\mathfrak{q}}$, i.e. it is the kernel of the morphism $\mathscr{T}_{\mathfrak{q}} \to \mathbf{F}_p$ that sends every Hecke operator to its degree (mod p). Consider, now, the sequence (4.5.7.15) completed at $\mathfrak{m}$.

- The first group becomes zero: Our assumption (as in the last paragraph of Theorem 4.5.1) implies, in particular, that $H_1(Y_0(1), R)_{\mathfrak{m}}$ is finite; that means again that the map $H_2(\partial Y_0(1), R)_{\mathfrak{m}} \to H_2(Y_0(1), R)_{\mathfrak{m}}$ is surjective, and we can proceed just as before.
- The middle group equals $K_2(\mathscr{O}_F) \otimes \mathbf{Z}_p$: We already saw that each $T \in \mathscr{T}_{\mathfrak{q}}$ acts on the middle group by its degree; thus completing at the ideal $\mathfrak{m}$ is the same as completing at p.
- The third group remains a quotient of $\mathcal{K}_{\mathfrak{q}}$: The group $\mathscr{K}_{\mathfrak{q}}$ is finite, and the completion of a finite module is isomorphic to a quotient of that module.

This concludes the proof of Theorem 4.5.1.

CHAPTER 5

The split case

This chapter is concerned with various issues, particularly of analytic nature, that arise when $\mathbb{G}$ is split.

One purpose of this Chapter is to generalize certain parts of the Chapter 4 to the noncompact case – these results are concentrated in § 5.5, § 5.6. and § 5.7. These parts contain some results of independent interest (e.g. the upper bound on Eisenstein torsion given in § 5.7).

However, the main purpose of the chapter is to prove Theorem 5.8.1; it compares Reidmeister and analytic torsion in the non-compact case. More precisely, we compare a *ratio* of Reidemeister torsions to a *ratio* of analytic torsions, which is a little easier than the direct comparison. This is the lynchpin of our comparison of torsion homology between a Jacquet–Langlands pair in the next chapter. The most technical part of the theorem is (5.8.1.1).

For the key analytic theorems we do not restrict to level structure of the form K_Σ. One technical reason for this is that, even if one is interested only in those level structures, the Cheeger-Müller theorem does not apply to orbifolds. Rather we need to apply the equivariant version of this theorem to a suitable covering $Y(K') \to Y(K_\Sigma)$.

Once the definitions are given the flow of the proof goes:

$$(5.8.1.1) \Longleftarrow \text{Theorem 5.8.3} \Longleftarrow \text{Theorem 5.9.1} \tag{5.0.7.18}$$

The proof of Theorem 5.9.1 is in § 5.9; the left- and right- implication arrows are explained in § 5.8.4 and § 5.10, respectively.

Here is a more detailed outline of some of the key parts of this Chapter:

- § 5.1 introduces the basic vocabulary for dealing with non-compact manifolds In particular § 5.1.1 introduces the notion of *height function* that measures how far one is in a cusp.
- § 5.2 discusses eigenfunctions of the Laplacian on non-compact manifolds. In particular, § 5.2.2 gives a quick summary of the theory of Eisenstein series, that is to say, the parameterization of the continuous spectrum of the Laplacian on forms and functions.
- § 5.3 is devoted to a definition of Reidemeister torsion and analytic torsion in the non-compact case. In particular:
  - § 5.3.1 discusses harmonic forms of polynomial growth and, in particular, specify an inner product on them.
  - § 5.3.2 gives a definition of Reidemeister torsion.
  - § 5.3.4 gives a definition of analytic torsion in the non-compact case.
- § 5.4 analyzes in more detail some of the foregoing definitions for arithmetic manifolds $Y(K)$. In particular,

  - § 5.4.1 specifies a height function on the $Y(K)$ (the same height function that appears for instance in the adelic formulation of the trace formula);
  - § 5.4.2 analyzes the difference between homology and Borel–Moore homology from the point of view of Hecke actions.
  - § 5.4.6 shows that, upon adding level structure at a further prime, thus replacing a subgroup K_Σ by $K_{\Sigma\cup\mathfrak{q}}$, the cusps simply double: the cusps of $Y(\Sigma\cup\mathfrak{q})$ are isometric to two copies of the cusps of $Y(\Sigma)$. (This isometry, however, does not preserve the height functions of § 5.4.1).
- § 5.5 extends certain results related to Ihara's lemma to the non-compact case.
- § 5.6 carries out explicit computations related to Eisenstein series on arithmetic groups, in particular, computation of scattering matrices on functions and forms. We strongly advise that the reader skip this section at a first reading.
- § 5.7 gives an upper bound on modular forms of "cusp–Eisenstein" type
- § 5.8 formulates the central result, Theorem 5.8.3: comparison of Reidemeister and analytic torsion in the non-compact case. We then explain why (5.8.1.1) follows from this Theorem.

  Our result here is adapted solely to our context of interest: we analyze the *ratio* of these quantities between two manifolds with isometric cusp structure.
- § 5.9 is the technical core of the Chapter, and perhaps of independent interest. It computes the "near to zero" spectrum of the Laplacian on a truncated hyperbolic manifold. In particular it shows that these near-zero eigenvalues are modelled by the roots of certain functions $f(s), g(s)$ related to the scattering matrices.
  - § 5.9.2 is purely real analysis: it analyzes zeroes of $f(s), g(s)$.
  - § 5.9.5 shows that *every zero of f or g gives rise to an eigenvalue of the Laplacian.*
  - § 5.9.8 goes in the reverse direction: *every eigenvalue of the Laplacian is near to a root of f or g.*
  - § 5.9.6 discusses issues related to eigenvalue 0.
- § 5.10 gives the proof of the main theorem (comparison between Reidemeister and analytic), using heavily the results of § 5.9.

5.1. Noncompact hyperbolic manifolds: height functions and homology

5.1.1. Hyperbolic manifolds and height functions. Let M be a finite volume hyperbolic 3-manifold — as usual, this may be disconnected– with cusps.

We think of the *cusps* $\mathcal{C}_1, \ldots, \mathcal{C}_k \subset M$ as being 3-manifolds with boundary, so that, firstly, $M - \bigcup \text{interior } (\mathcal{C}_i)$ is a compact manifold with boundary; and, moreover, each $\mathcal{C}_j$ is isometric to

$$\{(x_1, x_2, y) \in \mathbf{H}^3 : y \geq 1\}/\Gamma_{\mathcal{C}}, \tag{5.1.1.1}$$

where a finite index subgroup of $\Gamma_{\mathcal{C}}$ acts as a lattice of translations on (x_1, x_2). We call a cusp *relevant* if all of $\Gamma_{\mathcal{C}}$ acts by translations. Cusps which are not relevant have orbifold singularities.

We may suppose that the cusps are disjoint; for each cusp $\mathcal{C}_i$, fix an isometry σ_i onto a quotient of the form (5.1.1.1). For each $x \in M$, we define the "height" of x as a measure of how high it is within the cusps: set

$$\mathrm{Ht}(x) = \begin{cases} 1, & x \notin \bigcup \mathcal{C}_i \\ y\text{-coordinate of } \sigma_i(x), & x \in \mathcal{C}_i. \end{cases} \tag{5.1.1.2}$$

A *height function on M* is a function Ht that arises in the fashion above; such a function is far from unique, because the choice of the isometry σ_i is not unique. If, however, we fix a height function on M, then "compatible" σ_is (compatible in that (5.1.1.2) is satisfied for sufficiently large heights) are unique up to affine mappings in the (x_1, x_2)-coordinates. We will often use (x_1, x_2, y) as a system of coordinates on the cusp, with the understanding that it is (after fixing a height function) unique up to affine mappings in (x_1, x_2).

In what follows, we suppose that M is endowed with a fixed choice of height function. [1] In the case of $M = Y(K)$ an arithmetic manifold, we shall specify our choice of height function later. We shall often use the shorthand

$$y \equiv \mathrm{Ht}$$

e.g., writing "the set of points where $y \geq 10$" rather than "the set of points with height ≥ 10", since, with respect to a suitable choice of coordinates on M_B, Ht coincides with the y-coordinate on each copy of $\mathbf{H}^3$.

This being so, we make the following definitions:

(1) For any cusp $\mathcal{C}$, we define area($\mathcal{C}$) to be the area of the quotient $\{(x_1, x_2) \in \mathbf{R}^2\}/\Gamma_{\mathcal{C}}$ — i.e., if Γ' is a subgroup of $\Gamma_{\mathcal{C}}$ that acts freely, this is by definition $\mathrm{area}(\mathbf{R}^2/\Gamma') \cdot [\Gamma_{\mathcal{C}} : \Gamma']^{-1}$. The measure on $\mathbf{R}^2$ is the standard Lebesgue measure.

    Once we have fixed a height-function, this is independent of the choice of compatible isometries σ_i. We write area(∂M) for the sum $\sum_{\mathcal{C}} \mathrm{area}(\mathcal{C})$, the sum being taken over all cusps of M; we refer to this sometimes as the "boundary area" of M.

(2) For any cusp $\mathcal{C}$, we write vol($\mathcal{C}$) for the volume of the connected component of M that contains $\mathcal{C}$.

(3) Put

$$M_B = \coprod_i \mathbf{H}^3/\Gamma_{\mathcal{C}},$$

    a finite union of hyperbolic cylinders. Then "the geometry of M at ∞ is modelled by M_B." Note that M_B is independent of choice of height function.

    The y-coordinate on each copy of $\mathbf{H}^3$ descends to a function on M_B. Thus, for example, y^s defines a function on M_B for every $s \in \mathbf{C}$.

(4) The height Y truncation of M denoted $M_{\leq Y}$, is defined as

$$M_{\leq Y} = \{y \leq Y\} \subset M$$

    in words, we have "chopped off" the cusps. $M_{\leq Y}$ is a closed manifold with boundary, and its boundary is isometric to a quotient of a 2-torus by a finite group — or, more precisely, a union of such. We sometimes abbreviate $M_{\leq Y}$ to M_Y when there will be no confusion. We will similarly use the notation $M_{[Y',Y]}$ to denote elements of M satisfying $Y' \leq y \leq Y$.

    Convention on boundary conditions: When dealing with the Laplacian on forms or functions on a manifold with boundary, we shall *always suppose the boundary conditions to be absolute*: we work on the space of differential forms ω such that both ω and $d\omega$, when contracted with a normal vector, give zero.

[1] It is necessary to be precise about this because we will use the height function to truncate M, and then compute the Laplacian spectrum of the resulting manifold; this is certainly dependent on the choice of Ht.

(5) We set ∂M to be the boundary of $M_{\leq Y}$ for any sufficiently large Y; we are only interested in its homotopy class and not its metric structure, and as such this is independent of Y.

We have long exact sequences

$$\cdots \to H_c^j(M, \mathbf{Z}) \to H^j(M, \mathbf{Z}) \to H^j(\partial M, \mathbf{Z}) \to H_c^{j+1} \to \cdots$$

and we define as usual *cuspidal cohomology*

$$H_!^j(M, \mathbf{Z}) = \mathrm{image}(H_c^j(M, \mathbf{Z}) \longrightarrow H^j(M, \mathbf{Z})) \tag{5.1.1.3}$$

We denote correspondingly

$$H_{j,!}(M, \mathbf{Z}) = \mathrm{image}(H_j(M, \mathbf{Z}) \longrightarrow H_{j,\mathrm{BM}}(M, \mathbf{Z})),$$

where BM denotes Borel–Moore homology.

We have the following isomorphisms, where, if M has orbifold points, one needs to additionally localize $\mathbf{Z}$ away from any orbifold primes:

$$\begin{aligned} H_c^{3-n} M, \mathbf{Z}) &\simeq H_n(M, \mathbf{Z}), \\ H^{3-n}(M, \mathbf{Z}) &\simeq H_n^{\mathrm{BM}}(M, \mathbf{Z}), \\ H^{2-n}(\partial M, \mathbf{Z}) &\simeq H_n(\partial M, \mathbf{Z}). \end{aligned} \tag{5.1.1.4}$$

The first two are Poincaré duality and Borel–Moore duality respectively. The last comes from Poincaré duality for the closed manifold $\partial Y(K_\Sigma)$.

(6) We put $b_j(M) = \dim H_j(M, \mathbf{C})$, the jth complex Betti number. In particular, $b_0(M)$ is the number of connected components of M.

As mentioned, in the arithmetic case we shall endow $Y(K)$ with a certain, arithmetically defined, height function. In this case we write $Y(K)_T$ and $Y_B(K)$ for $M_{\leq T}$ and M_B respectively. We shall later (§ 5.4.1) describe $Y(K)_T$ and $Y_B(K)$ in adelic terms; the primary advantage of that description is just notational, because it handles the various connected components and cusps in a fairly compact way. The truncation $Y(K)_T$ is in fact diffeomorphic to the so-called Borel-Serre compactification of $Y(K)$ but all we need is that the inclusion $Y(K)_T \hookrightarrow Y(K)$ is a homotopy equivalence.

5.1.2. Linking pairings. We now discuss the analogue of the linking pairing (§ 3.4.1) in the non-compact case. Suppose that p does not divide the order of any orbifold prime for M.

We have a sequence:

(5.1.2.1)

$$H_1(M, \mathbf{Z}_p) \xrightarrow{\sim} H^1(M, \mathbf{Q}_p/\mathbf{Z}_p)^\vee \xrightarrow{\sim} H_c^2(M, \mathbf{Z}_p) \xleftarrow{g} H_c^1(M, \mathbf{Q}_p/\mathbf{Z}_p) = H_{1,\mathrm{BM}}(M, \mathbf{Z}_p)^\vee$$

where the morphisms are, respectively, duality between homology and cohomology, Poincaré duality, a connecting map (see below), and duality between Borel–Moore homology and compactly supported cohomology. The map g is a connecting map in the long exact sequence:

$$\cdots \to H_c^1(M, \mathbf{Q}_p) \to H_c^1(M, \mathbf{Q}_p/\mathbf{Z}_p) \xrightarrow{g} H_c^2(M, \mathbf{Z}_p) \to H_c^2(M, \mathbf{Q}_p) \to \ldots$$

which we may reformulate as an isomorphism

$$H_c^1(M, \mathbf{Q}_p/\mathbf{Z}_p)/\left(H_{c,\mathrm{div}}^1\right) \xrightarrow{\sim} H_c^2(M, \mathbf{Z}_p)_{\mathrm{tors}}.$$

From (5.1.2.1) above we now obtain an isomorphism

$$H_1(M, \mathbf{Z}_p)_{\mathrm{tors}} \xrightarrow{\sim} H_{1,\mathrm{BM}}(M, \mathbf{Z}_p)^\vee_{\mathrm{tors}}$$

By means of the map $H_1(M, \mathbf{Z}_p) \hookrightarrow H_{1,\mathrm{BM}}(M, \mathbf{Z}_p)$, we obtain a pairing on $H_{1,\mathrm{tors}}(M, \mathbf{Z}_p)$ but it is not perfect. There are, however, two situations of interest where one can deduce a perfect pairing from it:

- Localization: Suppose that $\mathbf{T}$ of correspondences acts on M (e.g. a Hecke algebra) and $\mathfrak{m}$ is an ideal of $\mathbf{T}$ with the property that the map $H_1(M, \mathbf{Z}_p) \to H_{1,\mathrm{BM}}(M, \mathbf{Z}_p)$ is an isomorphism when completed at $\mathfrak{m}$, then we do obtain a perfect pairing:
$$(H_1(M, \mathbf{Z}_p)_{\mathrm{tors}})_{\mathfrak{m}} \times (H_1(M, \mathbf{Z}_p)_{\mathrm{tors}})_{\mathfrak{m}} \longrightarrow \mathbf{Q}_p/\mathbf{Z}_p.$$
- Cusps have no homology: If we suppose that $H_1(\partial M, \mathbf{Z}_p) = 0$ (this will be often true for us, by § 5.4.3), then we have a sequence
$$H_1(M, \mathbf{Z}_p) \hookrightarrow H_{1,\mathrm{BM}}(M, \mathbf{Z}_p) \twoheadrightarrow \ker(H_0(\partial M, \mathbf{Z}_p) \to H_0(M, \mathbf{Z}_p))$$
The final group is torsion free, and so we obtain an isomorphism
$$H_1(M, \mathbf{Z}_p)_{\mathrm{tors}} \xrightarrow{\sim} H_{1,\mathrm{BM}}(M, \mathbf{Z}_p)_{\mathrm{tors}}.$$
Thus in this situation we obtain a perfect pairing on $H_{1,\mathrm{tors}}(M, \mathbf{Z}_p)$.

5.2. Noncompact hyperbolic manifolds: eigenfunctions and Eisenstein series

5.2.1. Constant terms. We continue with our prior notation: M is a noncompact hyperbolic 3-manifold endowed with a height function Ht.

We say that a function (or differential-form) f on M is of polynomial growth if $\|f(x)\| \leq A \cdot \mathrm{Ht}(x)^B$ for suitable A, B.

The *constant term* of the function f, denoted by f_N, is the function on $\mathcal{C}$ that is obtained by "averaging over toric cross-sections," characterized by the property:

$$f_N \circ \sigma_i^{-1}(z) = \frac{1}{\mathrm{area}(\mathcal{C}_i)} \int_{(x_1,x_2) \in \mathbf{R}^2/\Gamma_{\mathcal{C}_i}} f \circ \sigma_i^{-1}(x_1, x_2, y) dx_1 dx_2, \tag{5.2.1.1}$$

we shall again give an adelic formulation of this later. This has a natural analogue for f a differential form — one simply writes $f \circ \sigma_i^{-1}$ as a combination $Adx_1 + Bdx_2 + Cdy$ and replaces each of A, B, C by their averages A_N, B_N, C_N. In fact, although (5.2.1.1) defines f_N only on the cusp $\mathcal{C}$, this definition can be naturally extended to M_B by integrating on the corresponding horoball on M.

Recall that a function (or differential form) is called *cuspidal* if f_N is identically zero.

If f is an eigenfunction, then f is asymptotic to f_N — denoted $f \sim f_N$ — by which we mean the function $f - f_N$ on $\mathcal{C}$ decays faster than any power of $\mathrm{Ht}(x)$. This rapid decay takes place once the height is sufficiently large; intuitively the "wave" corresponding to f cannot penetrate into the cusp once the width of the cusp is shorter than its wavelength.

More precisely:

Lemma. *If f is an eigenfunction of eigenvalue $-T^2$, then there is an absolute constants b_0, A for which*

(5.2.1.2)
$$||f(x) - f_N(x)| \leq \|f\|_{L^2(M_{\leq Y})} \exp(-b_0 \cdot \mathrm{Ht}(x)), \quad A(1+T^2) \leq \mathrm{Ht}(x) \leq Y/2.$$

If f satisfies relative or absolute boundary conditions on $M_{\leq Y}$, then

$$|f(x) - f_N(x)| \leq \|f\|_{L^2(M_{\leq Y})} \exp(-b_0 \cdot \mathrm{Ht}(x)), \quad \mathrm{Ht}(x) \geq A(1+T^2). \tag{5.2.1.3}$$

that is to say, the prior estimate (5.2.1.2) *actually holds up to the boundary of* $M_{\leq Y}$.

In both cases, similar estimates hold on the right-hand side for any derivative — i.e., any monomial in $\partial_{x_1}, \partial_{x_2}, \partial_y$ *— of fixed order. A similar estimate holds if* f *is an eigenfunction of the Laplacian on* j*-forms.*

This Lemma is, in fact, quite important to us. We will later show that eigenfunctions on $M_{\leq Y}$ (for Y large) are well-approximated by restrictions of eigenfunctions on M (at least for small eigenvalue), and this plays an important role in showing that the eigenfunctions on M_Y are not too big near the boundary.

The proof is by considering the Fourier expansion; each Fourier coefficient satisfies a differential equation that forces it to be an explicit multiple of a Bessel function, and these can be analyzed by hand. The slightly unusual statement is due to the possibility of the Fourier expansion of f containing terms of exponential increase in y.

PROOF. This is proved by direct analysis of Fourier expansions. We give the proof only for zero-forms, the other cases being similar:

According to [**24**, Theorem 3.1] for instance (or rather the proof of *loc. cit.*), $f - f_N$ admits a Fourier expansion that is a sum of terms expressed, in local coordinates (x_1, x_2, y) on each cusp, as

$$(x_1, x_2, y) \mapsto \sum a_\mu F_s(2\pi|\mu|y) e^{2\pi i \langle \mu, (x_1, x_2)\rangle}$$

here μ varies over the dual lattice to the translation lattice of the cusp,[2] and $F \in \text{span}\{I_s, K_s\}$ is a linear combination of I_s and K_s-Bessel functions, and $1 - s^2$ is the eigenvalue of f. In what follows we are going to assume that $|\mu| \geq 1$ for every such μ. If this is not true it merely changes the constants b_0, A in the statement of the Lemma.

The desired relation (5.2.1.2) follows from the asymptotic expansion of I and K, namely, $I(y) \sim \frac{e^y}{\sqrt{2\pi y}}$ and $K(y) \sim \frac{e^{-y}}{\sqrt{2y/\pi}}$, this being true in the range $|y| \gg (1 + |s|)^2$. (See [**1**, page 378] for more precise asymptotic expansions.) Now this asymptotic expansion implies — under the assumption that $y \geq 10(1+T^2)$ — that

(5.2.1.4)
$$|F_s(2\pi|\mu|y)^2| \ll \exp(-\pi|\mu|y) \int_1^Y |F_s(2\pi|\mu|y)|^2 \frac{dy}{y^3} \quad (10(1+T^2) < y < Y/2)$$

One easily verifies (5.2.1.4) for $F = I$ or $F = K$ separately, using their asymptotic expansion. [3] Now (5.2.1.4) for F an arbitrary linear combination of K, I follows from the approximate orthogonality of K, I, more precisely from the inequality $\|aI + bK\|^2_{L^2} \geq \frac{1}{2}\left(\|aI\|^2 + \|bK\|^2\right)$, where, on both sides, the norm is on $[|\mu|, |\mu|Y]$ with respect to the measure dy/y^3. To see this we only need note that the form

[2] We can identify the translation lattice of a cusp with a sublattice of $\mathbf{C}$ via $(x_1, x_2) \mapsto x_1 + ix_2$; we can identify then its dual with a sublattice of $\mathbf{C}$ via the pairing $(z, w) \mapsto \text{Re}(z\overline{w})$, and therefore we may think of $\mu \in \mathbf{C}$ also, thus the notation $|\mu|$.

[3] For example, when $F = K$, the left-hand side is bounded by $e^{-4\pi|\mu|y} \cdot y^{-1}$, up to constant factors, whereas the integral on the right-hand side is bounded below by a constant multiple of $\int_{5(1+T^2)}^{10(1+T^2)} e^{-4\pi|\mu|x} x^{-4} dx \gg e^{-4\pi|\mu|(6(1+T^2))}(1 + T^2)^{-4} \gg e^{-2.5\pi|\mu|y} y^{-4} \gg e^{-3\pi|\mu|y} y^{-1}$. Similarly for $F = I$.

$\frac{1}{2}\|I\|^2x^2+2\langle I,K\rangle xy+\frac{1}{2}\|K\|^2y^2$ is positive definite, because $\langle I,K\rangle^2$ is much smaller than $\|I\|^2\cdot\|K\|^2$ so long as Y is large enough.

By Cauchy-Schwarz, we see that, if $b=\pi/2$ say,

$$\left|\sum_{\mu\neq 0} a_\mu F_s(2\pi|\mu|y)e^{2\pi i\langle\mu,(x_1,x_2)\rangle}\right|^2 \leq \left|\sum_{\mu\neq 0} a_\mu F_s(2\pi|\mu|y)e^{-\pi|\mu|y/2}e^{\pi|\mu|y/2}\right|^2$$
$$\ll \exp(-by)\sum_\mu |a_\mu|^2\int_A^Y F_s(2\pi|\mu|y)|^2\frac{dy}{y^3}$$
$$\leq \exp(-by)\|f\|_{L^2(M_Y)}.$$

To verify (5.2.1.3) we observe that the Fourier expansion of F must consist (for example, in the case of relative (Dirichlet) boundary conditions) in a linear combination of functions $G_{s,\mu}(y)$ where

$$G_{s,\mu} := K_s(2\pi|\mu|y) - \frac{K_s(2\pi|\mu|Y)}{I_s(2\pi|\mu|Y)}I_s(2\pi i|\mu|y),$$

this follows by noting that each Fourier coefficient must individually satisfy Dirichlet boundary conditions, and so restricts to zero at $y=Y$. Similarly, in the case of absolute (Neumann) boundary conditions, we obtain a linear combination of similar functions $G_s'(|\mu|y)$ with $G_s := K_s(2\pi i|\mu|y) - \frac{K_s'(2\pi i|\mu|Y)}{I_s'(2\pi i|\mu|Y}I_s(2\pi i|\mu|y)$. We proceed in the case of Dirichlet conditions, the Neumann case again being similar:

Then the asymptotic expansions imply – again, assuming that $Y\geq 10(1+T^2)$, but without the restrictive assumption that $y<Y/2$ – that

$$|G_s(|\mu|y)| \leq \exp(-\pi|\mu|y)\int_1^Y |G_s(|\mu|y)|^2\frac{dy}{y^3}, \tag{5.2.1.5}$$

hence (5.2.1.3) follows similarly.

To check (5.2.1.5), note that $\left|\frac{K_s(2\pi i|\mu|Y)}{I_s(2\pi i|\mu|Y)}I_s(2\pi i|\mu|y)\right| \leq |K_s(2\pi i|\mu|Y)| \leq |K_s(2\pi i|\mu|y)|$, at least for y larger than some absolute constant; that means, in particular, that $|G_{s,\mu}(y)|$ is bounded by $2|K_s(2\pi|\mu|y)|$. But the right-hand side integral exceeds a constant multiple of the same integral with $G_{s,\mu}$ replaced by $|K_s(2\pi i|\mu|y)|^2$, by means of the discussion after (5.2.1.4). Consequently, (5.2.1.5) results from the fact that, for $F_s\equiv K_s$, (5.2.1.4) is actually valid for $y<Y$, not merely for $y<Y/2$. □

We put

$$\wedge^Y f(x) = \begin{cases} f(x), & \mathrm{Ht}(x)\leq Y \\ f(x)-f_N(x), & \mathrm{Ht}(x)>Y. \end{cases}$$

5.2.2. Summary of results on Eisenstein series. We present in summarized form the parameterization of the continuous spectrum of the Laplacian on forms and functions on M. This parameterization is in terms of the continuous spectrum of the Laplacian on M_B.

We assume familiarity with some of the standard results on Eisenstein series; for example, the treatment in Iwaniec [**42**]; although this reference addresses hyperbolic two-space and only the case of functions, the generalizations necessary for hyperbolic 3-space and for i-forms are quite routine. One can also consult the much more general reference [**51**].

5.2.2.1. *Spaces of functions and forms on M_B.* We denote by $C^\infty(s)$ the space of functions on M_B that are a multiple of y^{1+s} on each component. We denote by $\Omega^+(s)$ resp. $\Omega^-(s)$ the space of 1-forms on M_B that are a multiple of $y^s(dx_1+idx_2)$ resp. $y^{-s}(dx_1-idx_2)$ (note the $-s$ in the second definition!) on each component.

If each cusp has no orbifold points, the dimension of all three spaces equals the number of cusps of M. We denote this number by h. In general the dimensions of $\Omega^\pm(s)$ equals the number of *relevant* cusps (denoted h_{rel}) and the dimension of $C^\infty(s)$ equals the number of cusps:

We sometimes write simply C^∞ (resp. Ω^+, Ω^-) for these respective spaces when $s=0$. Thus, for $f \in C^\infty$, we have $f \cdot y^s \in C^\infty(s)$, and our discussion of dimensions says:

$$\dim \Omega^+ = \dim \Omega^- = h_{\mathrm{rel}}, \dim C^\infty = h.$$

We introduce inner products on $C^\infty, \Omega^+, \Omega^-$ for purely imaginary s via the rule

$$\langle f, g\rangle = \int_{\mathrm{Ht}(x)=T} \langle f, g\rangle_x, \tag{5.2.2.1}$$

where the measure on the set $\mathrm{Ht}(x) = T$ (this set has an obvious structure of 2-manifold) is the measure induced by the hyperbolic metric, and $\langle f, g\rangle_x$ is the inner product on the space of functions/forms at x induced by the hyperbolic metric. Since s is purely imaginary, the resulting inner product is independent of the choice of T.

5.2.2.2. *Functions.* For $f \in C^\infty(0)$ and $s \neq 0$, the theory of Eisenstein series implies that there exists a unique eigenfunction $E(s, f)$ of the Laplacian on M, with eigenvalue $1-s^2$, and with the property that

$$E(s, f) \sim f \cdot y^s + g \cdot y^{-s},$$

for some $g \in C^\infty(0)$. Indeed, the association $f \mapsto g$ defines a meromorphic linear operator $\Psi(s) : C^\infty(0) \to C^\infty(0)$, the so-called *scattering matrix*; thus

$$f \cdot y^s + (\Psi(s) f)\, y^{-s}$$

is the asymptotic part of an eigenfunction. The uniqueness implies that $\Psi(s)\Psi(-s) =$ id. [4] We will write $\psi(s) := \det \Psi(s)$ for the determinant of Ψ.

Moreover, choosing an orthonormal basis $\mathcal{B} = \{f_1, \ldots, f_h\}$ for $C^\infty(0)$, the functions $E(s, f)$ for $s \in i\mathbf{R}_{\geq 0}$ span the continuous spectrum of the Laplacian — that is to say, for any square integrable 1-form F, we have the relation

$$\|F\|^2 = \sum_j \langle F, \psi_j\rangle^2 + \frac{1}{2\pi} \sum_{f \in \mathcal{B}} \int_{t=0}^{\infty} |\langle F, E(f, it)\rangle|^2, \tag{5.2.2.2}$$

where $\{\psi_j\}$ is an orthonormal basis for the discrete spectrum of the Laplacian on functions.

[4] These statements remain valid in slightly modified form at $s=0$; we leave the formulation to the reader.

The so-called Maass-Selberg relations assert that[5] for $s, t \in i\mathbf{R}$,

$$\langle \wedge^Y E(f,s), \wedge^Y E(f',t)\rangle = \langle f, f'\rangle \frac{Y^{s-t}}{s-t} + \langle \Psi(s)f, \Psi(t)f'\rangle \frac{Y^{t-s}}{t-s} + \langle \Psi(s)f, f'\rangle \frac{Y^{-s-t}}{-s-t} + \langle f, \Psi(t)f'\rangle \frac{Y^{s+t}}{s+t}. \tag{5.2.2.3}$$

In particular, taking the limit as $s \to t$ (and still assuming s is purely imaginary),

$$\| \wedge^Y E(f,s)\|^2 = 2\log Y \langle f, f\rangle - \langle \Psi(s)^{-1}\Psi'(s)f, f\rangle + \langle f, \Psi(s)f\rangle \frac{Y^{2s}}{2s} - \langle \Psi(s)f, f\rangle \frac{Y^{-2s}}{2s} \tag{5.2.2.4}$$

where $\Psi'(s) := \frac{d}{ds}\Psi(s)$; this implies in particular that

$$-\Psi(s)^{-1}\Psi'(s) \text{ is bounded below}, \quad s \in i\mathbf{R}, \tag{5.2.2.5}$$

i.e. $-\Psi(s)^{-1}\Psi'(s) + tI$ is positive definite for some $t > 0$ and *all* s.

The following observation will be of use later: if $h(s)$ is an analytic function satisfying $h(-s) = h(s)$, and $J(s) = \langle f, \Psi(s)f\rangle \frac{Y^{2s}}{2s} - \langle \Psi(s)f, f\rangle \frac{Y^{-2s}}{2s}$ is the second term of (5.2.2.4), then

$$\int_{t\in\mathbf{R}} h(t)J(it)dt - \pi h(0)\langle \Psi(0)f, f\rangle = \text{ decaying as } Y \to \infty$$

as one sees by shifting contours. [6]

5.2.2.3. *Forms.* For each $\omega \in \Omega^+(0)$, there exists unique $\omega' \in \Omega^-(0)$ with the property that there is an eigenfunction of the Laplacian on 1-forms, with eigenvalue $-s^2$, and asymptotic to

$$\omega \cdot y^s + \omega' \cdot y^{-s}. \tag{5.2.2.6}$$

The association $\omega \mapsto \omega'$ defines a meromorphic map $\Phi^+(s) : \Omega^+(0) \to \Omega^-(0)$. The operator $\Phi^+(s)$, for $s \in i\mathbf{R}$, is *unitary* for the unitary structure previously defined. The inverse of $\Phi^+(s)$ is denoted $\Phi^-(s)$. We denote the unique differential form asymptotic to (5.2.2.6) by $E(\omega, s)$ or $E(\omega', s)$:

$$E(\omega, s) \sim \omega \cdot y^s + \Phi^+(s)\omega \cdot y^{-s} \qquad (\omega \in \Omega^+(0)), \tag{5.2.2.7}$$

$$E(\omega', s) \sim \omega' \cdot y^{-s} + \Phi^-(s)\omega' \cdot y^{s} \qquad (\omega \in \Omega^-(0)). \tag{5.2.2.8}$$

Note the funny normalization of signs! In particular, $\Phi^-(s)\Phi^+(s) = 1$.

Fixing an orthonormal basis $\omega_1, \dots, \omega_r$ for $\Omega^+(s)$, the functions $E(\omega_i, s)$ form an orthonormal basis for *co-closed* 1-forms on $Y(K)$, i.e., forms satisfying $d^*\omega = 0$;

[5] The "short" mnemonic is that the right-hand side is the regularized integral

$$\int_{\mathrm{Ht}\geq Y}^{\mathrm{reg}} -E(f,s)_N \cdot \overline{E(f',t)_N}$$

[6] Write $A(s) = \langle \Psi(-s)f, f\rangle Y^{2s}$, so that $J(it) = \frac{A(it)-A(-it)}{2it}$. Shifting contours,

$$\int_{t\in\mathbf{R}} h(t)J(it) = \int_{\mathrm{Im}(t)=\delta} h(t)J(it) = \int_{\mathrm{Im}=\delta} \frac{h(t)A(it)}{2it}dt + \int_{\mathrm{Im}=-\delta} \frac{h(t)A(it)}{2it}dt.$$

But $A(it)$ decays rapidly with Y when $\mathrm{Im}(t) > 0$. Shift the second term to $\mathrm{Im} = \delta$ also, leaving a residue of $\pi h(0)A(0)$.

together, $E(\omega_i, s)$ and $dE(f, s)$ form an orthonormal basis for 1-forms, that is to say:

$$\|\nu\|^2 = \sum_{\psi} |\langle \nu, \psi \rangle|^2 + \int_0^{\infty} \sum_f |\langle \nu, \frac{dE(f, it)}{\sqrt{1+t^2}} \rangle|^2 \frac{dt}{2\pi} + \sum_{\omega} \frac{1}{2\pi} \int_{-\infty}^{\infty} |\langle \nu, E(\omega, it) \rangle|^2 dt,$$

where ψ ranges over an orthonormal basis for the spectrum of the form Laplacian acting on 1-forms that belong to L^2.

The Maass-Selberg relations assert that for $\omega_1, \omega_2 \in \Omega^+$ and $s, t \in i\mathbf{R}$,

$$\langle \wedge^Y \mathrm{Eis}(s, \omega_1), \wedge^Y \mathrm{Eis}(t, \omega_2) \rangle \tag{5.2.2.9}$$

$$= \langle \omega_1, \omega_2 \rangle \cdot (\frac{Y^{s-t}}{s-t}) + \langle \Phi^+(t)^{-1} \Phi^+(s) \omega_1, \omega_2 \rangle \frac{Y^{t-s}}{t-s}. \tag{5.2.2.10}$$

(one can remember this using the same mnemonic as before, but it is a little bit simpler.)

Taking the limit as $t \to s$, and a sum over an orthonormal basis for $\Omega^+)(s)$, we obtain

$$\sum_{\omega} \| \wedge^Y E(s, \omega) \|^2 = (2h_{\mathrm{rel}} \log Y) - \mathrm{trace}(\Phi(s)^{-1} \Phi'(s)). \tag{5.2.2.11}$$

Note that this reasoning also implies that, as in the previous case,

$-\frac{\Phi'(s)}{\Phi(s)}$ is bounded below,

i.e. $aI - \frac{\Phi'(s)}{\Phi(s)}$ is positive definite for all $s \in i\mathbf{R}$ and suitable $a > 0$.

5.2.2.4. *2-forms and 3-forms; the Hodge* $*$. The corresponding spectral decompositions for 2- and 3-forms follow by applying the * operator. For the reader's convenience we note that, on $\mathbf{H}^3$:

$$*dx_1 = \frac{dx_2 \wedge dy}{y}, \; dx_2 = \frac{dy \wedge dx_1}{y}, \; *dy = \frac{dx_1 \wedge dx_2}{y},$$
$$*(dx_2 \wedge dy) = y dx_1, \; *(dy \wedge dx_1) = y dx_2, \; *(dx_1 \wedge dx_2) = y dy$$

5.2.2.5. *Residues.* The function $\Psi(s) : C^{\infty} \to C^{\infty}$ (and, similarly, $s \mapsto E(s, f)$) has a simple pole at $s = 1$. We shall show that:

> The residue R of $\Psi(s)$ at $s = 1$ is, in a suitable basis, a diagonal matrix, with nonzero entries $\mathrm{area}(\partial N)/\mathrm{vol}(N)$, where N varies through connected components of M.

PROOF. For any $f \in C^{\infty}$, the residue of $s \mapsto E(s, f)$ at $s = 1$ is a function on M, constant on each connected component; we denote it by c_f. We also think of it as a function on cusps (namely, its constant value on the corresponding connected component). Alternately, $R(f)$ is a multiple of y on each connected component of M_B, and $R(f)/y$ coincides with c_f (considered as a function on cusps). Again, we refer to [**42**] for a treatment of the corresponding facts in hyperbolic 2-space, the proofs here being identical.

A computation with the Maass-Selberg relations shows that, if $f = y \cdot 1_{\mathcal{C}_i}$ (which is to say: the function which equals y on the component of M_B corresponding to $\mathcal{C}_i$, and zero on all other components) then c_f takes the value

$$\frac{\mathrm{area}(\mathcal{C}_i)}{\mathrm{vol}(\mathcal{C}_i)} \tag{5.2.2.12}$$

on the connected component of $\mathcal{C}_i$, and zero on all other components.

Indeed, it is enough to check the case of M connected. The Maass-Selberg relations show that $\|c_f\|^2_{L^2(M)} = \langle R(f), f\rangle_{C^\infty}$. In particular, if we write m_i for the value of c_f when f is the characterstic function of the cusp $\mathcal{C}_i$, we have, for any scalars x_i, $\mathrm{vol}(M)\,|\sum x_i m_i|^2 = \sum x_i m_i \overline{\sum x_i}\mathrm{area}(\mathcal{C}_i)$, whence (5.2.2.12). □

Observe for later reference that the kernel of the map $f \mapsto c_f$ (equivalently $f \mapsto R(f)$) consists of all $f = \sum x_i(y1_{\mathcal{C}_i})$ such that

$$\sum x_i \mathrm{area}(\mathcal{C}_i) = 0, \tag{5.2.2.13}$$

where the $\mathcal{C}_i$ vary through the cusps of any connected component of M.

5.3. Reidemeister and analytic torsion

We define carefully the regulator, Reidemeister torsion, and analytic torsion in the non-compact case. We shall first need to discuss harmonic forms of polynomial growth and how to equip them with an inner product. It is worth noting that for our purposes, when we are comparing two manifolds, the exact definition is not so important so long as one is consistent.

Let M be any hyperbolic 3-manifold of finite volume, equipped with a height function Ht as in § 5.1. Our main interest is in the case $M = Y(K)$, equipped with Ht as in § 5.4.1.

5.3.1. The inner product on forms of polynomial growth. We will define below a specific space $\mathscr{H}^j$ of harmonic forms ($0 \le j \le 2$) uch that:

The natural map $\mathscr{H}^j \to H^j(M, \mathbf{C})$ is an isomorphism.

- For $j = 0$ we take $\mathscr{H}^j$ to consist of constant functions.
- For $j = 1$, we take $\mathscr{H}^j$ to consist of cuspidal harmonic forms together with forms of the type $\mathrm{Eis}(\omega, 0)$, where $\omega \in \Omega^+(0)$.
- For $j = 2$, we take $\mathscr{H}^j$ to consist of cuspidal harmonic 2-forms together with forms of the type $*dE(f, 1)$ where $f \in C^\infty(0)$ is such that $s \mapsto E(f, s)$ is holomorphic at $s = 1$ (equivalently: f lies in the kernel of the residue of $\Psi(s)$ at $s = 1$).

One verifies by direct computation that the map to $H^j(M, \mathbf{C})$ from this subspace is an isomorphism. For $j = 1, 2$ we have a canonical decomposition

$$\mathscr{H} = \Omega^j_{\mathrm{cusp}}(M) \oplus \Omega^j_{\mathrm{Eis}}(M),$$

where $\Omega^j_{\mathrm{cusp}}(M)$ is the space of cuspidal j-forms (i.e. $\omega_N \equiv 0$), and $\Omega^j_{\mathrm{Eis}}(M)$ is the orthogonal complement[7] of Ω^j_{cusp}. Moreover, $\Omega^j_{\mathrm{cusp}}(M)$ maps isomorphically, under $\mathscr{H} \xrightarrow{\sim} H^j(M, \mathbf{C})$, to the cuspidal cohomology $H^j_!(M, \mathbf{C})$.

5.3.1.1. *An inner product on $\mathscr{H}^1$.* We introduce on $\Omega^1_{\mathrm{Eis}}(M)$ the inner product

$$\|\omega\|^2 = \lim_Y \frac{\int_{M_{\le Y}} \langle \omega, \omega\rangle}{\log Y}. \tag{5.3.1.1}$$

where the inner product $\langle \omega, \omega\rangle$ at each point of M is taken using the Riemannian structure. In explicit terms, if $\sigma_i^*\omega = a_i dx_1 + b_i dx_2$, then the norm of σ equals

[7]This makes sense even though elements of $\mathscr{H}$ are not square integrable, because elements in Ω^j_{cusp} are of rapid decay, and therefore can be integrated against a function of polynomial growth.

$\sum_i \text{area}(\mathcal{C}_i)\cdot(|a_i|^2+|b_i|^2)$. [8] This norm has fairly good invariance properties, e.g., the Hecke operators act self-adjointly, etc.

We now endow

$$\mathscr{H}^1 \xrightarrow{\sim} \Omega^1_{\text{cusp}} \oplus \Omega^1_{\text{Eis}}$$

with the direct sum of the two inner products defined.

5.3.1.2. *An inner product on $\mathscr{H}^2$.* For the definition of Reidemeister torsion later we will use only $H_{2!}$, in fact, but the general setup will be of use later.

Unike $\mathscr{H}^1$, there is no "good" inner product on Ω^2_{Eis} (in a suitable sense, $\mathscr{H}^1$ grow only logarithmically at ∞, but this is not so for $\mathscr{H}^2$). We make the more or less arbitrary definition:

$$\|\omega\|^2 = \lim_{Y\to\infty} Y^{-2}\int_{M_{\leq Y}}\langle\omega,\omega\rangle, \quad (\omega\in\Omega^2_{\text{Eis}}) \tag{5.3.1.3}$$

which, although analogous to the prior definition, has no good invariance properties — for exaple, the Hecke operators do not act self-adjointly. Again we endow

$$\mathscr{H}^2 \xrightarrow{\sim} \Omega^2_{\text{cusp}} \oplus \Omega^2_{\text{Eis}}$$

with the direct sum of the standard inner product on the first factor, and the inner product just defined, on the second factor.

The inner product on Ω^2_{Eis} is simple to compute explicitly: Supposing for simplicity that M is connected (the general case follows component by component). Let $\mathcal{C}_1,\dots,\mathcal{C}_h$ be the cusps of M. For $\mathbf{u}=(u_1,\dots,u_h)\in\mathbf{C}^h$ such that $\sum u_i=0$, let $\omega\in\Omega^2_{\text{Eis}}$ be such that $\int_{\mathcal{C}_i}\omega=u_i$. Then

$$\|\omega\|^2 = \frac{1}{2}\sum_i \text{area}(\mathcal{C}_i)^{-1}|u_i|^2 \tag{5.3.1.4}$$

Indeed $\sigma_i^*\omega = \frac{u_i}{\text{area}(\mathcal{C}_i)}dx_1\wedge dx_2+$ rapidly decaying, from where we deduce (5.3.1.4).

5.3.2. Definition of Reidemeister torsion. We *define*

$$\text{reg}(M) = \frac{\text{reg}(H_1)\text{reg}(H_{3,\text{BM}})}{\text{reg}(H_0)\text{reg}(H_{2!})} \tag{5.3.2.1}$$

where

$$\text{reg}(H_{i,?}) = \left|\det\int_{\gamma_i}\omega_j\right|,$$

and

- (for $i\in\{0,1\}$) the elements $\gamma_i\in H_i(M,\mathbf{Z})$ are chosen to project to a basis for $H_i(M,\mathbf{Z})_{\text{tf}}$, and the ω_j are an orthonormal basis for $\mathscr{H}^i$;
- (for $i=2$) as above, but replace $H_i(M,\mathbf{Z})_{\text{tf}}$ by the torsion-free quotient of $H_{2,!}(M,\mathbf{Z})$, and the ω_j are an orthonormal basis for the space of *cuspidal* harmonic 2-forms.

---

[8] An equivalent definition is

$$\|\omega\|^2 = \int_{\partial M_{\leq Y}}\omega_N\wedge\overline{\omega_N}, \tag{5.3.1.2}$$

this holding true for any Y large enough. Here, if $\sigma_i^*\omega_N = a dx_1 + b dx_2$, then $\overline{\omega_N}$ is defined to be the differential form on M_B so $\sigma_i^*\overline{\omega_N} = \bar{b}dx_1 - \bar{a}dx_2$ (this looks more familiar in coordinates $x_1 \pm ix_2$!) .

- (for $i = 3$) as above, but replace $H_i(M, \mathbf{Z})_{\mathrm{tf}}$ by the torsion-free quotient of $H_3^{\mathrm{BM}}(M, \mathbf{Z})$, and the ω_j are an orthonormal basis for the space of harmonic 3-forms (= multiples of the volume form on each component).

See also after (5.3.3.2) for some further discussion. Our ugly definition here pays off in cleaner theorems later.

If M is compact this specializes to (3.1.2.2). Note that

(i) $\mathrm{reg}(H_0) = \frac{1}{\sqrt{\mathrm{vol}(M)}}$ and $\mathrm{reg}(H_3) = \sqrt{\mathrm{vol}(M)}$. We follow our convention that vol denotes the product of volumes of all connected components.

(ii) If M were compact, $\mathrm{reg}(H_1)\mathrm{reg}(H_{2,!}) = 1$, which recovers our earlier definition

The definition of Reidemeister torsion given in § 3.1.2 – see equation (3.1.2.3) – now carries over to the non-compact case also, namely:

$$\mathrm{RT}(M) := |H_1(M, \mathbf{Z})_{\mathrm{tors}}|^{-1}\mathrm{reg}(M) \tag{5.3.2.2}$$

$$= |H_1(M, \mathbf{Z})_{\mathrm{tors}}|^{-1} \cdot \mathrm{vol}(M) \cdot \frac{\mathrm{reg}(H_1)}{\mathrm{reg}(H_{2!})}. \tag{5.3.2.3}$$

As always, vol here denotes the product of the volumes of all connected components.

5.3.3. Explication of the H_2-regulator. In this section, we relate $\mathrm{reg}(H_2)$ and $\mathrm{reg}(H_{2,!})$, which will be necessary later.

Begin with the sequence

$$H_3^{\mathrm{BM}}(M, \mathbf{Z}) \to H_2(\partial M, \mathbf{Z}) \to H_2(M, \mathbf{Z}) \twoheadrightarrow H_{2,!}(M, \mathbf{Z}),$$

where as before $H_{2,!}$ is the image of H_2 in H_2^{BM}. All of the groups above are torsion-free, at least away from any orbifold prime for M. In what follows we suppose there are no orbifold primes; if this is not so, our computation must be adjusted by a rational number supported only at those primes.

Fix generators[9] $\delta_1, \ldots, \delta_h$ for the free group $H_2(\partial M, \mathbf{Z})/\mathrm{image}(H_3^{\mathrm{BM}}(\mathbf{Z}))$. Fix elements $\gamma_1, \ldots, \gamma_r \in H_2(M, \mathbf{Z})$ whose images span $H_{2,!}$. Then:

$$H_2(M, \mathbf{Z}) \quad = \quad \bigoplus_j \mathbf{Z}\gamma_j \oplus \bigoplus_i \mathbf{Z}\delta_i, \tag{5.3.3.1}$$

where we have identified the δ_i with their image in $H_2(\partial M, \mathbf{Z})$.

Fix orthonormal bases $\omega_1, \ldots, \omega_r$ for $\Omega^2(M)_{\mathrm{cusp}}$ and $\eta_1, \ldots, \eta_h$ for $\Omega^2(M)_{\mathrm{Eis}}$. Note that *the ω_is are orthogonal to the δ_j*, that is to say

$$\int_{\delta_j} \omega_i \equiv 0,$$

because the ω_is are cuspidal and so decay in each cusp.

[9]For example, if M is connected without orbifold points and we enumerate the cusps $\mathcal{C}_1, \ldots, \mathcal{C}_h$, we may take $\delta_i = [\mathcal{C}_{i+1}] - [\mathcal{C}_1]$, where $[\mathcal{C}]$ denotes the fundamental class of a torus cross-section of $\mathcal{C}$.

Therefore, the "matrix of periods" with respect to these bases is block-diagonal, and so:

$$\text{(5.3.3.2)} \qquad \begin{aligned} \mathrm{reg}(H_2) &= \left|\det\left(\int_{\gamma_i}\omega_j\right)\cdot\det\left(\int_{\delta_i}\eta_j\right)\right| \\ &= \mathrm{reg}(H_{2,!})\cdot\left|\det\left(\int_{\delta_i}\eta_j\right)\right|. \end{aligned}$$

where we have used the definition: $\mathrm{reg}(H_{2,!}) = \det\left(\int_{\gamma_i}\omega_j\right)$.

The latter factor $\det\left(\int_{\delta_i}\eta_j\right)$ is easy to compute from (5.3.1.4): if M is connected with cusps $\mathcal{C}_i$, then there is an equality

$$\det\left(\int_{\delta_i}\eta_j\right)^{-2} = \prod(2\mathrm{area}(\mathcal{C}_i))^{-1}\left(\sum 2\mathrm{area}(\mathcal{C}_i)\right).$$

The computation on the right hand side is deduced from (5.3.1.4): the squared covolume of $\{(x_1,\dots,x_k)\in\mathbf{Z}^k : \sum x_i = 0\}$ in the metric $\sum a_i x_i^2$ is given by $\prod a_i\cdot\sum a_i^{-1}$; the inverse on the left arises because this is computing a covolume for *cohomology*.

More generally, if $N_1,\dots,N_k$ are the connected components of M, we deduce $\det\left(\int_{\delta_i}\eta_j\right)^2$ equals $\frac{(\prod_{\mathcal{C}} 2\mathrm{area}(\mathcal{C}))}{\prod_N 2\mathrm{area}(\partial N)}$, which, in view of the discussion after (5.2.2.13), can also be written:

$$\text{(5.3.3.3)} \qquad \det\left(\int_{\delta_i}\eta_j\right)^2 = \frac{(\prod_{\mathcal{C}} 2\mathrm{area}(\mathcal{C}))}{\mathrm{vol}(M)}\cdot{\det}'(2R)^{-1},$$

where R is the residue at $s=2$ of $\Psi(s)$, and $\det'$ denotes the product of nonzero eigenvalues. We have followed the convention that vol denotes the products of volumes of connected components.

Although (5.3.3.3) looks unwieldy, it will be convenient when comparing the left-hand side between two different manifolds.

5.3.4. Regularized analytic torsion. We now present the definition of the analytic torsion of a non-compact hyperbolic 3-manifold.

Notation as previous, the operator $e^{-t\Delta}$, and its analogue for i-forms, are not, in general, trace-class. Nonetheless there is a fairly natural way to regularize their traces, which we now describe:

Let $K(t;x,y)$ be the integral kernel of $e^{-t\Delta}$ acting on functions, i.e. the heat kernel, and let

$$k_t(x) = K(t;x,x).$$

(In the case of forms, $K(t;x,x)$ is an endomorphism of the space of forms at x, and we take the trace.) We also set $\wedge^Y k_t(x)$ to be $\wedge_x^T K(t;x,y)$ (the operator $\wedge^Y$ acts in the first variable x) specialized to $x=y$. By definition,

$$\wedge^Y k_t(x) = k_t(x) \text{ if } \mathrm{Ht}(x)\le Y,$$

and also $\wedge^Y k_t(x)$ is of rapid decay: If we fix t, the quantity $\sup_{\mathrm{Ht}(x)>Y}\wedge^Y k_t(x)$ decays faster than any polynomial in Y. In particular,

$$\int_{M_{\le Y}} k_t(x) = \int_M \wedge^Y k_t(x) + (\text{ rapidly decaying in } Y\ ).$$

In fact, either side is asymptotic to a linear polynomial in $\log Y$. This is a consequence of the Selberg trace formula, for example; see [**30**]. Assuming this, we may define the regularized trace:

DEFINITION 5.3.5. *The regularized trace* $\mathrm{tr}^*(e^{-t\Delta})$ *is the constant term of the unique linear function of* $\log Y$ *asymptotic to* $\int_M \wedge^Y k_t$*, i.e. it is characterized by the property*

$$\int_M \wedge^Y k_t(x)dx \sim k_0 \log Y + \mathrm{tr}^*(e^{-t\Delta}), \tag{5.3.5.1}$$

Here the notation $A(Y) \sim B(Y)$ *means that* $A(Y) - B(Y) \to 0$ *as* $Y \to \infty$.

Of course this definition of tr^* depends on the choice of height function. We can be more precise about $k_0 = k_0(t)$: e.g., in the case of 0-forms, $k_0 = \frac{h}{2\pi}\int_{t=0}^{\infty} e^{-t(1+s^2)}$, where h is the number of cusps.[10]

The Selberg trace formula may be used to compute $\mathrm{tr}^*(e^{-t\Delta})$ (see [**30**] for the case of functions); in particular, it has an asymptotic expansion near $t = 0$:

$$\mathrm{tr}^*(e^{-t\Delta}) \sim at^{-3/2} + bt^{-1/2} + ct^{-1/2}\log(t) + d + O(t^{1/2}), \tag{5.3.5.2}$$

We do not need the explicit values of the constants a, b, c, d, although they are computable. We define the regularized determinant of Δ as usual, but replacing the trace of $e^{-t\Delta}$ by tr^*, i.e.

$$\log\det{}^*(\Delta) = -\frac{d}{ds}\Big|_{s=0}\left(\Gamma(s)^{-1}\int_0^{\infty}(H(t) - H(\infty))t^s\frac{dt}{t}\right), \tag{5.3.5.3}$$
$$H(t) := \mathrm{tr}^*(e^{-t\Delta}), H(\infty) = \lim_{t\to\infty} H(t)$$

This definition requires a certain amount of interpretation: Split the integral into $I_1(s) = \int_0^1$ and $I_2(s) = \int_1^{\infty}$. The first term is absolutely convergent for $\Re(s)$ large and (by (5.3.5.2)) admits a meromorphic continuation to $\Re(s) > -1/2$ with at worst a simple pole at $s = 0$; in particular, $\Gamma(s)^{-1}I_1(s)$ is regular at $s = 0$ and so $\frac{d}{ds}\big|_{s=0}\Gamma(s)^{-1}I_1(s)$ is defined. The second term is absolutely convergent for $\Re(s) < 1/2$ (as follows, for example, from (5.3.5.4) below and elementary estimates) and indeed $\frac{d}{ds}\big|_{s=0}\Gamma(s)^{-1}I_2(s) = \int_1^{\infty}\frac{H(t)-H(\infty)}{t}dt$. We interpret (5.3.5.3) as *defining* $\log\det{}^*\Delta$ as the sum of these quantities.

This leads us to the definition of the regularized analytic torsion:

$$\log\tau_{\mathrm{an}}(Y) = \frac{1}{2}\sum_j(-1)^{j+1}j\log\det{}^*(\Delta_j),$$

where Δ_j denotes the regularized determinant on j-forms.

On the other hand, on the spectral side, we may write (for instance, in the case of functions, that is to say, 0-forms):

---

[10] This computation of k_0 follows easily from the spectral expansion

$$k_t(x) = \sum e^{-t\lambda}|\psi_\lambda(x)|^2 + \frac{1}{2\pi}\sum_{f\in\mathcal{B}}\int_{t=0}^{\infty} e^{-t(1+s^2)}|E(f, it)(x)|^2,$$

see (5.2.2.2).

(5.3.5.4)

$$H(t) - H(\infty) = \sum_{\lambda_j \neq 0} e^{-t\lambda_j} + \frac{1}{2\pi}\int_{u=0}^{\infty} -\frac{\psi'}{\psi}(iu)e^{-t(1+u^2)}du + \frac{1}{4}e^{-t}\text{trace}\Psi(0),$$

where $\psi(s)$ is, as before, the determinant of the scattering matrix, and λ_j are the nonzero eigenvalues of the Laplacian on functions. On the other hand one has

$$H(\infty) = b_0, \tag{5.3.5.5}$$

where b_0 is equal to the number of zero eigenvalues of the Laplacian in the discrete spectrum, i.e., the number of connected components of M. Both (5.3.5.4) and (5.3.5.5) follow from (5.2.2.4). They mean, in particular, that $H(t) - H(\infty)$ decays exponentially at ∞ to handle convergence issues in the discussion after (5.3.5.3). The case of 1-forms is slightly different: because 0 lies in the continuous spectrum, the decay of $H(t) - H(\infty)$ is only as $t^{-1/2}$, but that is still adequate for convergence.

5.3.6. Comparison of regularized trace for two hyperbolic manifolds. Suppose M, M' are a pair of hyperbolic manifolds with cusps $\mathcal{C}, \mathcal{C}'$. Suppose that there is an isometry $\sigma : \mathcal{C} \to \mathcal{C}'$.

Then one can compute the difference $\text{tr}^*(e^{-t\Delta_M}) - \text{tr}^*(e^{-t\Delta_{M'}})$ without taking a limit, which will be useful later. Indeed we show that

$$\begin{aligned} \text{tr}^*(e^{-t\Delta_M}) - \text{tr}^*(e^{-t\Delta_M}) \quad &= \quad \left(\int_{M-\mathcal{C}} k_t(x) - \int_{M'-\mathcal{C}'} k_t'(x)\right) \\ &+ \quad \left(\int_{\mathcal{C}} k_t(x) - k_t'(\sigma(x))\right), \end{aligned} \tag{5.3.6.1}$$

where all integrals are absolutely convergent. This holds for the Laplacian on j-forms, for any j; we do not include j explicitly in our notation below.

To verify (5.3.6.1) let $\mathcal{C}_Y = M_{\leq Y} \cap \mathcal{C}$. We may write

$$\text{tr}^*(e^{-t\Delta_M}) = \lim_{Y\to\infty}\left(\int_{M-\mathcal{C}} k_t(x) + \int_{\mathcal{C}_Y} k_t(x) - k_0(t)\log Y\right),$$

and similarly for M'. Here $k_0(t)$ is as discussed after Definition 5.3.5. The coefficients of $\log Y$, namely, $k_0(t)$ and $k_0'(t)$, are identical for M and M'; they depend only on the cusps, and were given after (5.3.5.1). Subtracting yields the result once we verify that

$$\lim_{Y\to\infty}\left(\int_{\mathcal{C}_Y} k_t(x) - \int_{\mathcal{C}_Y} k_t'(x)\right) = \int_{\mathcal{C}} k_t(x) - k_t'(\sigma(x)).$$

But it is easy to verify, by the "locality" of the heat kernel, that $k_t - k_t' \circ \sigma$ is of rapid decay on $\mathcal{C}$; in particular it is integrable. That completes the proof of (5.3.6.1).

5.4. Noncompact arithmetic manifolds

Thus far our considerations applied to any hyperbolic 3-manifold. In this section we specialize to the case of an arithmetic, noncompact 3-manifold.

Thus $\mathbb{G} = \text{PGL}_2$ and F is an imaginary quadratic field. Recall (§ 3.1.1) that $\mathbf{B}$ is the upper triangular Borel subgroup of $\mathbb{G} = \text{PGL}_2$ and $K_{\max}$ is a certain maximal compact subgroup.

5.4.1. The case of adelic quotients. We specify a height function for the manifolds $M = Y(K)$ and then give adelic descriptions of the objects $M_B, M_{\leq T}$, notation as in §5.1.1.

Our presentation is quite terse. We refer to [**35**, page 46] for a more complete discussion. Define the *height function* $\mathrm{Ht} : \mathbb{G}(\mathbb{A}) \to \mathbf{R}_{>0}$ via

$$\mathrm{Ht}(bk)^2 = |\alpha(b)|$$

for $b \in \mathbf{B}(\mathbb{A}), k \in K_{\max}$, where $\alpha : \mathbf{B} \to \mathbb{G}_m$ is the positive root, and $|\cdot|$ is the adelic absolute value. We denote the set $\mathrm{Ht}^{-1}(1)$ by $\mathbb{G}(\mathbb{A})^{(1)}$. *Warning* — this is nonstandard usage; also, the square on the left-hand side arises because of the difference between the "usual" absolute value on $\mathbf{C}$ and the "normalized" absolute value, cf. §3.2.1.

Recall from § 3.3.3 the definition of the arithmetic manifold

$$Y(K) := \mathbb{G}(F)\backslash\mathbf{H}^3 \times \mathbb{G}(\mathbb{A}_f)/K. \tag{5.4.1.1}$$

We define the height $\mathrm{Ht}(x)$ of $x \in Y(K)$ to be the maximal height of any lift $g_x \in \mathbb{G}(\mathbb{A})$ (where we think of the right-hand side of (5.4.1.1) as the quotient $\mathbb{G}(F)\backslash\mathbb{G}(\mathbb{A})/K_\infty K$). Reduction theory implies this is indeed a height function in the sense of § 5.1.1; more precisely, there exists a unique height function h_0 in the sense of § 5.1.1 such that $\mathrm{Ht}(x) = h_0(x)$ so long as either side is sufficiently large. In the notation of § 5.1.1 with $M = Y(K)$, the manifold M_B may be described as

$$Y_B(K) := \mathbf{B}(F)\backslash\mathbf{H}^3 \times \mathbb{G}(\mathbb{A}_f)/K$$

The function Ht descends (via the identification $\mathbb{G}(\mathbf{C})/K_\infty \cong \mathbf{H}^3$) to a proper function $\mathrm{Ht} : Y_B(K) \to \mathbf{R}_{>0}$. We write $Y_B(K)_{\geq T}$ for the preimage of $(T, \infty]$ under Ht. Then, for sufficiently large T, the natural projection

$$Y_B(K)_{\geq T} \to Y(K) \tag{5.4.1.2}$$

is a homeomorphism onto its image, and the complement of its image is compact. Fix such a T — call it T_0 — and define $\mathrm{Ht} : Y(K) \to [1, \infty]$ to be T_0 off the image of (5.4.1.2), and to coincide with Ht on the image.

5.4.2. Homology and Borel-Moore homology. Let H_c denote compactly supported cohomology. As on page 72, there is a long exact sequence as follows, for any (constant) coefficients:

$$H^{i-1}(\partial Y(K), -) \to H^i_c(Y(K), -) \to H^i(Y(K), -) \to H^i(\partial Y(K), -) \dots,$$

where we set $\partial Y = \partial Y_{\leq T}$ for any T; the homotopy class although not the isometry class is independent of T. Indeed, the inclusion $\partial Y_{\leq T} \hookrightarrow Y_B$ is a homotopy equivalence.

As before, we focus mainly on the case $K = K_\Sigma$, when we abbreviate the cohomology groups $H^i(Y(K_\Sigma), -)$ by $H^i(\Sigma, -)$, and similarly for H^i_c. Similarly, we denote $H^i(\partial Y(K_\Sigma), -)$ by $H^i(\partial\Sigma, -)$, and by $H^i_!(\Sigma, -)$ the image of the group $H^i_c(\Sigma, -)$ in $H^i(\Sigma, -)$.

For homology, we have groups H_i^{BM} and corresponding exact sequences and isomorphisms to the general case:

(5.4.2.1)

$$H_2(\partial Y(K), -) \to H_1(\partial Y(K), -) \to H_1(Y(K), -) \to H_{1,\mathrm{BM}}(\partial Y(K), -) \dots,$$

In particular, the cuspidal homology $H_{1,!}(\Sigma, \mathbf{Z})$ is defined to be the quotient of $H_1(\Sigma, \mathbf{Z})$ by the image of $H_1(\partial\Sigma, \mathbf{Z})$, or, equivalently, the image of $H_1(\Sigma, \mathbf{Z})$ in $H_1^{\mathrm{BM}}(\Sigma, \mathbf{Z})$.

5.4.3. The homology of the cusps. An often useful [11] fact is that the H_1 of cusps vanish for orbifold reasons:

LEMMA 5.4.4. *$H_1(\partial Y(K_\Sigma), \mathbf{Z}_p) = H^1(\partial Y(K_\Sigma), \mathbf{Z}_p) = 0$ for $p \neq 2$.*

This is certainly not true for general level structures, i.e., $H^1(\partial Y(K), \mathbf{Z}_p)$ can certainly be nonzero. This is simply a special, and convenient, property of the K_Σ-level structures. Geometrically, the cusps of $Y(K_\Sigma)$ are all quotients of a torus modulo its hyperelliptic involution (which is a sphere with four cone points of order 2), and in particular have trivial H_1 in characteristic larger than 2.

PROOF. Clearly,

$$Y_B(K) = \coprod_{z \in \mathbf{B}(F)\backslash\mathbb{G}(\mathbb{A}_f)/K} \Gamma_z \backslash \mathbf{H}^3, \tag{5.4.4.1}$$

where the union is taken over $\mathbf{B}(F)$-orbits on $\mathbb{G}(\mathbb{A}_f)/K$, and for $z \in \mathbb{G}(\mathbb{A}_f)/K$ representing such an orbit, Γ_z is the stabilizer of z in $\mathbf{B}(F)$.

Write $\alpha : \mathbf{B} \to \mathbb{G}_m$ for the natural map $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto ac^{-1}$. We claim that Γ_z always contains an element (call it b_0) such that $\alpha(b_0) = -1$.

In fact, choose any element $b \in \mathbf{B}(F)$ with $\alpha(b) = -1$. Then the set of $x \in F_v$ such that $bn(x)$ lies in gK_vg^{-1} is nonempty open for every v. Indeed, by a local computation, the image of $\mathbf{B}(F_v) \cap gKg^{-1}$ under α always contains, for any $g \in \mathbb{G}(\mathbb{A}_f)$, the units $\mathscr{O}_v^\times$. By strong approximation there exists $x_0 \in F$ such that $bn(x_0) \in gKg^{-1}$. Then $b_0 = bn(x_0)$ will do.

This is enough for our conclusion: The element b_0, acting by conjugation on $\Gamma_z^0 := \Gamma_z \cap \ker(\alpha) \cong \mathbf{Z}^2$, necessarily acts by negation; in particular, its co-invariants on $H_1(\Gamma_z^0, \mathbf{Z}_p)$ are trivial. □

Even in the general case, i.e. under *no* assumptions on the nature of the level structure K, the Hecke action on the boundary is simple to compute:

LEMMA 5.4.5. *For any Hecke eigenclass $h \in H_1(\partial K, \mathbf{C})$ there is a Grossen-characters $\psi : \mathbb{A}_F^\times/F^\times \to \mathbf{C}^\times$, such that:*

- *The restriction of ψ to $F_\infty^\times = \mathbf{C}^\times$ is given by $z \mapsto \sqrt{z/\bar{z}}$;*
- *ψ is unramified at any prime where K is conjugate to $\mathbb{G}(\mathscr{O}_v)$ or $K_{0,v}$;*
- *The eigenvalue $\lambda_\mathfrak{q}(h)$ of $T_\mathfrak{q}$ on h is*

$$\mathrm{N}(\mathfrak{q})^{1/2} \left(\psi(\mathfrak{q}) + \overline{\psi(\mathfrak{q})}\right). \tag{5.4.5.1}$$

[11]We do not make any *essential* use of this. For example, we do not assume it is so for our main analytical computations, and in the general case § 5.7 gives a good handle on $H_1(\partial Y(K))$. However, we often use it for essentially "cosmetic" reasons.

We omit the proof, which can be given by direct computation or via the theory of Eisenstein series.

Note the following consequence: Let S be the set of places at which the level structure K is not maximal. For any classes h_1, h_2 as in the Lemma, with corresponding characters $\psi_1 \neq \psi_2$, there exists $\mathfrak{q}$ such that

$$\lambda_{\mathfrak{q}}(h) \not\equiv \lambda_{\mathfrak{q}}(h_2) \bmod \mathfrak{l}.$$

whenever $\mathfrak{l}$ is any prime of $\overline{\mathbf{Q}}$ above a prime ℓ not dividing $h_F \cdot \prod_{q \in S} q(q-1)$.

5.4.6. Relationship between $Y_B(K)$ and $Y_B(K')$. The following discussion will be used in the proofs of Section § 5.8. It says, roughly, that the cusps of $Y(\Sigma \cup \mathfrak{q})$ are, metrically speaking, two copies of the cusps of $Y(\Sigma)$.

5.4.6.1. Suppose Ξ is a finite set of finite places disjoint from any level in K, and let $K' = K \cap K_\Xi$, so that K' is obtained from K by "adding level at Ξ."

Then there is an isometry

$$Y_B(K') \cong \{1,2\}^{\Xi} \times Y_B(K), \tag{5.4.6.1}$$

where $q = |\Xi|$, that is to say; $Y_B(K')$ is an *isometric* union of $2^{|\Xi|}$ copies of $Y_B(K)$. Moreover, these 2^q isometric copies are permuted simply transitively by the group of Atkin-Lehner involutions $W(\Xi)$ (see § 3.4.2).

In view of the definition,

$$Y_B(K) = \mathbf{B}(F)\backslash \mathbf{H}^3 \times \mathbb{G}(\mathbb{A}_f)/K$$
$$Y_B(K') = \mathbf{B}(F)\backslash \mathbf{H}^3 \times \mathbb{G}(\mathbb{A}_f)/K'$$

it suffices to construct a $\mathbf{B}(F)$-equivariant bijection

$$\{1,2\}^{\Xi} \times \mathbb{G}(\mathbb{A}_f)/K \xrightarrow{\sim} \mathbb{G}(\mathbb{A})/K'. \tag{5.4.6.2}$$

In fact, the map

$$\begin{aligned} \Lambda_v : \{1,2\} \times \mathbb{G}(F_v)/\mathbb{G}(\mathscr{O}_v) \quad &\longrightarrow \quad \mathbb{G}(F_v)/K_{0,v} \\ (i, bK_v) \quad &\longmapsto \quad \begin{cases} bK_{0,v}, & i=1 \\ bw_vK_{0,v}, & i=2 \end{cases} \end{aligned}$$

where $w_v = \begin{pmatrix} 0 & 1 \\ \varpi_v & 0 \end{pmatrix}$, defines a $\mathbf{B}(F_v)$-equivariant bijection,[12] and we take the product of Λ_v over all $v \in \Xi$ to get a map (5.4.6.2) that is even $\mathbf{B}(\mathbb{A}_f)$-equivariant.

REMARK 5.4.7. *Note that this bijection* does not *preserve the height function. In fact, since* $\mathrm{Ht}(bw_v) = \mathrm{Ht}(b)\sqrt{q_v}$*, we see that the height function on* $Y_B(K')$ *pulls back to the function on* $\{1,2\}^{\Xi} \times Y_B(K)$ *given by the product of the height on* $Y_B(K)$ *and the functions* $1 \mapsto 1, 2 \mapsto \sqrt{q_v}$ *for each* $v \in \Xi$.

[12]Geometrically speaking, the quotient $\mathbb{G}(F_v)/K_{\Xi,v}$ is identified with arcs on the Bruhat–Tits building; the Borel subgroup $\mathbf{B}(F_v)$ is the stabilizer of a point on the boundary $\partial\mathcal{B}$; the two orbits correspond to arcs that point "towards" or "away from" this marked point; and the Atkin-Lehner involution reverses arcs. Explicitly, in the action of $\mathbf{B}(F_v)$ on $G/K_{\Xi,v}$, the stabilizer of both $K_{\Xi,v}$ and $\begin{pmatrix} 0 & 1 \\ \varpi_v & 0 \end{pmatrix} K_{\Xi,v}$ is the same, namely, elements of $\mathbf{B}(F_v)$ that belong to $\mathrm{PGL}_2(\mathscr{O}_v)$.

REMARK 5.4.8. *Choose an auxiliary place v not contained in Ξ; and suppose $K_v^* \subset \mathbb{G}(F_v)$ is such that K_v^* is normal in K_v. Then K_v/K_v^* acts on both sides of* (5.4.6.1)*, where the action on $\{1,2\}^\Xi$ is trivial, and the identification is* equivariant *for that action.*

5.4.8.1. The topology of the projection $\partial Y(\Sigma \cup \mathfrak{q}) \to \partial Y(\Sigma)$ can be deduced similarly:

The preimage of each component P of $\partial Y(\Sigma)$ consists of two components P_1, P_2 of $\partial Y(\Sigma \cup \mathfrak{q})$, interchanged by the Atkin–Lehner involution. The induced map $H_2(P_i) \to H_2(P)$ are given by multiplication by $\mathrm{N}(\mathfrak{q})$ resp. trivial.

In particular, if $\mathcal{Q}$ is any set containing all but finitely many prime ideals, the cokernel of

$$\bigoplus_{\mathfrak{q}\in\mathcal{Q}} H_2(\Sigma \cup \mathfrak{q}, \mathbf{Z}_p)^{w_\mathfrak{q}-} \longrightarrow H_2(\Sigma, \mathbf{Z}_p),$$

where the maps are the difference of the two degeneracy maps, equals *zero* unless p divides w_F. Indeed $w_F = \gcd_{\mathfrak{q}\in\mathcal{Q}}(\mathrm{N}(\mathfrak{q}) - 1)$.

5.4.9. Congruence and essential homology, split case. The definitions of congruence homology of § 3.7.0.2 apply equally well to the compact and noncompact cases.

5.5. Some results from Chapter 4 in the split case

In this section we discuss the extension of various results from Chapter 4 to the split case. *The assumption on the congruence subgroup property in Lemma 4.1.3 is known unconditionally here for $p > 2$, making all the corresponding results unconditional.*

- Ihara's lemma (Lemma 4.1.3) applies: the original proof made no restriction on $\mathbb{G}$ being nonsplit.
- Theorem 4.2.3 still holds; the proof is given in § 5.5.1 below.
- The level-raising theorem (Theorem 4.3.1) applies, with the same proof, since it relies only on Lemma 4.1.3 and Theorem 4.2.3.

After we give the proof of Theorem 4.2.3 in the split case, we give results about comparison of regulators, of exactly the same nature as Lemma 4.2.1 in the nonsplit case.

5.5.1. Proof of Theorem 4.2.3 in the split case. Note that all the natural maps $H_c^1 \to H^1$ have become isomorphisms, since we have localized at a non-Eisenstein maximal ideal. Indeed, as long as the ideal is not cyclotomic-Eisenstein this is true: such a localization kills $H^0(\partial Y(K))$, and in this setting $H^1(\partial Y(K))$ vanishes by § 5.5.

Now the proof is exactly as the proof of Theorem 4.2.3, considering now the diagram:

$$\begin{array}{ccccccc}
0 & & \ker(\Phi^\vee_{\mathfrak{m}}) & & \ker(\Psi_{\mathfrak{m}}) & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^1_c(\Sigma, \mathbf{Q}_p)_{\mathfrak{m}} & \longrightarrow & H^1_c(\Sigma, \mathbf{Q}_p/\mathbf{Z}_p)_{\mathfrak{m}} & \xrightarrow[\delta_\Sigma]{} & H_1(\Sigma, \mathbf{Z}_p)_{\mathfrak{m}} & \longrightarrow & H_1(\Sigma, \mathbf{Q}_p)_{\mathfrak{m}} \\
\downarrow{\scriptstyle \Phi^\vee_{\mathbf{Q},\mathfrak{m}}} & & \downarrow{\scriptstyle \Phi^\vee_{\mathfrak{m}}} & & \downarrow{\scriptstyle \Psi_{\mathfrak{m}}} & & \downarrow \\
H^1_c(\Sigma/\mathfrak{q}, \mathbf{Q}_p)^2_{\mathfrak{m}} & \longrightarrow & H^1_c(\Sigma/\mathfrak{q}, \mathbf{Q}_p/\mathbf{Z}_p)^2_{\mathfrak{m}} & \xrightarrow[\delta_{\Sigma/\mathfrak{q}}]{} & H_1(\Sigma/\mathfrak{q}, \mathbf{Z}_p)^2_{\mathfrak{m}} & \longrightarrow & H_1(\Sigma/\mathfrak{q}, \mathbf{Q}_p)^2_{\mathfrak{m}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & \operatorname{coker}(\Phi^\vee_{\mathfrak{m}}) & & 0 & & 0
\end{array}$$

5.5.2. Comparison of regulators.

Lemma 5.5.3. *Suppose Σ is a level for which $H_1(\Sigma, \mathbf{C})^{\mathfrak{q}-\mathrm{new}} = 0$, i.e. the level raising map $\Psi^\vee_{\mathbf{C}} : H_1(\Sigma/\mathfrak{q}, \mathbf{C})^2 \to H_1(\Sigma, \mathbf{C})$ is an isomorphism. Then the cokernel of*

$$\Psi^\vee : H_{2,!}(\Sigma/\mathfrak{q}, \mathbf{Z})^2 \to H_{2,!}(\Sigma, \mathbf{Z})$$

is of order $h_{\mathrm{lif}}(\Sigma/\mathfrak{q}; \mathfrak{q})$, up to orbifold primes. The same conclusion holds with $H_{2!}$ replaced by Borel–Moore homology.

Here h_{lif} is as in § 3.7.1.2.

Proof. As in the proof of Lemma 4.2.1 the map $\Psi^\vee_{\mathrm{tf}} : H_1(\Sigma)_{\mathrm{tf}} \to H_1(\Sigma/\mathfrak{q})^2_{\mathrm{tf}}$ has cokernel of size $H_1(\Sigma/\mathfrak{q}; \mathfrak{q})$, up to orbifold primes. By dualizing (see (5.1.1.4)) $H_2^{\mathrm{BM}}(\Sigma/\mathfrak{q}, \mathbf{Z})^2 \xrightarrow{\Psi^\vee} H_2^{\mathrm{BM}}(\Sigma, \mathbf{Z})$ also has cokernel of order $h_{\mathrm{lif}}(\Sigma/\mathfrak{q}; \mathfrak{q})$, up to orbifold primes. The map $H_{2,!} \to H_{2,\mathrm{BM}}$ is an isomorphism with $\mathbf{Z}[\frac{1}{2}]$ coefficients, because of § 5.4.3. □

Remark 5.5.4. There is a more robust way to pass from $H_{2,\mathrm{BM}}$ to $H_{2,!}$, even if $H_1(\partial Y(K)$ did not vanish for trivial reasons: The Hecke action on the cokernel of $\Psi^\vee_{\mathrm{BM}}$ is congruence-Eisenstein, by our argument above; on the other hand, the Hecke action on $H_1(\partial Y(K))$ has been computed in (5.4.5.1), and it is easy to show these are incompatible for most primes p.

Theorem 5.5.5 (Comparison of regulators). *Suppose that $\mathfrak{q}$ is a prime in Σ and $H_1^{\mathfrak{q}-\mathrm{new}}(\Sigma, \mathbf{C}) = \{0\}$. Let b_0 be the number of connected components of the manifold $Y(\Sigma)$. Let h_{lif} be as in § 3.7.1.2 and put*

$$D = \det\left(T_{\mathfrak{q}}^2 - (1 + N(\mathfrak{q}))^2 \middle| H_1(\Sigma/\mathfrak{q}, \mathbf{C})\right).$$

Let D_{cusp} be the same determinant restricted to the cuspidal cohomology. Then up to orbifold primes:

$$\frac{\mathrm{reg}(H_1(Y(\Sigma/\mathfrak{q}) \times \{1,2\})}{\mathrm{reg}(H_1 Y(\Sigma))} = \frac{\sqrt{D}}{|h_{\mathrm{lif}}(\Sigma/\mathfrak{q}; \mathfrak{q})|}, \tag{5.5.5.1}$$

whereas

$$\frac{\mathrm{reg}(H_{2!}(Y(\Sigma/\mathfrak{q}) \times \{1,2\})}{\mathrm{reg}(H_{2!}Y(\Sigma))} = \frac{|h_{\mathrm{lif}}(\Sigma/\mathfrak{q};\mathfrak{q})|}{\sqrt{D_{\mathrm{cusp}}}} \tag{5.5.5.2}$$

Moreover, in the same notation,

$$\frac{\mathrm{reg}(H_0(Y(\Sigma/\mathfrak{q}) \times \{1,2\})}{\mathrm{reg}(H_0 Y(\Sigma))} = \frac{\mathrm{vol}(Y(\Sigma))^{1/2}}{\mathrm{vol}(Y(\Sigma/\mathfrak{q}))},$$

and the $H_{3,\mathrm{BM}}$ *regulators change by the inverse of these.*

Note in this theorem that, e.g. $\mathrm{reg}(H_1(M \times \{1,2\}))$ is simply $\mathrm{reg}(H_1(M))^2$; we phrase it in the above way for compatibility with some later statements.

Proof. We first give the proof for H_1, which is very close to that already given in Theorem 4.2.5. For simplicity we write the proof as if there are no orbifold primes; the general case, of course, only introduces "fudge factors" at these primes.

By Ihara's lemma, the cokernel of $\Psi : H_1(\Sigma, \mathbf{Z}) \to H_1(\Sigma/\mathfrak{q}, \mathbf{Z})^2$ is congruence homology (at least at primes away from 2). Note that, *just as in the compact case*, the map $\Psi \otimes \mathbf{Q} : H_1(\Sigma, \mathbf{C}) \to H_1(\Sigma/\mathfrak{q}, \mathbf{C})^2$ is still an isomorphism; this is *not so for* H_2, which is one reason why we work with $H_{2!}$ rather than H_2. In particular, the map $\Psi_{\mathrm{tf}} : H_1(\Sigma, \mathbf{Z})_{\mathrm{tf}} \to H_1(\Sigma/\mathfrak{q}, \mathbf{Z})^2_{\mathrm{tf}}$ is still injective, and, as before we have the diagram:

$$\begin{array}{ccccccc}
 & & (\ker\Psi)_{\mathrm{tors}} & \longrightarrow & \ker\Psi & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H_1(\Sigma,\mathbf{Z})_{\mathrm{tors}} & \underset{\Psi}{\longrightarrow} & H_1(\Sigma,\mathbf{Z}) & \twoheadrightarrow & H_1(\Sigma,\mathbf{Z})_{\mathrm{tf}} \\
 & & \downarrow{\scriptstyle \Psi_{\mathrm{tors}}} & & \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \Psi_{\mathrm{tf}}} \\
0 & \longrightarrow & H_1(\Sigma/\mathfrak{q},\mathbf{Z})^2_{\mathrm{tors}} & \underset{\Psi_{\mathrm{tf}}}{\longrightarrow} & H_1(\Sigma/\mathfrak{q},\mathbf{Z})^2 & \twoheadrightarrow & H_1(\Sigma/\mathfrak{q},\mathbf{Z})^2_{\mathrm{tf}} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathrm{coker}(\Psi_{\mathrm{tors}}) & \longrightarrow & \mathrm{coker}(\Psi) & \longrightarrow & \mathrm{coker}(\Psi_{\mathrm{tf}})
\end{array} \tag{5.5.5.3}$$

As before, the diagram shows that the order of $\mathrm{coker}(\Psi_{\mathrm{tf}})$ equals the order of the cokernel of $\mathrm{coker}(\Psi_{\mathrm{tors}}) \to \mathrm{coker}(\Psi)$, equivalently, the order of the cokernel of $H_1(\Sigma/\mathfrak{q}, \mathbf{Z})^2_{\mathrm{tors}} \to H_1(\Sigma/\mathfrak{q}, \mathbf{Z})_{\mathrm{cong}}$, that is to say $h_{\mathrm{lif}}(\Sigma/\mathfrak{q};\mathfrak{q})$.

In the sequence $H_1(\Sigma, \mathbf{Z})_{\mathrm{tf}} \stackrel{\Psi_{\mathrm{tf}}}{\to} H_1(\Sigma/\mathfrak{q}, \mathbf{Z})^2_{\mathrm{tf}} \stackrel{\Psi^\vee_{\mathrm{tf}}}{\to} H_1(\Sigma, \mathbf{Z})_{\mathrm{tf}}$, the composite map $\Psi \circ \Psi^\vee$ is given explicitly by Lemma 3.4.8; from that we deduce

$$|\mathrm{coker}(\Psi_{\mathrm{tf}})| \cdot |\mathrm{coker}(\Psi^\vee_{\mathrm{tf}})| = |\mathrm{coker}(\Psi_{\mathrm{tf}} \circ \Psi^\vee_{\mathrm{tf}})| = D.$$

Therefore the size of the cokernel of $\Psi^\vee_{\mathrm{tf}}$ equals $D/h_{\mathrm{lif}}(\Sigma/\mathfrak{q};\mathfrak{q})$.

Choose an orthonormal basis $\omega_1, \ldots, \omega_{2k}$ for $H^1(\Sigma, \mathbf{R})$, and also choose a basis $\gamma_1, \ldots, \gamma_{2k}$ for $H_1(\Sigma/\mathfrak{q}, \mathbf{Z})^2_{\mathrm{tf}}$. Then

$$\det \langle \gamma_i, \Phi^\vee_{\mathbf{R}} \omega_j \rangle = \det \langle \Psi^\vee(\gamma_i), \omega_j \rangle = \frac{D}{h_{\mathrm{lif}}(\Sigma/\mathfrak{q})} \cdot \mathrm{reg}(H_1(\Sigma))$$

Now $|\mathrm{coker}(\Psi^\vee)| = \frac{D}{|H_{1,\mathrm{lif}}(\Sigma/\mathfrak{q})|}$. Also $\|\Phi^\vee_{\mathbf{R}}\omega_1 \wedge \Phi^\vee_{\mathbf{R}}\omega_2 \wedge \ldots \Phi^\vee_{\mathbf{R}}\omega_{2k}\| = \sqrt{D}$ and consequently,

$$\mathrm{reg}(H_1(\Sigma/\mathfrak{q}))^2 = \frac{\det \langle \gamma_i, \Phi^\vee_{\mathbf{R}}\omega_j\rangle}{\|\Phi^\vee_{\mathbf{R}}\omega_1 \wedge \Phi^\vee_{\mathbf{R}}\omega_2 \wedge \ldots \Phi^\vee_{\mathbf{R}}\omega_{2k}\|} = \mathrm{reg}(H_1(\Sigma)) \cdot \frac{\sqrt{D}}{h_{\mathrm{lif}}(\Sigma/\mathfrak{q})},$$

which implies the desired conclusion

We now turn to H_2, or rather $H_{2!}$; the proof of (5.5.5.1) is very similar to that just given. Fix a basis γ_i for $H_{2,!}(\Sigma, \mathbf{Z}/\mathfrak{q})^{\oplus 2}$ and an orthonormal basis $\omega_1, \ldots, \omega_{2k}$ for $\Omega^2_{\mathrm{cusp}}(\Sigma)$. Then:

$$\det \langle \gamma_i, \Phi^\vee_{\mathbf{R}}\omega_j\rangle = \det \langle \Psi^\vee(\gamma_i), \omega_j\rangle = |\mathrm{coker}(\Psi^\vee_!)| \cdot \mathrm{reg}(H_{2!}(\Sigma))$$

The $\Psi^\vee_{\mathbf{R}}\omega_j$ do not form an orthonormal basis; indeed, the map $\Psi^*_{\mathbf{R}}$ multiplies volume by $\sqrt{D_{\mathrm{cusp}}}$, i.e. $\|\Phi^\vee_{\mathbf{R}}\omega_1 \wedge \Phi^\vee_{\mathbf{R}}\omega_2 \wedge \ldots \Phi^\vee_{\mathbf{R}}\omega_{2k}\| = \sqrt{D}$. Consequently,

$$\mathrm{reg}(H_{2,!}(\Sigma/\mathfrak{q})^2 = \frac{\det \langle \gamma_i, \Phi^\vee_{\mathbf{R}}\omega_j\rangle}{\|\Phi^\vee_{\mathbf{R}}\omega_1 \wedge \Phi^\vee_{\mathbf{R}}\omega_2 \wedge \ldots \Phi^\vee_{\mathbf{R}}\omega_{2k}\|} = \mathrm{reg}(H_{2!}(\Sigma))\frac{h_{\mathrm{lif}}(\Sigma/\mathfrak{q})}{\sqrt{D_{\mathrm{cusp}}}},$$

where we have used the fact (Lemma 5.5.3) that $\#\mathrm{coker}(\Psi^\vee_!) = h_{\mathrm{lif}}(\Sigma/\mathfrak{q})$. This implies (5.5.5.1).

Finally the H_0-statement is clear. □

5.6. Eisenstein series for arithmetic manifolds: explicit scattering matrices

We briefly recall those aspects of the theory of Eisenstein series. We need this to explicitly evaluate the "scattering matrices"; *if the reader is willing to accept this on faith, this section can be skipped without detriment.* What is needed from this section, for our later Jacquet–Langlands application, is *only* a comparison between the scattering matrices at different levels showing that, roughly speaking, "everything cancels out;" thus the details of the formulas do not matter, only that they match up at different levels.

In principle all of the computations of this section can be carried out in classical language; but the reason we have adelic language is not mathematical but aesthetic: it makes the various notational acrobatics involved in comparing the manifolds at two different levels $Y(K), Y(K')$ much easier. Recall that these manifolds are both disconnected and have multiple cusps in general!

The reader who prefers classical language may wish to study the paper of Huxley [**41**] which carries out a similar computation for arithmetic quotients of $\mathrm{SL}_2(\mathbf{Z})$.

Here is the main result of the section:

THEOREM 5.6.1. *Equip $Y(\Sigma)$ and $Y(\Sigma\cup\mathfrak{q})$ with the height function as in § 5.4.1; let $\Psi_\Sigma, \Psi_{\Sigma\cup\mathfrak{q}}$ be the scattering matrices on functions and $\Phi_\Sigma, \Phi_{\Sigma\cup\mathfrak{q}}$ the scattering matrices on* 1*-forms. (See § 5.2.2 for definitions.)*

Also equip $Y(\Sigma)\times\{1,2\}$ *with the height function induced by the identification* $Y_B(\Sigma\cup\mathfrak{q})\cong Y_B(\Sigma)\times\{1,2\}$ *of* (5.4.6.1)*, and let* $\Psi_{\Sigma\times\{1,2\}}$ *and* $\Phi_{\Sigma\times\{1,2\}}$ *be the corresponding scattering matrices.*

For $s\in\mathbf{C}, z\in\mathbf{C}^\times$ *write* $M(s)=\begin{pmatrix}1&0\\0&q^s\end{pmatrix}$ *and*

$$N(z,s)=\frac{1}{(q-z^2q^{-s})}\begin{pmatrix}z^2q^{-s}(q-1)&1-z^2q^{-s}\\q(1-z^2q^{-s})&q-1\end{pmatrix}.$$

Here $q=N(\mathfrak{q})$*. Note that* $N(z,s)N(1/z,-s)$ *is the identity and* $N(\pm1,1)$ *is a projection with eigenvalue* 1*.*

Then there exists a finite set X *of Grossencharacters*[13]*, unramified at* $\mathfrak{q}$*, and matrices* $A_\chi(s)\quad(\chi\in X)$ *such that:*

$$\Psi(s)\sim\bigoplus_{\chi\in X}A_\chi(s),\tag{5.6.1.1}$$

$$\Psi_{\Sigma\times\{1,2\}}(s)\sim\bigoplus_{\chi\in X}A_\chi(s)\otimes M(s),\tag{5.6.1.2}$$

$$\Psi_{\Sigma\cup\mathfrak{q}}(s)\sim\bigoplus_{\chi\in X}A_\chi(s)\otimes N(\chi(\mathfrak{q}),s),\tag{5.6.1.3}$$

where we write $\sim$ *for equivalence of linear transformations.*[14]

Exactly the same conclusion holds for Φ*, with a different set of Grossencharacters* X*.*

In particular the following hold:

(1) *If* A_χ *has size* $a_\chi\in\mathbb{N}$*, then*

$$\frac{\det\Psi_{\Sigma\cup\{1,2\}}(s)}{(\det\Psi_\Sigma)^2}=\prod_{\chi\in X}(q^s)^{a_\chi}\tag{5.6.1.4}$$

$$\frac{\det\Psi_{\Sigma\cup\mathfrak{q}}(s)}{(\det\Psi_\Sigma(s))^2}=\prod_{\chi\in X}\left((\chi(\mathfrak{q})^2q^{-s}\cdot\frac{q-\chi(\mathfrak{q})^{-2}q^s}{q-\chi(\mathfrak{q})^2q^{-s}}\right)^{a_\chi},\tag{5.6.1.5}$$

with exactly the same conclusion for Φ*, allowing a different set of Grossencharacters* X*.*

(2) *At* $s=0$*,*

$$\mathrm{tr}\Psi_{\Sigma\cup\{1,2\}}(0)=\mathrm{tr}\Psi_{\Sigma\cup\mathfrak{q}}(0)=2\mathrm{tr}\Psi(0).\tag{5.6.1.6}$$

(3) $R_\Sigma, R_{\Sigma\times\{1,2\}}, R_{\Sigma\cup\mathfrak{q}}$ *are the respective residues when* $s=1$*, then* $R_{\Sigma\cup\mathfrak{q}}\sim R_\Sigma\oplus0$ *whereas* $R_{\Sigma\times\{1,2\}}\sim R_\Sigma\oplus qR_\Sigma\oplus0$*, where* 0 *denotes a zero matrix of some size.*

(4) *Let* $\mathfrak{p}$ *be an auxiliary prime, not belonging to* $\Sigma\cup\{\mathfrak{q}\}$*. Then the set of Grossencharacters* X *that arise in the above analysis for* Ψ *is the same at levels* Σ *and* $\Sigma\cup\{\mathfrak{p}\}$*. The same is true for* Φ*. Moreover, the size of the corresponding matrices satisfies* $a_\chi^{\Sigma\cup\{\mathfrak{p}\}}=2a_\chi^\Sigma$*.*

Note that enunciations (1)–(3) are consequences of (5.6.1.1), and (4) will follow in the course of the proof of (5.6.1.1), so we will only prove that.

[13] That is to say: characters of $\mathbb{A}_F^\times/F^\times$.

[14] Thus, if $A:V\to V$ and $B:W\to W$, we write $A\sim B$ if there is an isomorphism $V\to W$ conjugating one to the other.

5.6.1.1. *Measure normalizations.* We have normalized previously measures on $\mathbb{A}$ and $\mathbb{A}^\times$ (see § 3.5.2). We equip $\mathbf{N}(\mathbb{A})$ with the corresponding measure arising from the map $y \mapsto \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$, and we denote by $d_l b$ the left Haar measure on $\mathbf{B}(\mathbb{A})$ given in coordinates $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ as $|x|^{-1} dx \cdot dy$.

We equip $\mathbb{G}(\mathbb{A})$ with the measure defined to be the push-forward of $\frac{1}{2} d_l b \cdot dk$ under $(b,k) \in \mathbf{B}(\mathbb{A}) \times \mathrm{PU}_2 \times \mathrm{PGL}_2(\widehat{\mathscr{O}}) \to bk \in \mathbb{G}(\mathbb{A})$, where dk is the Haar probability measure on $\mathrm{PU}_2 \times \mathrm{PGL}_2(\widehat{\mathscr{O}})$.

Push forward this measure to $\mathbb{G}(\mathbb{A})/\mathrm{PU}_2 \times \mathrm{PGL}_2(\widehat{\mathscr{O}})$. This is a union of copies of $\mathbf{H}^3$: for every $g_f \in \mathbb{G}(\mathbb{A}_f)/\mathrm{PGL}_2(\widehat{\mathscr{O}})$ we may identify

$$\mathbb{G}(\mathbf{C}) \times g_f \mathrm{PGL}_2(\widehat{\mathscr{O}})/\mathrm{PU}_2 \times \mathrm{PGL}_2(\widehat{\mathscr{O}})$$

with $\mathbb{G}(\mathbf{C})/\mathrm{PU}_2 \simeq \mathbf{H}^3$. On each such copy of $\mathbf{H}^3$ the induced measure is the same as that induced by the Riemannian structure.

We equip $\mathbb{G}(\mathbb{A})^{(1)}$ with the measure obtained as the "fibral" measure of $\mathbb{G}(\mathbb{A}) \stackrel{\mathrm{Ht}}{\to} \mathbf{R}_{>0}$, where the measure on $\mathbf{R}_{>0}$ is dx/x.

Push this measure forward to $\mathbb{G}(\mathbb{A})^{(1)}/\mathrm{PU}_2 \times \mathrm{PGL}_2(\widehat{\mathscr{O}})$. This intersects each component $\mathbf{H}^3$ from above in a level set $y = \text{const}$. The induced measure on this is $\frac{dx_1 dx_2}{y^2}$.

5.6.2. The induced space. Let ξ be any character of $\mathbf{C}^\times$. In what follows, the character $z \mapsto \frac{z}{|z|^{1/2}} = \sqrt{z/\bar{z}}$ and its inverse will play an important role, and consequently we denote it by η:

$$\eta : z \in \mathbf{C}^\times \mapsto \sqrt{z/\bar{z}}.$$

Define $V_\xi(s, K)$ to be:

$$\left\{ f \in C^\infty(\mathbb{G}(\mathbb{A})/K) : f\left(\begin{pmatrix} x & \star \\ 0 & 1 \end{pmatrix} g \right) = |x|^{1+s} \xi(x) f(g), \quad x \in \mathbf{C}^\times. \right\} \tag{5.6.2.1}$$

Note that $|x|$ denotes the usual absolute value on $\mathbf{C}$, so that, for example, $|2| = 2$; this is the *square root* of the "canonically normalized" absolute value that is defined for any local field, i.e., the effect of multiplication on Haar measure.

When the compact K is understood, we write simply $V_\xi(s)$. This is a representation of $\mathbb{G}(\mathbf{R})$ of finite length (in particular, any K_∞-isotypical component has finite dimension).

Multiplication by $\mathrm{Ht}(g)^s$ gives an identification $V_\xi \xrightarrow{\sim} V_\xi(s)$. We shall often use this isomorphism without explicit comment that it is multiplication by $\mathrm{Ht}(g)^s$.

We equip $V_\xi(s)$ with the unitary structure

$$\|f\|^2 = \int_{B(F)\backslash \mathbb{G}(\mathbb{A})^{(1)}} |f(g)|^2 dg.$$

where we normalize measures in a moment; this unitary structure is $\mathbb{G}(\mathbb{A})$-invariant for s purely imaginary.

5.6.2.1. We relate the spaces just introduced to to the standard presentation of Eisenstein series (see, for example, [**8**]):

Note that $V_\xi(s,K)$ decomposes according to Grossencharacters of F: If we denote by $X(s)$ the set of characters $\mathbb{A}_F^\times/F^\times$ whose restriction to $F_\infty^\times = \mathbf{C}^\times$ is $\xi \cdot |\cdot|^s$ — we sometimes abbreviate this simply to X when s is understood — then

$$V_\xi(s,K) = \bigoplus_{\chi \in X(s)} \mathcal{I}(\chi)^K, \tag{5.6.2.2}$$

where, for χ a character of $\mathbb{A}_F^\times$, we put

$$\mathcal{I}(\chi) = \{f : \mathbb{G}(\mathbb{A}) \to \mathbf{C} \text{ smooth} : f(bg) = \chi(b)|b|_{\mathbb{A}}^{1/2} f(g) \text{ for } b \in \mathbf{B}(\mathbb{A})\},$$

Here we write $\chi(b)$ and $|b|_{\mathbb{A}}$ as shorthand for $\chi(\alpha(b))$ and $|\alpha(b)|_{\mathbb{A}}$, where α is the positive root, as in (3.1.1.1). The set $X(s)$ of characters is infinite, but only finitely many $\chi \in X(s)$ have $\mathcal{I}(\chi)^K \neq 0$.

We will write X as a shorthand for $X(0)$. Note that there is a bijection between $X(0) = X$ and $X(s)$ given by twisting by $|\cdot|_{\mathbb{A}}^{s/2}$.

We make a similar definition if χ is a character of a localization $F_v^\times$, replacing $\mathbb{A}$ by F_v on the right-hand side. In that case $\mathcal{I}(\chi)$ is a representation of $\mathrm{PGL}_2(F_v)$, and there is a natural map $\bigotimes_v \mathcal{I}(\chi_v) \xrightarrow{\sim} \mathcal{I}(\chi)$, where we use the restricted tensor product (see [**28**]).

The theory of Eisenstein series gives a canonical intertwiner Eis : $V_\xi(s) \to C^\infty(\mathbb{G}(F)\backslash\mathbb{G}(\mathbb{A})/K)$ that is meromorphic — that is to say, the composition

$$V_\xi(0,K) \xrightarrow{\sim} V_\xi(s,K) \xrightarrow{\mathrm{Eis}} C^\infty(\mathbb{G}(F)\backslash\mathbb{G}(\mathbb{A})/K)$$

is meromorphic, where the first identification is multiplication by Ht^s. If ξ is unitary, this map is holomorphic when $\Re(s) = 0$. This map is uniquely characterized by the property

$$\mathrm{Eis}(v)(e) = \sum_{\gamma \in \mathbf{B}(F)\backslash\mathbb{G}(F)} v(\gamma)$$

whenever $\Re(s) > 2$.

There is a *standard intertwining operator* $V_\xi(s) \to V_{\xi^{-1}}(-s)$ given by (the meromorphic extension of)

$$f \mapsto \left(g \mapsto \frac{1}{\mathrm{vol}(\mathbf{N}(F)\backslash\mathbf{N}(\mathbb{A}))}\int_{n \in \mathbf{N}(\mathbb{A})} f(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} ng)dn\right). \tag{5.6.2.3}$$

Then

$$\mathrm{Eis}(f)_N = f + M(f). \tag{5.6.2.4}$$

where in this context the constant term $\mathrm{Eis}(f)_N$ is defined, as usual for automorphic forms, via $\frac{1}{\mathrm{vol}(\mathbf{N}(F)\backslash\mathbf{N}(\mathbb{A}))}\int_{n \in \mathbf{N}(F)\backslash\mathbf{N}(\mathbb{A})} \mathrm{Eis}(f)(ng)dn$; this will be compatible with our other usages of the notation.

5.6.3. V_ξ and the spaces $C^\infty(s), \Omega^\pm(s)$. Recall that — for M a hyperbolic manifold — we have defined spaces of functions $C^\infty(s)$ and of one-forms $\Omega^\pm(s)$ in § 5.2.2.1. In the case of an arithmetic $M = Y(K)$, we shall now relate these to certain $V_\xi(s,K)$, defined as in §5.6.2, specifically for $\xi = 1, \eta^{-1}, \eta$, where $\eta = \sqrt{z/\bar{z}}$ as before.

5.6.3.1. ξ *trivial.* When ξ is the trivial character, elements of $V_\xi(s,K)^{K_\infty}$ descend to the quotient $Y_B(K)$; this gives an isomorphism

$$V_\xi(s,K)^{K_\infty} \xrightarrow{\sim} C^\infty(s).$$

In this identification, the unitary structure on $V(s)$ induces the unitary structure on $C^\infty(s)$ given by

$$[\mathrm{PGL}_2(\widehat{\mathscr{O}}):K]^{-1}\int_{\mathrm{Ht}=Y}|f|^2,$$

where the integral is taken with respect to the measure induced by the hyperbolic metric. The result is independent of Y when $s \in i\mathbf{R}$. In particular it agrees up to scalars with the unitary structure previously described on $C^\infty(s)$.

5.6.3.2. $\xi = \eta$. Recall the notation $\mathfrak{p}$ from § 3.5.

There is an isomorphism

$$(V_\eta(s,K)\otimes \mathfrak{p}^*)^{K_\infty} \xrightarrow{\sim} \Omega^+(s) \tag{5.6.3.1}$$

whose inverse is described thus:

Firstly an element of $\Omega^+(s)$ can be regarded as a 1-form ω on $\mathbf{H}^3 \times \mathbb{G}(\mathbb{A}_f)/K$, left invariant by $\mathbf{B}(F)$, and satisfying

$$\begin{pmatrix} x & \star \\ 0 & 1 \end{pmatrix}^* \omega = |x|^{1+s}\eta(x)\omega = |x|^s x\cdot\omega. \tag{5.6.3.2}$$

Indeed, (5.6.3.2) is just the same as asking that ω is a multiple of $y^s(dx_1 + idx_2)$ on each $\mathbf{H}^3$ component (embedded into $\mathbf{H}^3 \times \mathbb{G}(\mathbb{A}_f)/K$ via $z \mapsto (z, g_f K)$ for some $g_f \in \mathbb{G}(\mathbb{A}_f)$).

Now we can construct the inverse to (5.6.3.1): Given a 1-form ω on $\mathbf{H}^3 \times \mathbb{G}(\mathbb{A}_f)/K$, satisfying (5.6.3.2) and left invariant by $\mathbf{B}(F)$ (so in fact by $\mathbf{N}(\mathbb{A})\mathbf{B}(F)$), we associate to it the $\mathfrak{p}^*$-valued function on $\mathbb{G}(\mathbb{A})$ with the property that

$$A(g) = (g^*\omega)|_e$$

for $g \in \mathbb{G}(\mathbb{A})$. This therefore descends to an element $(V_\eta(s)\otimes\mathfrak{p}^*)^{KK_\infty}$.

The inner product on $\Omega^\pm(s)$ for s purely imaginary that corresponds to the inner product on $V_\eta(s)$ is given up to scalars by

$$\|\omega\|^2 := \int_{\mathrm{Ht}=T}\omega\wedge\overline{\omega}. \tag{5.6.3.3}$$

In fact, the preimage $\mathrm{Ht}^{-1}(T)$ is a 2-manifold, and the result is independent of T because $\omega\wedge\overline{\omega}$ is closed and $[\mathrm{Ht}^{-1}(T)] - [\mathrm{Ht}^{-1}(T')]$ a boundary. The space $(V^+(s)\otimes\mathfrak{p}^*)^{K_\infty}$ is equipped with a natural inner product, derived from that on $V_\eta(s)$ and the Riemannian metric on $\mathfrak{p}$. Up to scalars, this coincides with the norm defined in (5.6.3.3).[15]

---

[15] Indeed, suppose that $\omega \in \Omega^+(s)$ corresponds to $A \in (V_\eta(s)\otimes\mathfrak{p}^*)^{K_\infty}$; then

$$\|A\|^2 = \int_{g\in B(F)\backslash\mathbb{G}(\mathbb{A})^{(1)}} \|g^*\omega(1)\|^2 dg$$

But the signed measure $\|g^*\omega(1)\|^2 dg$, pushed down to $\partial Y_B(K) = B(F)\backslash\mathbb{G}(\mathbb{A})^{(1)}/K_\infty K$, coincides with the measure $\omega\wedge\overline{\omega}$. Indeed, $\|g^*\omega(1)\|^2 = \|\omega(g)\|^2$; on the other hand, we have noted earlier that dg pushes down to $\frac{dx_1 dx_2}{y^2}$ on each component of $\partial Y_B(K)$.

5.6.3.3. $\xi = \eta^{-1}$. Similarly, if we denote by $\Omega^-(s)$ the space of one forms on $Y_B(K)$ that are multiples of $y^{-s}(dx_1 - idx_2)$ on each component, then

$$(V_{\eta^{-1}}(-s, K) \otimes \mathfrak{p}^*)^{K_\infty} \xrightarrow{\sim} \Omega^-(s).$$

The corresponding inner product on $\Omega^-(s)$ is given (up to scalars) again by (5.6.3.3).

5.6.4. Local intertwining operators. For a nonarchimedean local field k, with ring of integers $\mathscr{O}$, uniformizer ϖ, residue field $\mathscr{O}/\pi$ of size p, and $z \in \mathbf{C}^\times$, let χ_z be the character $x \in k^\times \mapsto z^{\mathrm{val}(x)}$, and consider the local induced representation $\mathcal{I}(\chi_z)$:

$$I(z)(= I(\chi_z)) := \{f : \mathrm{PGL}_2(k) \to \mathbf{C} : f\left(\begin{pmatrix} x & * \\ 0 & 1 \end{pmatrix} g\right) = |x|^{1/2} z^{\mathrm{val}(x)} f(g).\}$$

Let $M : I(z) \to I(1/z)$ be defined by the local analogue of (5.6.2.3), taking the measure on $N(k)$ to be the one that assigns mass one to $N(\mathscr{O})$.

Let $f_0 \in I(z)$ be the $\mathrm{PGL}_2(\mathscr{O})$-fixed vector that takes value 1 on $\mathrm{PGL}_2(\mathscr{O})$. Set $K_0 \subset \mathrm{PGL}_2(\mathscr{O})$ to be the inverse image of the upper triangular matrices in $\mathrm{PGL}_2(\mathscr{O}/\varpi)$, and let $f_1, f_2 \in I(z)$ be so that $f_1|\mathrm{PGL}_2(\mathscr{O})$ is the characteristic function of K_0 and $f_1 + f_2 = f_0$. Then

$$M(z) f_0 = \frac{1 - z^2/p}{1 - z^2} f_0 \tag{5.6.4.1}$$

— the constant of proportionality here may be more familiar as $\frac{L(0,\chi_z)}{L(1,\chi_z)}$ — and the matrix of $M(z)$ in the basis $\{f_1, f_2\}$ is given by:

$$\begin{aligned} M(z) &= \frac{1}{p(1-z^2)} \begin{pmatrix} z^2(p-1) & 1-z^2 \\ p(1-z^2) & p-1 \end{pmatrix}. \\ &= \frac{1-z^2/p}{1-z^2} \frac{1}{p-z^2} \begin{pmatrix} z^2(p-1) & 1-z^2 \\ p(1-z^2) & p-1 \end{pmatrix}. \\ \det M(z) &= z^2 \left(\frac{1-z^2/p}{1-z^2}\right)^2 \cdot \left(\frac{p-1/z^2}{p-z^2}\right) \end{aligned}$$

Moreover the eigenvalues of $\frac{1-z^2}{1-z^2/p} M(z)$ are $\frac{pz^2-1}{p-z^2}$ and 1.

5.6.5. Proof of theorem 5.6.1. Let $\xi \in \{1, \eta, \eta^{-1}\}$. Let Ξ be a finite set of places, disjoint from any level in K. Let $K' = K \cap K_\Xi$. We first analyze the scattering matrix for $Y(K')$, then for $Y(K)$, and then by comparing them we obtain the statements of the Theorem (the Theorem follows, in fact, from the case $\Xi = \{\mathfrak{q}\}$ and $K = K_\Sigma$).

5.6.5.1. *Analysis of the scattering matrix for $Y(K')$.* The scattering matrix for $Y(K')$ can be computed by computing the composite:

$$\alpha(s) : V_\xi(K', 0) \xrightarrow{\mathrm{Ht}^s} V_\xi(K', s) \xrightarrow{M} V_{\xi^{-1}}(K', -s) \xrightarrow{\mathrm{Ht}^{-s}} V_{\xi^{-1}}(K', 0).$$

After splitting V_ξ into spaces $\mathcal{I}(\chi)$ according to the decomposition (5.6.2.2), and decomposing each $\mathcal{I}(\chi)$ into a tensor products of local representations $\mathcal{I}(\chi_v)$ (defined as in the prior section), computation of $\alpha(s)$ above reduces to computing

$$\mathcal{I}(\chi_v)^{K_{0,v}} \xrightarrow{\mathrm{Ht}^s} \mathcal{I}(\chi_v|\cdot|_v^{s/2})^{K_{0,v}} \xrightarrow{M} \mathcal{I}(\chi_v^{-1}|\cdot|_v^{-s/2})^{K_{0,v}} \xrightarrow{\mathrm{Ht}^{-s}} \mathcal{I}(\chi_v^{-1})^{K_{0,v}}$$

in particular for places $v \in \Xi$. Here, Ht_v is the local analogue of the function Ht, so that $\mathrm{Ht} = \prod_v \mathrm{Ht}_v$, and χ_v is the local constituent of a character $\chi \in X$.

This composite – with respect to the standard basis (f_1, f_2) for both the source and target prescribed in the prior subsection — is given by the matrix $M(z)$ of the prior section, where z is the value of χ_v on a uniformizer of F_v, and $p = q_v$, the size of the residue field of F_v.

This proves (after interpretation) the assertions of the Theorem concerning $\Sigma \cup \{\mathfrak{q}\}$.

5.6.5.2. *Analysis of the scattering matrix for* $Y(K) \times \{1,2\}^{\Xi}$. Now let Ht' be the pullback of Ht to $Y_B(K) \times \{1,2\}^{\Xi}$ under the isometry specified in § 5.4.6.1.

We analyze the scattering matrix for $Y(K) \times \{1,2\}^{\Xi}$ with height Ht' under (5.4.6.1). For this write $H = (\mathbf{C}^2)^{\otimes \Xi}$, and regard it as the space of functions on $\{1,2\}^{\Xi}$; in particular H has an algebra structure by pointwise multiplication.

We may identify $V_1(s,K) \otimes H$ with functions on $\{1,2\}^{\Xi} \times Y_B(K)$. In particular, Ht'^s may be considered as an element of $V_1(s,K) \otimes H$, and mutiplication by Ht'^s defines a map $V_\xi(K,0) \otimes H \xrightarrow{\mathrm{Ht}'^s} V_\xi(K,s) \otimes H$.

With these identifications, the scattering matrix for $Y(K) \times \{1,2\}^{\Xi}$ is given by the composite

$$\alpha(s) : V_\xi(K,0) \otimes H \xrightarrow{\mathrm{Ht}'^s} V_\xi(K,s) \otimes H \xrightarrow{M \otimes \mathrm{id}} V_{\xi^{-1}}(K,-s) \otimes H \xrightarrow{\mathrm{Ht}'^s} V_{\xi^{-1}}(0) \otimes H.$$

Now we have an identification, as in (5.6.2.2),

$$V_\xi(K,0) \otimes H = \bigotimes_{\chi \in X(s)} \mathcal{I}(\chi)^K \otimes H = \bigoplus_{\chi} \left(\bigotimes_v \mathcal{I}(\chi_v)^{K_v} \otimes H_v \right),$$

where $H_v = \mathbf{C}^{\{1,2\}}$ for $v \in \Xi$ and $H_v = \mathbf{C}$ for $v \notin \Xi$. With respect to this factorization, Ht' factors as the product $\bigotimes \left(\mathrm{Ht}_v \otimes (1, \sqrt{q_v})\right)$, where $(1, \sqrt{q_v}) \in H$ is the vector that has value 1 on the first $\mathbf{C}$ factor and value $\sqrt{q_v}$ on the second $\mathbf{C}$ factor (See Remark 5.4.7).

Accordingly, computing $\alpha(s)$ reduces to computing the local maps

$$\alpha_v(s) : \mathcal{I}(\chi_v)^{K_v} \otimes \mathbf{C}^2 \xrightarrow{\mathrm{Ht}'^s} \mathcal{I}(\chi_v |\cdot|_v^{s/2})^{K_v} \otimes \mathbf{C}^2 \xrightarrow{M \otimes \mathrm{id}} \mathcal{I}(\chi_v^{-1} |\cdot|_v^{-s/2})^{K_v} \otimes \mathbf{C}^2 \xrightarrow{\mathrm{Ht}'^s} \mathcal{I}(\chi_v^{-1})^{K_v} \otimes \mathbf{C}^2$$

for χ_v a local constituent of some character $\chi \in X$, and $v \in \Xi$.

Recall that $K_v = \mathrm{PGL}_2(\mathscr{O}_v)$ for $v \in \Xi$, for we are supposing that Ξ is disjoint from any level in K. Write f_0 for the element of $\mathcal{I}(\chi_v)$ that takes value 1 on K_v. Let e_1, e_2 be the standard basis of $\mathbf{C}^2$. Then $\alpha_v(s)$, expressed with respect to the standard basis $f_0 \otimes e_1, f_0 \otimes e_2$ for the source and target — is expressed by the matrix $M \begin{pmatrix} 1 & 0 \\ 0 & q_v^s \end{pmatrix}$, where M is now the scalar by which the intertwining operator acts on f_0 (see (5.6.4.1)).

This proves (after interpretation) the assertions of the Theorem concerning $\Sigma \times \{1,2\}$.

5.7. Modular symbols, boundary torsion, and the Eisenstein regulator

In this section, we work with an arbitrary open compact subgroup $K \subset \mathbb{G}(\mathbb{A}_f)$, and discuss the issue of "integrality" of Eisenstein series. In other words: the Eisenstein series gives an explicit harmonic form on $Y(K)$; which multiple of that form is cohomogically integral?

This is actually not necessary for our results on Jacquet–Langlands correspondence, and also is vacuous in the case of $K = K_\Sigma$ level structure in view of Lemma 5.4.4. However, it is interesting from two points of view:

- The methods of this section should allow a complete understanding of the Eisenstein part of the regulator.
- The methods lead to a bound on the cusp-Eisenstein torsion (notation of § 3.8.1), i.e., the torsion in $H_1(Y(K), \mathbf{Z})$ that lies in the image of $H_1(\partial Y(K), \mathbf{Z})$.

We do not strive for the sharpest result; we include this section largely to illustrate the method: We use "modular symbols" (that is to say: paths between two cusps, which lie a priori in Borel–Moore homology) to generate homology. When one integrates an Eisenstein series over a modular symbol, one obtains (essentially) a linear combination of L-values, but this linear combination involves some denominators. The main idea is to trade a modular symbol for a sum of two others to avoid these denominators. Since working out the results we have learned that similar ideas were used by a number of other authors, in particular Sczech in essentially the same context. We also note the related work [**25, 45**] as well as Harder's paper on Eisenstein cohomology [**34**]. We include it for self-containedness and also give a clean adelic formulation. We also would like to thank G. Harder and J. Raimbault who pointed to errors in the original version.

One pleasant feature of the current situation is that the harmonic forms associated to Eisenstein series have absolutely convergent integral over modular symbols.

Denote by $\overline{\mathbf{Z}}$ the ring of algebraic integers. In this section we shall define a certain integer e in terms of algebraic parts of certain L-values (§ 5.7.3) and prove:

THEOREM 5.7.1. *Let*

$$s \in H^1(\partial Y(K), \overline{\mathbf{Q}})$$

be a Hecke eigenclass which lies in the image of $H^1(Y(K), \overline{\mathbf{Q}})$. *If* s *is integral (i.e., lies in the image of* $H^1(\partial Y(K), \overline{\mathbf{Z}})$*) then* s *is in fact in the image of an Hecke eigenclass* $\tilde{s} \in H^1(Y(K), \overline{\mathbf{Z}}[\frac{1}{e}])$.

The integer e is the product of certainly easily comprehensible primes (as mentioned, these could probably be removed with a little effort) and a more serious factor: the numerator of a certain L-value. We have not attempted for the most precise result. In particular, the method gives a precise bound on the "denominator" of $\tilde{s}$, rather than simply a statement about the primes dividing this denominator.

In any case, let us reformulate it as a bound on torsion:

> The image of $H_1(\partial Y(K), \mathbf{Z})$ in $H_1(Y(K), \mathbf{Z})$ contains no ℓ-torsion unless ℓ divides e or is an orbifold prime.

PROOF. It is equivalent to consider instead the image of $H^1(\partial Y(K), \mathbf{Z})$ in $H^2_c(Y(K), \mathbf{Z})$.

Let ℓ be a prime not dividing e. Take $s \in H^1(\partial Y(K), \mathbf{Z})$ whose image in $H^2_c(Y(K), \mathbf{Z})$ is ℓ-torsion. Let $s_{\mathbf{Q}}$ be the image of s in $H^1(\partial Y(K), \overline{\mathbf{Q}})$. The theorem applies to $s_{\mathbf{Q}}$. Although $s_{\mathbf{Q}}$ may not be a Hecke eigenclass, we may write $s_{\mathbf{Q}} = \sum a_i s_i$, where the $s_i \in H^1(\partial Y(K), \overline{\mathbf{Q}})$ are Hecke eigenclasses and the a_i have denominators only at e, as follows from the discussion after Lemma 5.4.5. Then, according to the theorem, there is N such that $e^N s_i$ lifts to $H^1(Y(K), \overline{\mathbf{Z}})$. In particular, $e^N s$ maps to zero inside $H^2_c(Y(K), \overline{\mathbf{Z}})$. That proves the claim: the image of s in H^2_c could not have been ℓ-torsion. $\square$

REMARK. *One can in some cases obtain a much stronger result towards bounding the image of* $H_1(\partial Y(K), \mathbf{Z})$. *One method to do so is given by Berger* [**4**] *who uses the action of involutions, related to the Galois automorphism, on the manifold in question. Another method which is apparently different but seems to have the same range of applicability, is to construct enough cycles in* $H_1(\partial Y(K), \mathbf{Z})$ *as the boundary of Borel–Moore classes in* $H_{2,\mathrm{BM}}(Y(K), \mathbf{Z})$*, by embedding modular curves inside* $Y(K)$*. We do not pursue these methods further here; in any case the latter method, and likely also the former, is fundamentally limited to the case when all Eisenstein automorphic forms are invariant by the Galois automorphism of* $F/\mathbf{Q}$.

5.7.2. Let $X = X_0$ be the set of characters defined after (5.6.2.2), corresponding to $s = 0$, and let $X(K)$ be the subset with $\mathcal{I}(\chi)^K \neq 0$. Let $\eta(z) = \sqrt{z/\bar{z}}$ as before. For $\chi \in X$ let $f(\chi)$ be the norm of the conductor of χ, so that $f(\chi)$ is a natural number.

The formulation of the theorem involves an algebraic L-value. In order to make sense of this, we need to define first the transcendental periods that make it algebraic.

5.7.3. Periods and integrality of L-values. To each imaginary quadratic field F we shall associate a complex number $\Omega \in \mathbf{C}^*$ that is well-defined up to units in the ring of algebraic integers:

Let $E = \mathbf{C}/\mathscr{O}_F$, considered as an elliptic curve over $\mathbf{C}$. Then there exists a number field $H \subset \mathbf{C}$ such that E is defined over H, and has everywhere good reduction there. Let $\pi : \mathscr{E} \to \mathscr{O}_H$ be the Néron model of E over $\mathscr{O}_H$. Then $\pi_* \Omega^1_{\mathscr{E}}$ defines a line bundle on $\mathscr{O}_H$. *After extending H, if necessary, we may suppose this line bundle is trivial.* Having done so, we choose a generator ω for the global sections of that bundle; this is defined up to $\mathscr{O}_H^\times$. On the other hand the first homology group of the complex points of $E(\mathbf{C})$ are free of rank one as an $\mathscr{O}_F$-module. Choose a generator γ for $H_1(E(\mathbf{C}), \mathbf{Z})$ as $\mathscr{O}_F$-module, and set

$$\Omega := \int_\gamma \omega.$$

Let $\chi \in X$. According to a beautiful result of Damerell [**21**],

$$L^{\mathrm{alg}}(1/2, \chi) := \frac{L(1/2, \chi)}{\Omega}$$

is algebraic, and, in fact, integral away from primes dividing $6f(\chi)$. The same is true for

$$L^{\mathrm{alg}}(1/2, \chi^2) := \frac{L(1/2, \chi^2)}{\Omega^2}.$$

If S is a finite set of places, we denote by $L^{\mathrm{alg},S}(1,\chi)$ the corresponding definition but omitting the factors at S from the L-function.

5.7.4. Definition of the integer e. If α is algebraic, we denote by $\mathrm{num}(\alpha)$ the product of all primes p "dividing the numberator of α": that is, such that there exists a prime $\mathfrak{p}$ of $\mathbf{Q}(\alpha)$ of residue characteristic p so that the valuation of α at $\mathfrak{p}$ is positive.

Let S be the set of finite places at which the compact open subgroup K is not maximal, i.e. the set of v such that $\mathrm{PGL}_2(\mathscr{O}_v)$ is not contained in K. Set $e_1 = 30h_F \cdot \mathrm{disc}(F) \cdot \prod_{v \in S} q_v(q_v - 1)$. Now put

$$e = e_1 \prod \mathrm{num}(L^{\mathrm{alg},S}(1,\chi^2)); \tag{5.7.4.1}$$

this involves the much more mysterious (and important) factor of the numerator of an L-function.

As mentioned, we have not been very precise in our subsequent discussion; we anticipate most of the factors in e could be dropped except the numerator of the L-value.

5.7.5. Modular symbols. Let $\alpha, \beta \in \mathbf{P}^1(F)$ and $g_f \in \mathbb{G}(\mathbb{A}_f)/K$. Then the geodesic from α to β (considered as elements of $\mathbf{P}^1(\mathbf{C})$, the boundary of $\mathbf{H}^3$), translated by g_f, defines a class in $H_{1,\mathrm{BM}}(Y(K))$ (a "modular symbol") that we denote by $\langle \alpha, \beta; g_f \rangle$. Evidently these satisfy the relation

$$\langle \alpha, \beta; g_f \rangle + \langle \beta, \gamma; g_f \rangle + \langle \gamma, \alpha; g_f \rangle = 0.$$

the left-hand side being the (translate by g_f of the) boundary of the Borel–Moore chain defined by the ideal triangle by α, β, γ.

If there exists $\epsilon \in \mathbb{G}(F)$ with the property that $\epsilon.(\alpha, g_f K) = (\beta, g_f K)$ – that is to say, $\epsilon\alpha = \beta$ and $\epsilon g_f K = g_f K$ – we term the triple $\langle \alpha, \beta; g_f \rangle$ *admissible*; for such the image of $\langle \alpha, \beta, K \rangle$ under $H_{1,\mathrm{BM}} \to H_0(\partial)$ is *zero*, and consequently we may lift this to a class $[\alpha, \beta, K] \in H_1(Y(K), \mathbf{Z})$; of course we need to make a choice at this point; the lifted class is unique only up to the image of $H_1(\partial Y(K), \mathbf{Z})$.

LEMMA 5.7.6. *Classes $[\alpha, \beta, K]$ associated to admissible triples, together with the group $H_1(\partial Y(K), \mathbf{Z})$, generate $H_1(Y(K), \mathbf{Z})$.*

PROOF. Put $\Gamma = \mathbb{G}(F) \cap g_f K g_f^{-1}$. Then the connected component of $Y(K)$ containing $1 \times g_f$ is isomorphic to $M := \Gamma \backslash \mathbf{H}^3$.

Fix $z_0 \in \mathbf{H}^3$. Then $H_1(M, \mathbf{Z})$ is generated by the projection to M of geodesic between z_0 and γz_0. The basic idea is that "taking z_0 towards the point ∞ yields the result:" the geodesic between $z_0, \gamma z_0$ defines an equivalent class in $H_{1,\mathrm{BM}}$ to the infinite geodesic $\infty, \gamma\infty$.

More precisely, let C be a horoball around ∞ inside $\mathbf{H}^3$. Let $D = \gamma.C$. Let $\mathcal{G}$ denote the geodesic from ∞ to $\gamma\infty$.

Let $z_0 \in C$, so that $\gamma z_0 \in D$. Let w_0 be a point in $\mathcal{G} \cap C$ and let w_1 be a point in $\mathcal{G} \cap D$.

Now let:

- P_1 be a path starting at z_0 and ending at w_0, entirely within C;
- Let P_2 be the geodesic path from w_0 to w_1 along $\mathcal{G}$;
- Let P_3 be a path from w_1 to γz_0, remaining entirely within D.

The composite path $P_1+P_2+P_3$ gives a path from z_0 to γz_0, and its projection to M represents the class of $[\gamma]$ inside $H_1(M) \simeq \Gamma^{\mathrm{ab}}$. But P_1, P_3 lie entirely within cusps of M, and P_2 differs from the geodesic path $\mathcal{G}$ from ∞ to $\gamma\infty$ only by segments lying within the cusps of M. So $P_1+P_2+P_3$ and the admissible symbol $\langle \infty, \gamma\infty; g_f \rangle$ represent the same class in Borel–Moore homology.

In particular, the classes of admissible triples generate a subgroup $L \subset H_1(M, \mathbf{Z})$ such that the image of L in $H_1^{\mathrm{BM}}(M, \mathbf{Z})$ contains the image of $H_1(M, \mathbf{Z})$ in $H_1^{\mathrm{BM}}(M, \mathbf{Z})$. In other words, $H_1(M, \mathbf{Z})$ is generated by L together with $H_1(\partial M, \mathbf{Z})$, as claimed.

Our proof has also shown that modular symbols generate the full Borel–Moore homology $H_1^{\mathrm{BM}}(M, \mathbf{Z})$: We showed that the group L' thus generated contains, at least, the image of $H_1(M, \mathbf{Z}) \to H_1^{\mathrm{BM}}(M, \mathbf{Z})$; on the other hand, $L' \twoheadrightarrow H_0(\partial M)$ because one can join any two cusps by a geodesic.

We thank G. Harder for his feedback on the earlier (unclear) version of the proof. □

5.7.7. Denominator and denominator avoidance. We now introduce the notion of the "denominator" of a triple, which will correspond to "bad primes" when we compute the integral of an Eisenstein series over it:

We say a finite place v is in the "denominator" of the triple $\langle \alpha, \beta; g_f \rangle$ if the geodesic between $\alpha_v, \beta_v \in \mathbf{P}^1(F_v)$ inside the Bruhat-Tits building of $\mathbb{G}(F_v)$ does not passes through $g_f \mathrm{PGL}_2(\mathscr{O}_v)$. This notion is invariant by $\mathbb{G}(F)$, i.e., if we take $\gamma \in \mathbb{G}(F)$, then v is in the denominator of $\langle \alpha, \beta; g_f \rangle$ if and only if v is in the denominator of $\gamma\langle \alpha, \beta; g_f \rangle$.

Note that, because $\mathbb{G}(F)$ acts 2-transitively on $\mathbf{P}^1(F)$, we may always find $\gamma \in \mathbb{G}(F)$ such that $\gamma(\alpha, \beta) = (0, \infty)$; thus any modular symbols is in fact equivalent to one of the form $\langle 0, \infty; g_f \rangle$ for suitable g_f. Now v does not divide the denominator of the symbol $\langle 0, \infty; g_f \rangle$ if and only if $g_f \in \mathbf{A}(F_v) \cdot \mathrm{PGL}_2(\mathscr{O}_v)$.

The key point in our proof is the following simple fact about denominator avoidance. Let $p > 5$ be a rational prime.

> (♡) We may write $\langle \alpha, \beta; g_f \rangle \in H_{1,\mathrm{BM}}$ as a sum of symbols whose denominators do not contain any place v with either q_v or $q_v - 1$ divisible by p.

The original version of this proof ([**16**]) contained minor errors. A much sharper statement with a corrected proof appears in [**5**, §7.4] and we refer the reader there.

5.7.8. Differential forms from automorphic representations. For any automorphic representation π, denote by Ω the natural map

$$\Omega : \mathrm{Hom}_{K_\infty}(\mathfrak{g}/\mathfrak{k}, \pi)^K \to \Omega^1(Y(K)) \tag{5.7.8.1}$$

Indeed, $\Omega^1(Y(K))$ can be considered as functions on $\mathbb{G}(F)\backslash(\mathbb{G}(\mathbb{A}) \times \mathfrak{g}/\mathfrak{k})/K_\infty K$ that are linear on each $\mathfrak{g}/\mathfrak{k}$-fiber.

Explicitly, for $X \in \mathfrak{g}/\mathfrak{k}$ and $g \in \mathbb{G}(\mathbb{A})$, we can regard (g, X) as a tangent vector to $\mathbb{G}(F)gK_\infty K \in Y(K)$, corresponding to the 1-parameter curve $\mathbb{G}(F)ge^{tX}K_\infty K$. Denote this vector by $[g, X]$. Then the map Ω is normalized by the requirement that, for $f \in \mathrm{Hom}_{K_\infty}(\mathfrak{g}/\mathfrak{k}, \pi)$, we have

$$\Omega(f)([g, X]) = f(X)(g). \tag{5.7.8.2}$$

Recall that $\mathbf{A}$ denotes the diagonal torus in PGL_2. There is a natural isomorphism

$$\mathbf{A}(F_\infty)/\mathbf{A}(F_\infty) \cap \mathrm{PU}_2 \xrightarrow{\sim} \mathbf{R}_+ \tag{5.7.8.3}$$

descending from the map $\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \mapsto |z|$. The isomorphism (5.7.8.3) normalizes an element X in the Lie algebra of the left hand side, namely, the element corresponding to the one-parameter subgroup $t \in \mathbf{R} \mapsto \exp(t) \in \mathbf{R}_+$.

We will also regard X as an element of $\mathfrak{g}/\mathfrak{k}$. In fact explicitly $X = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$ (where we identify $\mathfrak{pgl}_2(\mathbf{C})$ with trace-free matrices).

5.7.9. Eisenstein integrals over modular symbols. Let $\chi \in X(0)$ and $\mathcal{I}(\chi)$ as in (5.6.2.2) and $\upsilon \in \mathrm{Hom}_{K_\infty}(\mathfrak{g}/\mathfrak{k}, \mathcal{I}(\chi))^K$. We denote by $\bar{\upsilon} \in \mathrm{Hom}_{K_\infty}(\mathfrak{g}/\mathfrak{k}, \mathcal{I}(\chi^{-1}))$ the composition of υ with the standard intertwiner $\mathcal{I}(\chi) \longrightarrow \mathcal{I}(\chi^{-1})$ (defined by the analogue of formula (5.6.2.3)). Note that, for $\mathcal{I}(\chi)^K$ to be nonempty,

$$\chi \text{ must be unramified at places } v \text{ for which } K_v \text{ is maximal.} \tag{5.7.9.1}$$

Applying the Eisenstein intertwiner (cf. §5.6.2.1): We define a 1-form $\Omega(\mathrm{Eis}\ \upsilon)$ on $Y(K)$ as in § 5.7.8. We also denote the automorphic representation which is the image of $\mathcal{I}(\chi)$ under Eis, by the letter Π.

Put $n(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. The Lie algebra of $\mathbf{N}$ as an F-algebraic group is naturally identified with F, with trivial Lie bracket (identify $x \in F$ with the one-parameter subgroup $t \mapsto n(xt)$). With this convention, for $x \in F$, we write $\upsilon_x \in \mathcal{I}(\chi)$ for the evaluation of υ at x (considered in $\mathrm{Lie}(N) \to \mathfrak{g}/\mathfrak{k}$; recall that we have always fixed an embedding of F into $\mathbf{C}$.) Therefore, for example, υ_1 makes sense as an element of $\mathcal{I}(\chi)$; it is in particular a function on $\mathbb{G}(\mathbb{A})$.

Lemma. *If the restriction of $\Omega(\mathrm{Eis}\ \upsilon)$ to $\partial Y(K)$ is integral, i.e. takes $\overline{\mathbf{Z}}$-values on 1-cycles, then $\upsilon_1(k)$ and $\bar{\upsilon}_1(k)$ takes values in $N^{-1}\overline{\mathbf{Z}}$ whenever $k \in \mathrm{PGL}_2(\widehat{\mathscr{O}})$ where N is divisible only by primes that are nonmaximal*[16] *for K and primes dividing the discriminant of F.*

Proof. Let $g_f \in \mathbb{G}(\mathbb{A}_f)$. Let $t \in F$ be such that $n(t) \in N(F) \cap g_f K g_f^{-1}$; we identify t with an element of the Lie algebra of N. The pair (t, g_f) defines a 1-cycle $\mathcal{C}$ in the cusp of $Y(K)$: namely the projection of $X \to n(t_\infty X).g_f : 0 \le X \le 1$ to $Y(K)$; here t_∞ is the image of t under $F \hookrightarrow \mathbf{C}$.

We compute

$$\int_{\mathcal{C}} \Omega(\mathrm{Eis}\ \upsilon) = \upsilon_t(g_f) + \bar{\upsilon}_t(g_f);$$

this is a routine computation from (5.6.2.4). Thus $\upsilon_t(g_f) + \bar{\upsilon}_t(g_f) \in \overline{\mathbf{Z}}$ whenever $n(t) \in g_f K g_f^{-1}$.

For fixed $g_f \in \mathbb{G}(\mathbb{A}_f)$, the map $x \mapsto \upsilon_x(g_f)$, considered as a map $F \to \mathbf{C}$, is F-linear, whereas $x \mapsto \bar{\upsilon}_x(g_f)$ is F-conjugate-linear.

For each place v at which K is not maximal, choose $m_v > 0$ such that the principal congruence subgroup of level m_v is contained in K_v. Then $\{\lambda \in F_v : n(\lambda) \in g_f K g_f^{-1}\}$ is stable by the order $\mathbf{Z}_v + \varpi_v^{m_v} \mathscr{O}_v$ (here $\mathbf{Z}_v$ is the closure of $\mathbf{Z}$

[16] By this we mean: primes p such that there exists a finite place w of residue characteristic p, with $K_w \neq \mathrm{PGL}_2(\mathscr{O}_{F,w})$.

in $\mathscr{O}_v$). For each place v at which K is maximal, the same conclusion holds with $m_v = 0$: we may suppose, by Iwasawa decomposition, that the projection of g_f to $\mathbb{G}(F_v)$ actually belongs to $\mathbf{B}(F_v)$, and there the result is clear.

Let $\mathcal{O} = \{x \in \mathscr{O}_F : x \in \mathbf{Z}_v + \varpi_v^{m_v}\mathscr{O}_v \text{ for all } v\}$; it's an order in $\mathscr{O}_F$, and its discriminant is divisible only by the discriminant of F and by primes at which K is not maximal.

Then $\{x \in F : n(x) \in g_f K g_f^{-1}\}$ is always $\mathcal{O}$-stable; let $1, t$ be a $\mathbf{Z}$-basis of $\mathcal{O}$.

Now, if $g_f \in \mathrm{PGL}_2(\widehat{\mathscr{O}})$, then we may find $x \in F$ which is actually an *integer*, divisible only by non-maximal primes for K, and moreover $n(x) \in g_f K g_f^{-1}$. Then $\upsilon_x(g_f) + \bar{\upsilon}_x(g_f) \in \overline{\mathbf{Z}}$ and $t\upsilon_x(g_f) + \bar{t}\bar{\upsilon}_x(g_f) \in \overline{\mathbf{Z}}$ implies that $\upsilon_x(g_f) \in (t-\bar{t})^{-1}\overline{\mathbf{Z}} \subset \mathrm{disc}(\mathcal{O})^{-1}\overline{\mathbf{Z}}$. That implies the desired result about $\upsilon_1(g_f)$. □

5.7.10. An integrality result, and deduction of the Theorem from it. In what follows we suppose υ to be integral in the sense above. We shall verify in the coming subsections the following:

> (*) Suppose that Eisυ is integral in the sense of the Lemma from §5.7.9. Suppose that p does not divide e and does not divide $q_v(q_v-1)$ for any place v in the denominator of the modular symbol $\langle \alpha, \beta; g_f\rangle$. Then $\int_{\langle \alpha,\beta;g_f\rangle} \Omega(\mathrm{Eis}\ \upsilon) \in \overline{\mathbf{Z}}_{(p)}$, where $\overline{\mathbf{Z}}_{(p)}$ denotes the algebraic numbers that are integral at all primes above p.

As we now explain, this implies Theorem 5.7.1.

In order to deduce Theorem 5.7.1 from (*), we need to pay attention carefully about the explicit way of lifting $\langle x, y; g_f\rangle$ to homology. The tricky point was overlooked in the earlier version [**16**] and we thank Jean Raimbault for pointing it out to us.

Now take s as in the statement of the Theorem. Then s is in fact the image of some $\Omega(\mathrm{Eis}\upsilon)$ for suitable υ; moreover, υ is integral in the sense of §5.7.9. Fix p a prime number that does not divide the integer e from Theorem 5.7.1. We are going to check that, for each $\gamma \in H_1(Y(K), \mathbf{Z})$, that

$$\int_\gamma \Omega(\mathrm{Eis}\upsilon) \in \overline{\mathbf{Z}}_{(p)}.$$

For each cusp, we fix a "reference modular symbol" emanating from it. More precisely, we choose a subset $\mathscr{S}$ of the space of modular symbols

$$\mathscr{S} \subset \mathrm{PGL}_2(F)\backslash \left(\mathbf{P}^1(F) \times \mathbf{P}^1(F) \times \mathbb{G}(\mathbb{A}_f)/K\right)$$

such that

(a) $\mathscr{S}$ projects bijectively via $(x, y, g_f) \mapsto (x, g_f)$ to $\mathrm{PGL}_2(F)\backslash \left(\mathbf{P}^1(F) \times \mathbb{G}(\mathbb{A}_f)/K\right)$. In other words, "for each cusp, there is unique element of $\mathscr{S}$ starting at it."

(b) All modular symbols in $\mathscr{S}$ have denominator relatively prime to p.

Such a $\mathscr{S}$ exists: We may choose a set of representatives for the $\mathrm{PGL}_2(F)$ action on $\mathbf{P}^1(F) \times \mathbb{G}(\mathbb{A}_f)/K$ of the form $(\infty, g_{f,i})$, where $g_{f,i}$ varies through a set of representatives of $B(F)$ on $\mathrm{PGL}_2(\mathbb{A}_f)/K$. Now, for each i, choose $y_i \in \mathbf{P}^1(F)$ such that $(\infty, y_i; g_{f,i})$ has denominator relatively prime to p. That is certainly possible: it is simply a local constraint on y_i for each prime above p. We then define $\mathscr{S}$ to be the collection of $\mathrm{PGL}_2(F)$-orbits of $(\infty, y_i, g_{f,i})$.

We are now ready to fix a precise lifting of modular symbols to homology:

Fix a large parameter T. For each $x \times g_f \in \mathbf{P}^1(F) \times \mathbb{G}(\mathbb{A}_f)$ let $s(x, g_f)$ be such that $(x, s(x, g_f); g_f) \in \mathscr{S}$. (Note this is not unique, but the $\mathrm{PGL}_2(F)$-class of $(x, s(x, g_f); g_f)$ is uniquely determined.) Now, for any $x, y \in \mathbf{P}^1(F)$ and $g_f \in \mathrm{PGL}_2(\mathbb{A}_f)$, we define the path $[x, y, g_f]^*$ as the concatenation of the following three paths:

(a) In the cusp $x \times g_f$, draw a path from $\langle x, s(x, g_f); g_f\rangle \cap \partial Y(K)_T$ to $\langle x, y; g_f\rangle \cap \partial Y(K)_T$ along $\partial Y(K)_T$.
(b) The segment of the geodesic $\langle x, y; g_f\rangle$ that has height $\leq T$, i.e. lies inside $Y(K)_{\leq T}$.
(c) In the cusp $y \times g_f$, draw a path from $\langle x, y; g_f\rangle \cap \partial Y(K)_T$ to $\langle s(y, g_f), y; g_f\rangle \cap \partial Y(K)_T$ by a path along $\partial Y(K)_T$.

Note that steps (a) and (c) are only well-defined up to an element of $H_1(\partial Y(K), \mathbf{Z})$; make an arbitrary choice. In any case, the resulting path starts at

$$P_{x,g_f} := \langle x, s(x, g_f); g_f\rangle \cap \partial Y(K)_T \tag{5.7.10.1}$$

and ends at $P_{y,g_f} = \langle s(y, g_f), y; g_f\rangle \cap \partial Y(K)_T$. Note that P_{x,g_f} doesn't depend on the choice of $s(x, g_f)$.

We claim that, if $\langle x, y; g_f\rangle$ has denominator relatively prime to p, then, as $T \to \infty$,

$$\int_{[x,y;g_f]^*} \Omega(\mathrm{Eis}\upsilon) - \int_{\langle x,y;g_f\rangle} \Omega(\mathrm{Eis}\upsilon) \longrightarrow a, \quad a \in \overline{\mathbf{Z}}_{(p)}. \tag{5.7.10.2}$$

To verify (5.7.10.2) it is necessary to analyze the integral of $\Omega(\mathrm{Eis}\upsilon)$ along segments (a) and (c) of the path described above. In what follows, write for short $x' = s(x, g_f)$. Let U_x be the unipotent radical of the stabilizer of x in PGL_2. There's a unique element $u \in U_x(F)$ satisfying

$$uy = x' \quad (\text{equality in } \mathbf{P}^1(F)).$$

Let v be a place above p. Now x, y, x' define points on the boundary of the Bruhat–Tits tree of $\mathrm{PGL}_2(\mathscr{O}_v)$. By virtue of assumption (b) in the definition of $\mathscr{S}$, the geodesic from x to x' passes through $g_f\mathscr{O}_v^2$. Since we are assuing that $\langle x, y; g_f\rangle$ has denominator relatively prime to p, the geodesic from x to y passes through $g_f\mathscr{O}_v^2$, too. So the element u fixes $g_f\mathscr{O}_v^2$, that is to say, at the place v,

$$u \in U_x(F_v) \cap g_f K_v g_f^{-1}.$$

But the element u determines the relative position of the geodesics $\langle x, y; g_f\rangle$ and $\langle x, x'; g_f\rangle$ inside the cusp associated to x. What we have just shown implies that the order of the coset

$$\bar{u} \in U_x(F)/U_x(F) \cap g_f K g_f^{-1}$$

is actually prime-to-p. That, and the fact that we are supposing that $\Omega(\mathrm{Eis}\upsilon)$ has integral periods over $H_1(\partial Y(K), \mathbf{Z})$, shows that the integral of $\Omega(\mathrm{Eis}\upsilon)$ over segment (a) of the path approaches an element of $\overline{\mathbf{Z}}_{(p)}$, as $T \to \infty$. For segment (c), we reason identically but replacing $x, y, x' = s(x, g_f)$ by $y, x, y' = s(y, g_f)$.

This concludes the verification of (5.7.10.2).

Take $\gamma \in H_1(Y(K), \mathbf{Z})$. By Lemma 5.7.6 we can write $\gamma = \sum\langle x_i, y_i; g_{f,i}\rangle$ modulo $H^1(\partial Y(K), \mathbf{Z})$, where the image of $\sum\langle x_i, y_i; g_{f,i}\rangle$ in $H_0(\partial Y(K)$ is zero:

$$\sum\langle x_i, g_{f,i}\rangle = \sum\langle y_i, g_{f,i}\rangle$$

in the free abelian group on $\mathrm{PGL}_2(F)\backslash(\mathbf{P}^1(F)\times\mathbb{G}(\mathbb{A}_f)/K)$. By (♡) we may assume that, if q is any prime divisor of the denominator of any $\langle x_i, y_i; g_{f,i}\rangle$, p does not divide $q(q-1)$.

Now the boundary of $[x, y; g_f]$ is given by $P_{x,g_f} - P_{y,g_f}$, where P_{x,g_f} is defined in (5.7.10.1). In our case, we have

$$\sum P_{x_i,g_{f,i}} = \sum P_{y_i,g_{f,i}},$$

and in particular $\sum[x_i, y_i; g_{f,i}]^*$ is a cycle and we have in fact

$$\gamma \equiv \sum[x_i, y_i; g_{f,i}]^* \text{ modulo } H_1(\partial Y(K), \mathbf{Z}).$$

By (*) and (5.7.10.2) we see that

$$\int_\gamma \Omega(\mathrm{Eis}\upsilon) \in \overline{\mathbf{Z}}_{(p)}$$

as claimed.

5.7.11. We proceed to the proof of (*). To recall, the statement is

> (*) Suppose that $\mathrm{Eis}\upsilon$ is integral in the sense of the Lemma from §5.7.9. Suppose that p does not divide e and does not divide $q_v(q_v - 1)$ for any place v in the denominator of the modular symbol $\langle\alpha, \beta; g_f\rangle$. Then $\int_{\langle\alpha,\beta;g_f\rangle}\Omega(\mathrm{Eis}\ \upsilon) \in \overline{\mathbf{Z}}_{(p)}$.

5.7.11.1. *Some measure normalizations.* Let ψ be the additive character of $\mathbb{A}_F/F$ given by the composition of the standard character of $\mathbb{A}_\mathbf{Q}/\mathbf{Q}$ with the trace. Fix the measure on $\mathbb{A}_F$ that is self-dual with respect to ψ, and similarly on each F_v. This equips $\mathbf{N}(F_v), \mathbf{N}(\mathbb{A})$ with measures via $x \mapsto n(x)$; the quotient measure on $\mathbf{N}(\mathbb{A})/\mathbf{N}(F)$ is 1.

For each v we denote by d_v the absolute discriminant of the local field F_v, so that the self-dual measure of the ring of integers of F_v equals $d_v^{-1/2}$. We denote by q_v the size of the residue field of F_v.

On the multiplicative group we use a different measure normalization. We put on $\mathbf{A}(\mathbb{A}_f) \simeq \mathbb{A}_f^\times$ the measure that assigns the maximal compact subgroup volume 1, and similarly for $\mathbf{A}(F_v) \simeq F_v^\times$ for v finite. We put on $\mathbf{A}(F_\infty)$ the measure which, when projected to $\mathbf{A}(F_\infty)/\mathbf{A}(F_\infty)\cap K_\infty \simeq \mathbf{R}_+$, is defined by a differential form dual to X (see (5.7.8.3)). If dx is the self-dual measure on $F_\infty \simeq \mathbf{C}$, then this measure can alternately be described as $(4\pi)^{-1}\frac{dx}{|x|_\mathbf{C}}$.

5.7.11.2. *Normalization of $F(X)$.* Let X be as in defined after (5.7.8.3). Now put $\upsilon_X := \upsilon(X) \in \mathcal{I}(\chi)$; we may suppose without loss of generality that it is a factorizable vector $\bigotimes f_v$ where:

- For finite v, each f_v takes $\overline{\mathbf{Z}}_{(p)}$-values on $\mathrm{PGL}_2(\mathscr{O}_v)$.
- f_∞ lies in the unique $\mathbf{A}(F_\infty)\cap K_\infty$-fixed line in the unique K_∞ subrepresentation of π_∞ that is isomorphic to $\mathfrak{g}/\mathfrak{k}$. We normalize it so that the K_∞-equivariant map $\mathfrak{g}/\mathfrak{k} \to \mathcal{I}(\chi_\infty)$ that carries X to f_∞ carries the image of $1 \in \mathrm{Lie}(N)$ to a function taking value 1 at 1.

Indeed, write $\upsilon_X = f_\infty \otimes g$; the Lemma from §5.7.9 shows that g takes $\overline{\mathbf{Z}}_{(p)}$-values on $\mathrm{PGL}_2(\widehat{\mathscr{O}})$, and thus can be factored as a sum of pure tensors with the same property.

We can be completely explicit about f_∞: We can decompose $\mathfrak{g}/\mathfrak{k}$ as the direct sum $\mathfrak{h} \oplus \mathfrak{n}$, where these are (respectively) the Lie algebras of $\mathbf{A}(F_\infty)$ and the image of the Lie algebra of $\mathbf{N}(\mathbf{C})$. Then f_∞ can be taken to be given on PU_2 by the function

$$k \in \mathrm{PU}_2 \mapsto \mathfrak{n}\text{-component of } \mathrm{Ad}(k)X.$$

and the image f_∞^Z of any other $Z \in \mathfrak{g}/\mathfrak{k}$ is given by the corresponding function wherein we replace X by Z.

5.7.11.3. *The Whittaker function of υ_X.* The Whittaker function of $\mathrm{Eis}(\upsilon_X)$ is given by

$$\int \mathrm{Eis}(\upsilon_X)(ng)\psi(n)dn = \prod_v W_v(g_v),$$

where $W_v = \int_{x\in F_v} f_v(wn(x)g)\psi(x)dx$. This integral defining W_v is "essentially convergent," i.e. it can be replaced by an integral over a sufficiently large compact set without changing its value. Thus W_v is valued in $d_v^{-1} \cdot \overline{\mathbf{Z}}_{(p)}[q_v^{-1}]$, as follows from the fact that ψ is $\overline{\mathbf{Z}}$-valued and the self-dual measure gives measure $d_v^{-1/2}$ to the maximal compact subring (see § 5.7.11.1). If f_v is spherical one may compute more precisely that W_v takes values in $\frac{1}{d_v L_v(1,\chi_v^2)}\overline{\mathbf{Z}}_{(p)}[q_v^{-1}]$.

5.7.11.4. *Integral of $\Omega(\mathrm{Eis}\ \upsilon)$ over a modular symbol.* We are now ready to attack (*) stated in §5.7.10.

Without loss of generality we may suppose $\alpha = 0, \beta = \infty$. Let S_2 be the set of finite places v at which K is not maximal. Let S_1 be the set of finite places v such that K is maximal at v, but v lies in the denominator of the modular symbol. Set $S = S_1 \cup S_2$. For ω_v a unitary character of $F_v^\times$ define the local integral $I_v(\omega_v) := \int_{y\in F_v^\times} W(a(y)g_v)\omega_v(y)d^\times y$, where the measure on $F_v^\times$ is as normalized in §5.7.11.1.

The integral of $\Omega(\mathrm{Eis}\ \upsilon)$ over the modular symbol $\langle 0, \infty; g_f\rangle$ can be expressed as (see [**5**, §7.6.3] for essentially the same computation):

(5.7.11.1)
$$\int_{\langle 0,\infty;g_f\rangle} \Omega(\mathrm{Eis}\ \upsilon) \overset{(a)}{=} \frac{w'_F}{h_F}\sum_\omega \int_{\mathbb{A}(F)\backslash\mathbb{A}(\mathbb{A})} \mathrm{Eis}(\upsilon_X)(tg)\omega(t)d\mu$$

(5.7.11.2)
$$\overset{(b)}{=} \frac{w'_F}{h_F}\sum_\omega \prod I_v(\omega_v)$$

(5.7.11.3)
$$\overset{(c)}{=} \frac{w'_F}{h_F}\sum_\omega I_\infty \cdot \frac{L(\frac{1}{2}, \Pi\times\omega)}{L^{S_2}(1,\chi^2)}$$

(5.7.11.4)
$$\cdot \left(\prod_{v\in S_1} \frac{I_v}{L_v(\frac{1}{2},\Pi\times\omega)/L_v(1,\chi^2)} \cdot \prod_{v\in S_2}\frac{I_v}{L_v(\frac{1}{2},\Pi\times\omega)}\right)$$

$$= \frac{w'_F}{h_F}\sum_\omega I_\infty \cdot \frac{L^{\mathrm{alg}}(\frac{1}{2},\chi\times\omega)L^{\mathrm{alg}}(\frac{1}{2},\chi^{-1}\times\omega)}{L^{S_2,\mathrm{alg}}(1,\chi^2)} \cdot$$

(5.7.11.5)
$$\cdot \left(\prod_{v\in S_1} \frac{I_v}{L_v(\frac{1}{2},\Pi\times\omega)/L_v(1,\chi^2)} \cdot \prod_{v\in S_2}\frac{I_v}{L_v(\frac{1}{2},\Pi\times\omega)}\right)$$

Here, at step (a) and below, w'_F is an integer: At step (a), the summation is taken over characters ω of $\mathbf{A}(\mathbb{A})/\mathbf{A}(F)$ that are trivial on $\mathbf{A}(\mathbb{A}) \cap \left(\mathbf{A}(F_\infty) \cdot g_f K g_f^{-1}\right)$. In particular, any such ω is unramified at places $v \notin S$. At step (b) we unfolded the integral; the product is not convergent but is understood via the expression given in (c). Finally, the various L^{alg} are defined as in §5.7.3.

Now consider the terms: We will prove in §5.7.11.5 that I_∞ and the final bracketed term is p-integral. The second term $\dfrac{L^{\text{alg}}(\frac{1}{2}, \chi \times \omega) L^{\text{alg}}(\frac{1}{2}, \chi^{-1} \times \omega)}{L^{S_2,\text{alg}}(1, \chi^2)}$ is integral, by Damerell's theorem quoted in §5.7.3, after being multiplied by the numerator of $L^{S_2,\text{alg}}(1, \chi^2)$ and a suitable power of $6f(\chi)f(\omega)$. Now p doesn't divide $6f(\chi)$ by definition of e, and it doesn't divide $f(\omega)$ because, as noted above, ω is ramified only at $v \in S$, i.e. the conductor of ω is divisible only by primes dividing e and by factors q_v where v is in the denominator of the modular symbol.

This concludes the proof of (*), and so also of the Theorem.

5.7.11.5. *Local integrals.* We assume that the defining integral is absolutely convergent.

(a) For finite $v \notin S$, f_v is spherical and $g_v \in \mathbf{A}(F_v) \cdot \mathrm{PGL}_2(\mathscr{O}_v)$ From that we deduce that

$$I_v = f_v(e) \frac{L(\frac{1}{2}, \Pi_v \times \omega_v)}{L(1, \chi_v^2)} \in \overline{\mathbf{Z}}_{(p)} \cdot \frac{L(\frac{1}{2}, \Pi_v \times \omega_v)}{L(1, \chi_v^2)} \tag{5.7.11.6}$$

for $v \notin S$.

(b) For $v \in S_1$, we have also

$$I_v \in \frac{1}{q_v - 1} \overline{\mathbf{Z}}_{(p)}[q_v^{-1}] \cdot \frac{L(\frac{1}{2}, \Pi_v \times \omega_v)}{L(1, \chi_v^2)} \subset \overline{\mathbf{Z}}_{(p)} \cdot \frac{L(\frac{1}{2}, \Pi_v \times \omega_v)}{L(1, \chi_v^2)} \tag{5.7.11.7}$$

where the latter inclusion follows since p doesn't divide $q_v(q_v - 1)$.

To see this, split into two cases:

(b1) ω_v is unramified. In this case, we consider I_v as a function of g_v; it is a function on $\mathrm{PGL}_2(F_v)/\mathrm{PGL}_2(\mathscr{O}_v)$ that is left $\mathbf{A}(F_v)$-equivariant. By examining the linear recurrence satisfied by this function, arising from the Hecke operators, we see that $I_v(g_v)$ is a multiple of $I_v(e)$, where only denominators of q_v and $q_v - 1$ arise in this process.

(b2) ω_v is ramified. In this case, write $g = ank$ with $a \in \mathbf{A}(F_v), n \in \mathbf{N}(F_v), k \in \mathrm{PGL}_2(\mathscr{O}_V)$; we see that I_v can be rewritten (up to algebraic units) as

$$\int_{y \in F_v^\times} \psi_v(ty) W(a(y)) \omega_v(y) d^\times y_v$$

for some $t \in F_v$. The integral is locally constant on cosets of a sufficiently small subgroup of $\mathscr{O}_v^\times$. Moreover, since f_v is spherical, the integrand is valued in $\frac{1}{d_v L_v(1,\chi_v^2)} \overline{\mathbf{Z}}_{(p)}[q_v^{-1}]$. Finally, p does not divide d_v by assumption, and $L(\frac{1}{2}, \Pi_v \times \omega_v) = 1$ because ω_v is ramified and Π_v unramified.

(c) For $v \in S_2$, we have

$$I_v \in \frac{1}{d_v \cdot (q_v - 1)} L_v(\frac{1}{2}, \Pi_v \times \omega_v) \overline{\mathbf{Z}}_{(p)}[q_v^{-1}] \subset \overline{\mathbf{Z}}_{(p)} L_v(\frac{1}{2}, \Pi_v \times \omega_v). \tag{5.7.11.8}$$

Indeed, $L_v(\frac{1}{2}, \Pi_v \times \omega_v)^{-1} \int_{y \in F_v^\times} W_v(a(y))\omega_v(y) d^\times y$ is in effect a finite sum, with coefficients in $\frac{1}{q_v - 1}\overline{\mathbf{Z}}_{(p)}[q_v^{-1}]$, of values of $W_v \omega_v$, each of which lie in $d_v^{-1}\overline{\mathbf{Z}}_{(p)}[q_v^{-1}]$. (The multiplication by L_v has the effect of rendering the sum finite.) Again, p does not divide d_v or $q_v(q_v - 1)$ by assumption, whence the last inclusion.

(d) For $v = \infty$, we explicitly compute as follows. (We compute formally; it can easily be justified.)

$$wn(x) = \begin{pmatrix} 0 & 1 \\ -1 & -x \end{pmatrix} = \begin{pmatrix} 1 & -\bar{x} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1+|x|^2}} & 0 \\ 0 & \sqrt{1+|x|^2} \end{pmatrix} k$$

and we compute that the projection to $\mathfrak{n}$ of $\mathrm{Ad}(k)X$ is given by simply $2\bar{x}$. Thus $f_\infty(wn(x)) = \bar{x}(1 + |x|^2)^{-2}$, and we wish to compute:

$$I_\infty = \int_{x \in F_\infty, y \in F_\infty^\times} f_\infty(wn(x)a(y))\psi(x) dx d^\times y \tag{5.7.11.9}$$

$$= \int_{x,y} f_\infty(a(y^{-1})wn(x))|y|_{\mathbf{C}}\psi(xy) dx d^\times y \tag{5.7.11.10}$$

$$= \int_x \bar{y} \frac{\bar{x}}{(|x|^2 + 1)^2} \psi(xy) dx d^\times y \tag{5.7.11.11}$$

Now, the distributional Fourier transform of $1/y = \bar{y}/|y|^2$ i.e. $\int \frac{\bar{y}}{|y|^2}\psi(xy)dy$ is $i/\bar{x}$. Since the measure on $F_\infty^\times$ is given by $\frac{1}{4\pi}\frac{dy}{|y|^2}$ our integral becomes

$$I_\infty = \frac{i}{4\pi} \int_x \frac{dx}{(|x|^2 + 1)^2} = i/2.$$

5.8. Comparing Reidmeister and analytic torsion: the main theorems

We have previously defined notions of analytic torsion $\tau_{\mathrm{an}}(Y)$ and Reidemeister torsion $\mathrm{RT}(Y)$ in the non-compact case. It is an interesting question to relate them, but it is not quite necessary for us to completely carry this out.

We will consider the case where we compare levels $\Sigma, \Sigma \cup \{\mathfrak{p}\}, \Sigma \cup \{\mathfrak{q}\}$ and $\Sigma \cup \{\mathfrak{p}, \mathfrak{q}\}$. More generally, we do not restrict to the case of $K = K_\Sigma$; let K be an arbitrary compact open subgroup of $\mathbb{G}(\mathbb{A}_f)$ which is maximal at $\mathfrak{p}, \mathfrak{q}$, and set

$$Y = Y(K) \times \{1,2\}^2, \ \ Y_\mathfrak{q} = Y(K \cap K_{\{\mathfrak{q}\}}) \times \{1,2\}; \tag{5.8.0.12}$$
$$Y_\mathfrak{p} = Y(K \cap K_{\{\mathfrak{p}\}}) \times \{1,2\}, \ \ Y_{\mathfrak{pq}} = Y(K \cap K_{\{\mathfrak{p},\mathfrak{q}\}}).$$

Note that all statements and proofs will carry over to the case when $\{\mathfrak{p}, \mathfrak{q}\}$ is an arbitrary set of primes of size ≥ 2.

These will always be equipped with height functions normalized as follows: equip $Y(K_\mathfrak{q})$ and $Y(K_{\mathfrak{pq}})$ with the heights of (5.4.1), and $Y_\mathfrak{q} = Y(K_\mathfrak{q}) \times \{1,2\}$ with the height that restricts to this on each component, and similarly for $Y_{\mathfrak{pq}}$; this specifies a height on $Y_\mathfrak{q}, Y_{\mathfrak{pq}}$. Now recall that we have specified (§ 5.4.6) an isometry

$$Y(K_\mathfrak{q})_B \ \cong \ Y(K)_B \times \{1,2\}, Y(K_{\mathfrak{pq}}) \cong Y(K_\mathfrak{p})_B \times \{1,2\}$$

and now pull back heights via this isometry to obtain heights on Y and $Y_\mathfrak{p}$. This definition seems somewhat idiosyncratic, but it has been chosen so that:

> The cusps of Y and $Y_{\mathfrak{q}}$ are isometric in a fashion preserving the height function; the cusps of $Y_{\mathfrak{p}}$ and $Y_{\mathfrak{pq}}$ are isometric in a fashion preserving the height function.

What we will actually show is:

THEOREM 5.8.1. *Notation and choices of height function as above,*

$$\frac{\mathrm{RT}(Y_{\mathfrak{pq}})}{\mathrm{RT}(Y_{\mathfrak{p}})}\frac{\mathrm{RT}(Y)}{\mathrm{RT}(Y_{\mathfrak{q}})} = \frac{\tau_{\mathrm{an}}(Y_{\mathfrak{pq}})}{\tau_{\mathrm{an}}(Y_{\mathfrak{p}})}\frac{\tau_{\mathrm{an}}(Y)}{\tau_{\mathrm{an}}(Y_{\mathfrak{q}})}, \tag{5.8.1.1}$$

up to orbifold primes.

(5.8.1.1) will prove to be easier than understanding the relation between each separate term $\tau_{\mathrm{an}}/\mathrm{RT}$. Indeed, it is well-known that it is much easier to handle *ratios* of regularized determinants than the actual determinants.

The next Corollary properly belongs in the next Chapter, but we give it here to simplify matters there:

COROLLARY 5.8.2. *Let Y' be the arithmetic manifold associated to the division algebra D' ramified precisely at $\mathfrak{p}, \mathfrak{q}$ and the corresponding open compact subgroup K' of $\mathbb{G}'(\mathbb{A}_f)$. Then, up to orbifold primes,*

$$\frac{\mathrm{RT}(Y_{\mathfrak{pq}})}{\mathrm{RT}(Y_{\mathfrak{p}})}\frac{\mathrm{RT}(Y)}{\mathrm{RT}(Y_{\mathfrak{q}})} = \mathrm{RT}(Y'). \tag{5.8.2.1}$$

By "corresponding open compact subgroup K'" we mean that K'_v is the image of units in a maximal order for v dividing $\mathfrak{p}, \mathfrak{q}$, and K'_v corresponds to K_v for v not dividing $\mathfrak{p}, \mathfrak{q}$. In particular, if K is simply a level structure of type K_Σ, then the corresponding K' is also the level structure K'_Σ, and the Y, Y' are a Jacquet–Langlands pair in the sense of § 6.1.

PROOF. Fix any $0 \le j \le 3$. If $\phi, \phi_{\mathfrak{p}}, \phi_{\mathfrak{q}}, \phi_{\mathfrak{pq}}$ are the determinants of the scattering matrices for j-forms on $Y, Y_{\mathfrak{p}}, Y_{\mathfrak{q}}, Y_{\mathfrak{pq}}$ then our prior computations (specifically (5.6.1.4) and (5.6.1.5), as well as assertion (4) of Theorem 5.6.1) show that

$$\left(\frac{\phi'}{\phi}\right) - \left(\frac{\phi'_{\mathfrak{p}}}{\phi_{\mathfrak{p}}}\right) = \left(\frac{\phi'_{\mathfrak{q}}}{\phi_{\mathfrak{q}}}\right) - \left(\frac{\phi'_{\mathfrak{pq}}}{\phi_{\mathfrak{pq}}}\right). \tag{5.8.2.2}$$

Also, we have seen that, if Ψ is the scattering matrix on functions, then, in similar notation,

$$\mathrm{tr}\ \Psi(0) - \mathrm{tr}\ \Psi_{\mathfrak{p}}(0) = \mathrm{tr}\ \Psi_{\mathfrak{q}}(0) - \mathrm{tr}\ \Psi_{\mathfrak{pq}}(0). \tag{5.8.2.3}$$

Now consider our definition (5.3.5.3), (5.3.5.4) of analytic torsion in the noncompact case. (5.8.2.2) and (5.8.2.3) show that all the terms involving scattering matrices cancel out, and we are left with:

$$\log \frac{\tau_{\mathrm{an}}(Y_{\mathfrak{pq}})}{\tau_{\mathrm{an}}(Y_{\mathfrak{p}})}\frac{\tau_{\mathrm{an}}(Y)}{\tau_{\mathrm{an}}(Y_{\mathfrak{q}})} = \frac{1}{2}\sum_j (-1)^{j+1} j \log \det{}^*(\Delta_j^{\mathrm{new}}), \tag{5.8.2.4}$$

where Δ_j denote the Laplacian on j-forms on $Y(K_{\mathfrak{pq}})$ but restricted to the *new subspace* of j-forms on $Y(K_{\mathfrak{pq}})$. The Jacquet–Langlands correspondence (skip ahead to § 6.2 for a review in this context) asserts the spectrum of Δ_j^{new} coincides with the spectrum of the Laplacian Δ_j on j-forms, on $Y'(K)$. Thus the right hand side equals $\log \tau_{\mathrm{an}}(Y'(K))$, and applying the Cheeger-Müller theorem to the compact manifold $Y'(K)$ we are done. ∎

We deduce Theorem 5.8.1 from the following result.

Suppose we are given two hyperbolic manifolds M, M' such that M_B and M'_B are isometric; we suppose that we have an isometry $\sigma : M_B \to M'_B$ and height functions on M, M' that match up with respect to σ. We suppose that M, M' are of the form $Y(K)$ for some K; *this arithmeticity assumption is almost surely unnecessary to the proof*, but it allows us to be a bit lazy at various points in the proof (e.g., we may appeal to standard bounds for the scattering matrix.)

THEOREM 5.8.3. *(Invariance under truncation) As $Y \to \infty$,*

$$\begin{aligned} &\log \tau_{\mathrm{an}}(M) - \log \tau_{\mathrm{an}}(M_Y) - (\log \mathrm{RT}(M) - \log \mathrm{RT}(M_Y)) \\ -&(\log \tau_{\mathrm{an}}(M') - \log \tau_{\mathrm{an}}(M'_Y)) + (\log \mathrm{RT}(M') - \log \mathrm{RT}(M'_Y)) \longrightarrow 0, \end{aligned}$$

where we understand the computation of analytic torsion on M_Y to be with respect to absolute boundary conditions.

When dealing with manifolds with boundary, we *always* compute analytic torsion with respect to absolute boundary conditions in what follows.

Indeed, the following observation was already used in Cheeger's proof: when studying the effect of a geometric operation on analytic torsion, it can be checked more easily that this effect is *independent of the manifold on which the geometric operation is performed.* In Cheeger's context, the surgery is usual surgery of manifolds. In our context, the geometric operation will be to truncate the cusp of a hyperbolic manifold. Cheeger is able to obtain much more precise results by then explicitly computing the effect of a particular surgery on a simple manifold; we do not carry out the analogue of this step, because we do not need it. This missing step does not seem too difficult: the only missing point is to check the short-time behavior of the heat kernel near the boundary of the truncated manifold; at present we do not see the arithmetic consequence of this computation.

The actual proof is independent of the Cheeger–Müller theorem, although it uses ideas that we learned from various papers of both of these authors. It will be given in § 5.10 after the necessary preliminaries about small eigenvalues of the Laplacian on M_T, which are given in § 5.9. We warn that the corresponding statement is not true for the $\log\det^*$ or the RT terms alone, although we precisely compute both: indeed, both terms individually diverge as $Y \to \infty$.

For now we only say the main idea of the proof:

> Identify the continuous parts of the trace formula for M, M' as arising from "newly created" eigenvalues on the truncated manifolds M_Y, M'_Y.

During the proof we need to work with j-forms for $j \in \{0, 1, 2, 3\}$. Our general policy will be to treat the case $j = 1$ in detail. The other cases are usually very similar, and we comment on differences in their treatment where important.

5.8.4. Proof that Theorem 5.8.3 implies Theorem 5.8.1. For $M = Y(K), Y(K_{\mathfrak{p}}), \ldots$ or any compact manifold with boundary, put

$$\alpha(M) := \frac{\tau_{\mathrm{an}}(M)}{\mathrm{RT}(M)};$$

in the cases of $Y(K)$ etc. the definitions of τ_{an} and RT are those of § 5.3 and depend on the choice of height function. Recall that we understand τ_{an} to be computed with reference to absolute boundary conditions in the case where M is a manifold with boundary.

For the moment we suppose that all these manifolds are genuine manifolds, i.e., do not have orbifold points. We discuss the necessary modifications in the general case in § 5.8.5.

According to Cheeger [**17**, Corollary 3.29] $\alpha(M)$ depends only on the germ of the metric of M near the boundary. The metric germs at the boundary being identical for $Y, Y_{\mathfrak{q}}$ (see comment before (5.8.1.1)), and similarly for $Y_{\mathfrak{p}}, Y_{\mathfrak{pq}}$, Cheeger's result shows that:

$$\frac{\alpha(Y_T)}{\alpha(Y_{\mathfrak{q},T})} = 1 \text{ and } \frac{\alpha(Y_{\mathfrak{pq},T})}{\alpha(Y(K_{\mathfrak{p},T}))} = 1. \tag{5.8.4.1}$$

Here, we understand Y_T to be the manifold with boundary obtained by truncating at height T, etc. Now Theorem 5.8.3 implies, after taking the limit,

$$\frac{\alpha(Y)}{\alpha(Y_{\mathfrak{q}})} \cdot \frac{\alpha(Y_{\mathfrak{pq}})}{\alpha(Y_{\mathfrak{p}})} = 1; \tag{5.8.4.2}$$

this proves (5.8.1.1).

5.8.5. Modifications in the orbifold case. Our discussion in the prior section § 5.8.4 concerned only the case when $Y(K)$ is strictly a manifold, i.e. has no orbifold points. Let us see why (5.8.1.1) remains true in the general case — i.e., when $Y(K)$ has orbifold points – up to a factor in $\mathbf{Q}^\times$ "supported at orbifold primes", i.e. whose numerator and denominator is divisible by' primes dividing the order of some isotropy group.

First of all, we note that the proof of Theorem 5.8.3 is entirely valid in the orbifold case. Indeed, the proof of this theorem is in fact entirely about properties of Laplacian eigenvalues and harmonic forms, and is insensitive to whether or not M and M_Y are manifolds or orbifolds.

Now the reasoning of the prior section § 5.8.4 goes through word-for-word in the orbifold case, *except* for (5.8.4.1); this remains valid only when 1 is replaced by a rational number supported only at orbifold primes, according to ($\diamondsuit$) below. Note that that indeed the orbifolds $Y(K)$ as well as any orbifolds obtained by truncating it are indeed global quotients, i.e., admit finite covers which are manifolds.

We now check

> ($\diamondsuit$) Suppose given two manifolds with boundary $\tilde{M}_1, \tilde{M}_2$, isometric near the boundary, and whose boundaries have zero Euler characteristic. Suppose also given a finite group Δ acting on $\tilde{M}_i$ preserving this isometry. Define M_i as the orbifold quotient $\tilde{M}_i/\Delta$.
>
> Then the ratios of analytic and Reidemeister torsion for M_1, M_2 differ by a rational number supported only at orbifold primes.

We proceed similarly to the previous discussion (§ 3.10.5), but use the work of Lück [**47**]. Note that Lück's torsion is the square of that of [**46**], which causes no problem for us; this is stated in (4.8) of [**47**]. In fact, Lück studies such equivariant torsion under the following assumptions:

(i) the metric near the boundary is a product;

(ii) A certain assumption of *coherence*; the latter is automatically satisfied when the flat bundle under consideration is trivial (or indeed induced by a G-representation).

Indeed we may first compatibly deform the metrics on the M_i so the metric near the boundary is a product. The ratio of Reidemeister and analytic torsion changes in the same way for M_1, M_2 when we do this (this argument is just as in [**17**, Corollary 3.29]).

This does not change the purely topological statement the M_i can be expressed as global quotients, and we then apply Lück's results.

In this case, the equivariant Reidemeister and equivariant analytic torsions are not literally *equal*; they differ by two terms, one measuring the failure of equivariant Poincaré duality, and the second proportional to the Euler characteristic of the boundary. The latter term vanishes in our case, as we will be dealing with certain truncated hyperbolic 3-manifolds; their boundaries are unions of 2-dimensional tori. As for the former term: the term measuring the failure of equivariant Poincaré duality is supported entirely at primes dividing the order of some isotropy group, that is to say, orbifold primes for $\tilde{M}/\Delta$. That follows from [**47**, Proposition 3.23(a) and Proposition 5.4] and the fact that Poincare duality holds "away from orbifold primes" in the sense of § 3.4.1.1.

5.9. Small eigenvalues

We are in the situation of Theorem 5.8.3. We denote by Φ^+, Φ^- the scattering matrices for 1-forms on M, and by Ψ the scattering matrix for functions; these notations are as in § 5.2.2.

Write Δ_{M_Y} for the Laplacian on 1-forms, on the truncated manifold M_Y, with absolute boundary conditions. The key part of the proof of Theorem 5.8.3 is the following result, which describes the small eigenvalues of Δ_{M_Y} precisely. Through the proof we will make the abbreviation

$$T_{\max} := (\log Y)^{100}.$$

This determines the upper threshhold to which we will analyze eigenvalues, i.e. we will not attempt to compute eigenvalues of M_Y directly that are larger than $T_{\max}^2$.

THEOREM 5.9.1. *[Small eigenvalues.] Put*

$$f(s) = \det(\mathrm{Id} - Y^{-4s}\Phi^-(-s)\Phi^+(s)),$$

Let $0 \le a_1 \le a_2 \le \dots$ be the union (taken with multiplicity) of the non-negative real roots of $t \mapsto f(it)$ and the set

$$\{t \in \mathbf{R}_{>0} : t^2 \text{ is the eigenvalue of a cuspidal co-closed } 1\text{-form on } M\},$$

both sets taken themselves with multiplicity. Set $\bar{\lambda}_j = a_j^2$.

Let $0 \le \lambda_1 \le \dots$ be the eigenvalues, with multiplicity, of

$$\Delta_{M_Y} | \ker d^*,$$

i.e. Laplacian eigenvalues on co-closed 1-forms on M_Y satisfying absolute boundary conditions.

There is $a > 0$ such that, with, $\delta = \exp(-aY)$ and for Y sufficiently large (this depending only on M):

(a) *(This is trivial:) If* $b = \dim H^1(M, \mathbf{C})$*, then* $\lambda_1 = \cdots = \lambda_b = 0$ *and* $\lambda_{b+1} > 0$*. The same is true for* $\bar{\lambda}_i$*.*
(b) $|\lambda_j - \bar{\lambda}_j| \le \delta$ *for any* j *with* $\sqrt{|\lambda_j|} \le T_{\max}$.
(c) *The same assertion holds for the eigenvalues of* Δ *on* 0*-forms (=co-closed* 0*-forms) with absolute boundary conditions, replacing* $f(s)$ *by the function*

$$g(s) = \det(\mathrm{Id} + \frac{1-s}{1+s} Y^{-2s} \Psi(s)),$$

but taking only roots of $g(it)$ *where* t *is strictly positive; also replace* b *by* $H^0(M, \mathbf{C})$ *(and* $\bar{\lambda}_j$ *by* $1 - a_j^2$*.)*
(d) *A similar assertion holds for eigenvalues on co-closed* 2*-forms with absolute boundary conditions (equivalently: for eigenvalues on closed* 1*-forms with relative boundary conditions*[17]*) where we now replace the function* $f(s)$ *by the function*

$$g'(s) = \det(\mathrm{Id} + Y^{-2s}\Psi(s))$$

we replace b *by* $\dim H^2(M, \mathbf{C})$*, and we make the following modification to the definition of* $\bar{\lambda}_j$*:*

Let $0 < u_1 \le \cdots \le u_h < 1$ *be the roots, with multiplicity, of* $g'(t)$ *for* $t \in (0, 1]$ *together with parameters of cusp forms* $\{t \in [0, 1] : 1 - t^2$ *is the eigenvalue of a cuspidal* 3*-form on* $M\}$*, and let* $a_j = iu_j$ *for* $1 \le j \le h$*.*

Let $0 < a_{h+1} \le a_{h+2} \le \dots$ *be the positive real roots of* $t \mapsto g'(it)$ *together with parameters of cusp forms; and put* $\bar{\lambda}_j = 1 + a_j^2$*.*

Warning: There exist roots a_j of $g' = 0$ very close to $t = 1$ related to the residue of Ψ; see discussion of § 5.9.10.

What is perhaps not obvious from the statement is *that this description "matches" in a very beautiful way part of the trace formula for* M. The proof of the theorem follows along lines that we learned from work of W. Müller (see [**54**]) and we present it in § 5.9.4. We first give the proof of Theorem 5.8.3, using Theorem 5.9.1. In fact, we will prove in detail only (a), (b), the other parts being similar; we will however discuss in some detail the "extra" eigenvalues of case (e). These eigenvalues play an important role in accounting for the failure of Poincaré duality for the non-compact manifold M.

For the moment we try to only give the main idea by describing:

5.9.1.1. *Plausibility argument for (b) of the Theorem.* Consider, for $\omega \in \Omega^+(0)$ and $\bar{\omega} \in \Omega^-(0)$, the 1-form $F = E(s, \omega) + E(-s, \bar{\omega})$; we shall attempt to choose $\omega, \bar{\omega}$ so it almost satisfies absolute boundary conditions when restricted to M_Y. Then in any cusp

$$F \sim y^s \omega + \Phi^+(s) y^{-s} \omega + y^s \bar{\omega} + y^{-s} \Phi^-(-s) \bar{\omega}.$$

The sign $\sim$ means, as before, that the difference is exponentially decaying. The contraction of F with a boundary normal is automatically of exponential decay; on the other hand, up to exponentially decaying factors,

$$s^{-1} \cdot dF \sim (y^s \omega - y^{-s} \Phi^-(-s) \bar{\omega}) \wedge \frac{dy}{y} + (y^s \bar{\omega} - y^{-s} \Phi^+(s) \omega) \wedge \frac{dy}{y}.$$

[17] Note that the nonzero such eigenvalues are just the nonzero eigenvalues of the Laplacian on 0-forms with relative boundary conditions. This is why Ψ intervenes.

Note that the s^{-1} on the left-hand side suggests that we will have to treat the case when s is near zero separately, and indeed we do this (§ 5.9.6).

In order that the contraction of this form with a normal vector at the boundary be zero, we require that both the "holomorphic part" $(y^s\omega - y^{-s}\Phi^-(s)\bar{\omega})$ and the "anti-holomorphic part" $(y^s\bar{\omega} - y^{-s}\Phi^{-s}\omega)$ be zero at the boundary, that is to say,

$$\omega = Y^{-2s}\Phi^-(-s)\bar{\omega} \text{ and } Y^s\bar{\omega} = Y^{-s}\Phi^+(s)\omega, \tag{5.9.1.1}$$

which imply that ω is fixed by $Y^{-4s}\Phi^-(-s)\Phi^+(s)$. For such ω to exist, we must indeed have $f(s) = \det(\mathrm{Id} - Y^{-4s}\Phi^-(-s)\Phi^+(s)) = 0$.

Of course, this analysis is only approximate, and we need to both verify that a zero of $f(s)$ gives rise to a nearby eigenfunction satisfying absolute boundary conditions, and, conversely, that any eigenfunction satisfying absolute boundary conditions "came from" a zero of $f(s)$.

For later use, it will also be helpful to compute the inner product $\langle \wedge^Y F, \wedge^Y F\rangle$ notations as above. We have:

$$\begin{aligned} \| \wedge^Y F\|^2_{L^2(M)} = \| \wedge^Y E(s,\omega)\|^2_{L^2(M)} + \| \wedge^Y E(-s,\bar{\omega})\|^2_{L^2(M)} \\ +\mathrm{Re}\, \langle \wedge^Y E(s,\omega), \wedge^Y E(-s,\bar{\omega})\rangle. \end{aligned} \tag{5.9.1.2}$$

Assuming that (5.9.1.1) holds, the second line *vanishes identically.* Indeed, according to the Maass-Selberg relation (see before (5.2.2.11)), the inner product in question equals $\frac{Y^{2s}}{2s}\langle \omega, \Phi^-(s)\bar{\omega}\rangle + \frac{Y^{-2s}}{-2s}\langle \Phi^+(s)\omega, \bar{\omega}\rangle$, and (5.9.1.1) means that this is purely imaginary . Therefore (after some simplifications) one obtains

$$\| \wedge^Y F\|^2 = \langle (4\log Y - A(s)^{-1}\frac{dA}{ds})\omega, \omega\rangle, \tag{5.9.1.3}$$

where we have abbreviated $A(s) := \Phi^-(-s)\Phi^+(s)$. This expression will recur later.

5.9.2. Analysis of zeroes of $f(s)$ and $g(s)$. Before we proceed (and also to help the reader's intuition) we analyze the zeroes of $f(s)$ and $g(s)$. We discuss $f(s)$, the case of g and g' being similar.

Write

$$A(s) = \Phi^-(-s)\Phi^+(s) : \Omega^+(s) \to \Omega^+(s).$$

Recall $f(s) = \det(1 - Y^{-4s}A(s))$. Let h_{rel} be the size of the matrix $A(s)$, i.e., the total number of relevant cusps of M.

Recall that a continuously varying family of matrices has continuously varying eigenvalues (in the "intuitively obvious" sense; formally, every symmetric function in the eigenvalues is continuous). Consequently, we may (non-uniquely) choose continuous functions $\nu_i : \mathbf{R} \to \mathbf{R}/\mathbf{Z}$ with the property that $\{\exp(i\nu_i(t))\}_{1\le i\le h}$ is the set of eigenvalues of the unitary matrix $A(it)$, where the eigenvalues are counted with multiplicity, for every $t \in \mathbf{R}$. [18] At $s = 0$, we have $A(0) = 1$ so every eigenvalue is 1; we normalize, accordingly, so that $\nu_i(0) = 0$ for every i.

In what follows, we write $\frac{A'}{A}$ for $A^{-1} \cdot \frac{dA(s)}{ds}$ when A is a matrix-valued function of s.

---

[18] Indeed, it is sufficient to check that this is so in a neighbourhood of every point; shrinking the neighbourhood appropriately, we may suppose (multiplying by a constant if necessary) that -1 is never an eigenvalue, and now simply take the ν_i to be the arguments of the eigenvalues, taken so that $\nu_1 \le \nu_2 \le \cdots \le \nu_h$.

The functions ν_i are real analytic away from a discrete set of points. Moreover, $-\nu_i' = -\frac{d\nu_i}{dt}$ is bounded below whenever differentiable, as follows from the almost-positive definiteness (see (5.2.2.5) and its analogue for 1-forms after (5.2.2.11)) of $-\frac{A'(s)}{A(s)}$: At any point at which the eigenvalues are distinct, $-\nu_i'$ is computable by perturbation theory: if v_i is an eigenvalue corresponding to $\lambda_i = \exp(i\nu_i t)$, then

$$\lambda_i' \left(= \frac{d\lambda_i}{dt} \right) = i \frac{\langle A' v_i, v_i \rangle}{\langle v_i, v_i \rangle}.$$

Note that λ_i' (and also ν_i') denote the derivative with respect to the variable t, where A' denotes the derivative with respect to $s = it$. This gives $i\nu_i' = \frac{\lambda_i'}{\lambda_i} = i\frac{\langle A'/A \ v_i, v_i\rangle}{\langle v_i, v_i \rangle}$. So $-\nu_i' = \langle -\frac{A'}{A} v_i, v_i \rangle / \|v_i\|^2$, but the Maass-Selberg relations (see (5.2.2.5) and page 78) imply that this is bounded below. Note for later use that

$$\sum -\nu_i' = \text{trace} \frac{-A'}{A} = -\frac{(\det A)'}{\det A}. \tag{5.9.2.1}$$

We may also give *upper* bounds for $|-\nu_i'|$, although these are more subtle. In the arithmetic case, at least, there exists for every j constants A, B such that

$$\|(\Phi^{\pm})^{(j)}(s)\| \leq A |\log(2 + |s|)|^B, \quad s \in i\mathbf{R}. \tag{5.9.2.2}$$

See [**62**] for a proof of the corresponding fact about L-functions. (It is most unlikely that we really need this estimate, but it is convenient in writing the proof with more explicit constants.) In particular,

$$\begin{aligned} \frac{d}{dt}(-\nu_i + 4t\log Y) &= 4\log Y + O(a(\log\log Y)^b) \\ &\asymp (\log Y) \end{aligned} \tag{5.9.2.3}$$

(and thus a similar bound for A), for suitable constants a, b, whenever

$$|t| \leq T_{\max} := (\log Y)^{100}. \tag{5.9.2.4}$$

We will often use $T_{\max}$ as a convenient upper bound for the eigenvalues that we consider. Its precise choice is unimportant (although if we only had weaker bounds for the size of ν_i' we would use a smaller size for $T_{\max}$). As usual the notation $\asymp$ means that the ratio is bounded both above and below.

There are corresponding estimates for the Eisenstein series itself. What we need can be deduced from the fact that the Eisenstein series $E(s, f)$ and $E(s, \nu)$ have no poles around $s = it$ in a ball of radius $\geq \frac{C}{\log(1+|t|)}$ (see [**62**] for a proof of the corresponding fact about L-functions) and also the bound there:

$$\|E(\sigma + it, f)\|_{L^\infty(M_Y)} \leq a((1+|t|)Y)^b \quad \left(|\sigma| \leq \frac{C}{\log(1+|t|)} \right), \tag{5.9.2.5}$$

for suitable absolute a, b. Similar bounds apply to any s-derivative of $E(s, f)$. To obtain, for instance, an estimate for the derivative of E, one uses Cauchy's integral formula and (5.9.2.5).

We are interested in the solutions $s = it$ ($t \in \mathbf{R}$) to

$$f(it) := \det(1 - Y^{-4it} A(it)) = 0,$$

that is to say, those $s = it$ for which there exists j with

$$-\nu_j(t) + 4t\log(Y) \in 2\pi\mathbf{Z}. \tag{5.9.2.6}$$

Now — for sufficiently large Y — the function $-\nu_j(t) + 4t\log(Y)$ is monotone increasing, with derivative bounded away from zero. This implies, speaking informally, that the solutions to (5.9.2.6) are very regularly spaced (see figure for an example which was actually computed with the scattering matrix for functions on $\mathrm{SL}_2(\mathbf{Z}[i])$.)

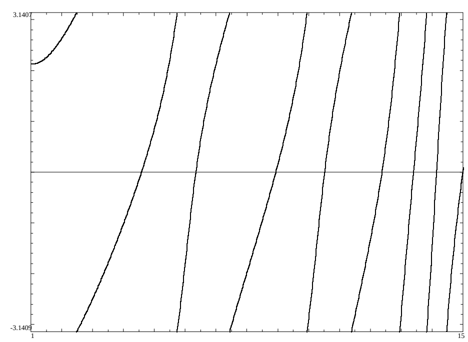


FIGURE 1. What the function $-\nu_j(t) + 4t\log(Y)$ looks like, modulo 2π

LEMMA 5.9.3. *There exists Y_0 such that, for $Y \geq Y_0$:*

(i) *The number of solutions to* (5.9.2.6) *in the interval* $0 < t \leq T$ *is given by* $\left[\frac{4T\log(Y)-\nu_j(T)}{2\pi}\right]$.

(ii) *Every small value of $f(s)$ is near a root:*
If $|f(it)| < \epsilon$ and $|t| \leq T_{\max}$, there exists $t' \in \mathbf{R}$ with $f(it') = 0$ and $|t' - t| \ll \epsilon^{1/h_{\mathrm{rel}}}$.

(iii) *Nearby solutions are almost orthogonal:*
Suppose $t_0 \in \mathbf{R}$ with $|t_0| \leq T_{\max}$; let $[x, y]$ be the positive definite inner product on $\Omega^+(0)$ defined as

$$[x, y]_{it_0} := \langle \left(4\log Y - \frac{A'(it_0)}{A(it_0)}\right) x, y\rangle;$$

we will sometimes omit the subscript it_0 where clear.

(As previously remarked, we interpret the fraction as $A(it_0)^{-1}A'(it_0)$. It is an endomorphism of $\Omega^+(0)$. This inner product arises naturally from the Maass-Selberg relations; see (5.9.1.3)*. In particular one verifies that it is positive definite so long as Y is large enough.)*

Suppose $t_1 \neq t_2$ satisfy $|t_i - t_0| < \epsilon$, and we are given v_i satisfying the relations $Y^{-4it_i}A(it_i)v_i = v_i$ for $i = 1, 2$. Then

$$[v_i, w_i]_{it_0} = O(\|v_i\|\|w_i\|(\log Y)^2\epsilon).$$

(iv) *Every almost-solution to $Y^{-4s}A(s)v = v$ is near (a linear combination of) exact solutions:*

Suppose that $|t| \leq T_{\max}, \|v\| = 1$ *are such that*

$$\|Y^{-4it}A(it)v - v\| < \epsilon. \tag{5.9.3.1}$$

Then there exists pairs $(t_i \in \mathbf{R}, w_i \in \Omega^+(0)) : 1 \leq i \leq m$, *with* $m \leq h_{\text{rel}} = \text{size}(A)$, *and an absolute constant* M, *such that:*

- $Y^{-4it_i}A(it_i)w_i = w_i$;
- $\|t - t_i\| \ll \epsilon^{1/M}$;
- $\|v - \sum w_i\| \leq (\log Y)^C \epsilon^{1/M}$;
- $\|w_i\| \leq (\log Y)^C$.

Proof. We note throughout the proof that the assumption that we are considering $\nu_i(t)$ when $|t| \leq T_{\max}$ implies $|\nu_i'(t)| = o(\log Y)$: indeed $|A'(s)| + |A''(s)| = o(\log Y)$ for such s by choice of $T_{\max}$. So for sufficiently large Y_0 we may certainly assume:

$$(-\nu_i + 4t\log Y)' \geq \log Y \tag{5.9.3.2}$$

under the same conditions.

(i) follows from monotonicity and our assumption that $-\nu_i(0) = 0$.

(ii): if $|f(it)| \leq \epsilon$, then there exists an eigenvalue λ of $A(it)$ satisfying $|1 - Y^{-4it}\lambda| \leq \epsilon^{1/h}$. Thus $\lambda = \exp(2\pi i \nu_i(t))$ for some i. The result follows from the lower bound (5.9.3.2).

(iii): Set $A_s := Y^{-4s}A(s)$ and consider

$$\begin{aligned} 0 &= \\ \langle A_{it_1}v, A_{it_2}w\rangle - \langle v, w\rangle &= \langle (A_{it_2}^{-1}A_{it_1} - 1)v, w\rangle \\ &= i(t_2 - t_1)[v, w]_{it_0} + O(|t_1 - t_2|^2(\log Y)^2\|v\|\|w\|). \end{aligned}$$

where the factor of $\log Y$ results from estimating the second derivative of $t \mapsto A_{it}$, cf. (5.9.2.2). In any case the claim follows immediately by rearranging this equation.

(iv): This proof is a little bit ugly.

Notice, first of all, that we may (by adjusting the constants C, M) always suppose that ϵ is smaller than any fixed power of $(\log Y)^{-1}$. Otherwise the statement simply asserts that there exists t_i close to t such that $Y^{-4it_i}A(it_i)$ has a fixed vector — the assertions about the position of that vector becomes vacuous — and this follows easily from what we showed in (ii).

Choose $\delta > \sqrt{\epsilon}$ such that $Y^{-4it}A(it)$ has no eigenvalues satisfying $|\lambda - 1| \in [\delta, \sqrt{\delta})$. Clearly we may do this with $\delta < \epsilon^{2^{-h-1}}$ by considering the intervals $[\epsilon^{1/2}, \epsilon^{1/4}], [\epsilon^{1/4}, \epsilon^{1/8}], \ldots$ in turn, and there are at most h_{rel} eigenvalues.

Let W be the span of all eigenvectors of $Y^{-4t}A(it)$ whose eigenvalue satisfy $|\lambda - 1| \leq \delta$; suppose that these eigenvalues correspond to $\nu_{i_1}(t), \ldots, \nu_{i_m}(t) \in \mathbf{R}/\mathbf{Z}$. Then the orthogonal projection of v to W — call it v_W – satisfies $\|v - v_W\| < \epsilon\delta^{-1} \ll \sqrt{\epsilon}$. Here v is as in the statement (5.9.3.1).

By the monotonicity and (5.9.3.2), there exist solutions $t_1, \ldots, t_h$ to

$$-\nu_j(x) + 4x\log(Y) \in 2\pi\mathbf{Z}$$

satisfying $|t_i - t| \ll \frac{\delta}{\log Y}$. Let $v_{i_1}, \ldots, v_{i_h}$ be corresponding fixed vectors for $Y^{-4t_i}A(it_i)$, normalized so that $\|v_?\| = 1$.

We have seen in (iii) that $[v_{i_a}, v_{i_b}] \ll \delta$ for $a \neq b$. This implies in particular that the $v_?$ are linearly independent. Even more precisely, there is an absolute constant C such that

$$\det[v_i, v_j] \gg (\log Y)^{-C}, \quad \det\langle v_i, v_j\rangle \gg (\log Y)^{-C}, \tag{5.9.3.3}$$

where the second inner product is the standard one introduced on $C^\infty(0)$ in (5.2.2.1). For the first inequality: the diagonal entries of the matrix are in size $\gg (\log Y)$, its off-diagonal entries are $\ll \delta$, and we may suppose, as we stated at the start of the proof, that δ smaller than any fixed power of $(\log Y)^{-1}$. For the second inequality: The ratio $[v, v]/\langle v, v\rangle$ is bounded above and below by constant multiples of $(\log Y)$. Therefore the passage between the $\langle -, -\rangle$ metric and $[-, -]$ metric distorts volumes by a factor that is bounded above and below by constant multiples of $(\log Y)^h$.

Let us now compute the projection of each v_{i_a} onto the orthogonal complement of W. Let $u = v_{i_a}$ for some a, and write

$$u = u_W + u'_W \quad (u_W \in W, u'_W \perp W).$$

Since $|t - s_i| \ll \delta/\log Y$, it follows from the remarks after (5.9.3.2) concerning the choice of $T_{\max}$ that $\|Y^{-4it}A(it)u - u\| \ll \delta$. But, all the eigenvalues of $Y^{-4it}A(it) - 1$ on $W^\perp$ are $\geq \sqrt{\delta}$, and thus it follows that $\|u'_W\| \ll \sqrt{\delta}$.

Let $v_{i_a,W}$ be the projection of v_{i_a} to W with respect to the standard metric $\langle -, -\rangle$. Then

$$\det\langle v_{i_a,W}, v_{i_b,W}\rangle \gg (\log Y)^{-C} + C\sqrt{\delta}, \tag{5.9.3.4}$$

That follows from the second inequality of (5.9.3.3) together with the fact we have just proved: $\|v_{i_a,W} - v_{i_a}\| \ll \sqrt{\delta}$. Indeed, all the entries of the matrix $\langle v_{i_a}, v_{i_b}\rangle$ are at most 1 in absolute value, and we then are modifying each entry by at most a constant multiple of $\sqrt{\delta}$, whence the determinant changes by at most a constant multiple of $\sqrt{\delta}$.

We may suppose that δ is smaller than any fixed power of $(\log Y)$, as before; so we can suppose that the right-hand side of (5.9.3.4) is $\gg (\log Y)^{-C}$. Thus $v_{?,W}$ span W. Let $v^*_{?,W} \in W$ denote the dual basis with respect to the $\langle -, -\rangle$ inner product. It also follows from (5.9.3.4) that $\|v^*_{i_a}\| \ll (\log Y)^{C'}$.

We may now write:

$$\begin{aligned} v &= (v - v_W) + v_W \\ &= (v - v_W) + \sum_a [v_W, v^*_{i_a,W}] v_{i_a,W} \\ &= O(\sqrt{\epsilon}) + \sum_a \langle v_W, v^*_{i_a,W}\rangle v_{i_a} - \sum_a \langle v_W, v^*_{i_a,W}\rangle (v_{i_a} - v_{i_a,W}) \\ &= \sum_a \langle v_W, v^*_{i_a,W}\rangle v_{i_a} + O((\log Y)^{C'}\sqrt{\delta}). \end{aligned}$$

We have shown that there exists absolute constants C, M and constants $|c_a| \leq (\log Y)^C$ such that:

$$\|v_W - \sum c_a v_{i_a}\| \ll (\log Y)^C \epsilon^{1/M}.$$

□

5.9.4. Beginning of the proof. ***Important notational warning.*** In this section, when we write *exponentially decaying* we shall mean: bounded by $a\exp(-bY)$ where a, b are constants depending only on M. For example, when we say "F is exponentially close to G" for two functions $F, G \in L^2(M)$, we mean, unless otherwise stated, that they are close in L^2-norm:

$$\|F - G\|_{L^2(M)} \leq a\exp(-bY),$$

for absolute constants a, b. The precise values of a, b, however, *may vary from one instance of this phrase to the next.* This convention, we hope, improves the readability of the text. Similarly, it will often be necessary for our argument that a quantity that is exponentially decaying be less than (say) Y^{-1}. Such an inequality is automatically valid for large enough Y. We will often proceed implicitly assuming that Y is large enough. This is certainly no loss, since the statement to be proved allows us to suppose that Y is larger than a fixed constant that we may choose. We apologize for the implicit imprecision, but again we hope this improves the readability of the text.

We shall carry out the proof in the case of co-closed 1-forms, proving assertion (b) of the theorem; the other cases (c)—(e) are handled in the same way and are discussed briefly in § 5.9.10.

Denote by $N(x)$ the number of eigenvalues of the Laplacian on co-closed 1-forms for M_Y with eigenvalue in $[0, x^2]$, and $\bar{N}(x) = M_1(x) + M_2(x)$, where

$M_1(x)$: the number of zeroes of $t \mapsto f(it)$ in $[0, x]$;
$M_2(x)$: the number of eigenvalues in $[0, x^2]$ of the Laplacian on co-closed cuspidal 1-forms on M.

Then we shall prove that, for certain absolute constants a, b we have

$$\bar{N}(T - ae^{-bY}) \leq N(T) \leq \bar{N}(T + ae^{-bY}), \quad x \leq T_{\max}. \tag{5.9.4.1}$$

where we interpret $T - ae^{-bY}$ as 0 if $T < ae^{-bY}$. This proves Theorem 5.9.1 (b).

The first inequality will come in §5.9.5 ('there is an eigenfunction of M_Y near any root of f) and the second equality will be proved in §5.9.8 ("any eigenvalue of M_Y is near a root of $f(s)$.") And the intervening section §5.9.6 explains what happens when treating eigenvalues very close to zero.

Recall (§ 5.2.2.3) that to each $\omega \in \Omega^+(0)$, we have associated an "Eisenstein series" $E(s, \omega)$, which is a 1-form on M with eigenvalue $-s^2$ under the form Laplacian (and eigenvalue $-is$ under the operator $*d$). Moreover, there exists inverse linear operators $\Phi_\pm(s) : \Omega^\pm \longrightarrow \Omega^\mp$ so that $\Phi_+(s)\Phi_-(s) = \mathrm{id}$, and so that the restriction of $E(s)$ to the cusps looks like:

$$E(s) \sim y^s\omega + y^{-s}\Phi^+(s)\omega.$$

We write, as in the statement of the Theorem,

$$f(s) = \det(\mathrm{Id} - Y^{-4s}\Phi^-(-s)\Phi^+(s)).$$

For short we write $\Phi(s) = \Phi^+(s)$.

In Lemma 5.9.3 we introduced an inner product on $\Omega^+(0)$, which is

$$[\omega_1, \omega_2]_{it_0} = \langle \big(4\log Y - A^{-1}A'(it_0)\big)\,\omega_1, \omega_2\rangle \tag{5.9.4.2}$$

and the inner product on the right hand side is the standard one introduced in (5.2.2.1). Again, this inner product arises naturally when computing the norm of the truncated Eisenstein series, as in (5.2.2.3).

REMARK. *Some remarks on small eigenvalues will be useful:*

At $s = 0$, $Y^{-4s}\Phi^{-}(-s)\Phi^{+}(s)$ is the identity transformation, and consequently $f(s) = 0$. Indeed, the derivative of $\mathrm{Id} - Y^{-4s}\Phi^{-}(-s)\Phi^{+}(s)$ *with respect to s is easily seen to be invertible for large enough s, and thus $f(s)$ has a zero of order equal to the size of Φ, which also equals the number of zero eigenvalues for the Laplacian on co-closed* 1*-forms.*

It will be useful to note that smallest nonzero eigenvalue λ_1 of the Laplacian on zero-forms on M_Y, satisfying absolute boundary conditions (i.e. the smallest Neumann eigenvalue) is bounded below as Y varies, as we see by an elementary argument; all we need is the weak bound

$$\lambda_1 \geq \mathrm{const} \cdot (\log Y)^{-1}. \tag{5.9.4.3}$$

5.9.5. There is an eigenfunction of M_Y near any root of f. We show the first inequality of (5.9.4.1), based on the idea sketched in § 5.9.1.1 of constructing "approximate eigenfunctions."

Let $T \leq T_{\max}$. For each $s \in i\mathbf{R}_{\geq 0}$ with $f(s) = 0, |s| \leq T$, we choose an orthonormal basis $\mathcal{B}_s$ for the set of solutions to

$$\omega = Y^{-2s}\Phi^{-}(-s)\bar{\omega} \text{ and } Y^{s}\bar{\omega} = Y^{-s}\Phi^{+}(s)\omega \tag{5.9.5.1}$$

where orthonormality is taken with respect to the inner product $[-,-]_s$ (see (5.9.4.2) for definition).

Let $\mathcal{B}_{\mathrm{cusp}}$ be a basis for the set of coclosed cuspidal 1-forms on M with eigenvalue $\leq T^2$, and set

$$\mathcal{B}_{eis} = \{(s,\omega) : f(s) = 0, \omega \in \mathcal{B}_s\};$$
$$\mathcal{B} = \mathcal{B}_{eis} \bigcup \mathcal{B}_{\mathrm{cusp}}$$

We shall associate to each element of $\mathcal{B}$ an element of $L^2(M_Y)$ that lies in the subspace spanned by co-closed 1-forms of eigenvalue $\leq T + a\exp(-bY)$, for some absolute positive constants a, b, and then show that the resulting set $\mathcal{F}$ is linearly independent.

Let $(s,\omega) \in \mathcal{B}_{eis}$, and let $\bar{\omega}$ be the other component of the solution to (5.9.1.1). We normalize so that $\|\omega\| = 1$; the same is true for $\bar{\omega}$ because $\Phi^{\pm}$ is unitary. Set $F = E(s,\omega) + E(-s,\overline{\omega})$. We saw in § 5.9.1.1 that F "almost" satisfies absolute boundary conditions, in the sense that, for example, the contraction of dF with a boundary normal is exponentially small in Y.

REMARK. *When $s = 0$, then the set of solutions to* (5.9.5.1) *consists of all of $\Omega^{+}(0)$; moreover, $\overline{\omega} = \Phi^{+}(0)\omega$ and $F = 2E(0,\omega)$.*

The next step is to modify F slightly so as to make it satisfy absolute boundary conditions, but losing, in the process, the property of being an exact Δ-eigenfunction. This can be done in many ways; for definiteness one can proceed as follows: fix a smooth monotonically increasing function $h : \mathbf{R} \to \mathbf{C}$ such that $h(x) = 1$ for $x > 1$ and $h(x) = 0$ for $x < 0$, and let $h_Y : y \mapsto h(Y/2 - y)$. Now, near each cusp , considered as isometric to a quotient of $\mathbf{H}^3 : y \leq T$, write

$$F = y^s\omega + \Phi^{+}(s)y^{-s}\omega + y^s\bar{\omega} + y^{-s}\Phi^{-}(s)\bar{\omega} + F_0,$$

and replace F_0 by $F_0 \cdot h_Y$; thus we replace F by the form F' which is given, in each cusp, by

$$F' = y^s\omega + \Phi^{+}(s)y^{-s}\omega + y^s\bar{\omega} + y^{-s}\Phi^{-}(s)\bar{\omega} + F_0 \cdot h_Y,$$

Then F' satisfies absolute boundary conditions. On the other hand, it is clear that

$$\|(\Delta + s^2)F'\| \ll \exp(-bY) \tag{5.9.5.2}$$

for some absolute constant b. (Note that F' is no longer co-closed, but it is almost so, as we shall check in a moment.)

We now see there is an co-closed eigenfunction close to F': Set $\epsilon = \exp(-bY)$, where b is the constant of the estimate (5.9.5.2), and let W be the space spanned by co-closed 1-forms on M_Y with eigenvalue satisfying $\|\lambda + s^2\| \leq \sqrt{\epsilon}$ and let $F = F_{s,\omega}$ be the projection of F' onto W. We refer to $F_{s,\omega}$ as *the eigenfunction associated to* (s, ω); this is a slight abuse of terminology as $F_{s,\omega}$ need not in general be a Laplacian eigenfunction, but it behaves enough like one for our purposes.

Then:

(i) $F_{s,\omega}$ is exponentially close to $E(s, \omega) + E(-s, \bar{\omega})$, where $\bar{\omega}$ is defined by (5.9.1.1); in particular, $\|F_{s,\omega}\|_{L^2(M_Y)} \asymp (\log Y)$.
(ii) $F_{s,\omega}$ is a linear combination of eigenfunctions with eigenvalue exponentially close to $-s^2$.

In order to verify these, we need to check that F' is "almost co-closed", so that the projection onto co-closed forms does not lose too much.

Indeed, first of all, the orthogonal projection of F' onto the orthogonal complement of co-closed forms has exponentially small norm. This orthogonal complement is spanned by df, for f a function satisfying absolute (Neumann) boundary conditions and orthogonal to 1; now

$$\langle F', df \rangle_{M_Y} = \langle d^* F', f \rangle_{M_Y} = \langle d^*(F' - F), f \rangle_{M_Y},$$

where we used the fact that $d^* F = 0$. But $F' - F$ is supported entirely "high in cusps;" we may express the last quantity as a sum, over cusps, of terms bounded by $\|d^* F_0(1 - h_Y)\| \|f\|$. Now $\|df\| \gg \|f\| / \log Y$ because of (5.9.4.3). So,

$$\langle F', df \rangle \ll (\log Y) \|d^* F_0(1 - h_Y)\| \cdot \|df\|,$$

Since $d^* F_0(1 - h_Y)$ is certainly exponentially small, we arrive at the conclusion that the projection of F' onto the orthogonal complement of co-closed forms has exponentially small norm. Similarly, the orthogonal projection of F' onto the span of eigenvalues *not in* $[-s^2 - \sqrt{\epsilon}, -s^2 + \sqrt{\epsilon}]$ has, by (5.9.5.2), a norm $\ll \sqrt{\epsilon}$.

This concludes the proof of claim (i) about $F_{s,\omega}$; claim (ii) follows directly from the definition.

In an exactly similar way, we may associate to every co-closed cusp 1-form ψ a co-closed F_ψ on M_Y satisfying corresponding properties:

(i) F_ψ is exponentially close to $\psi|_{M_Y}$;
(ii) F_ψ is a linear combination of eigenfunctions on M_Y with eigenvalue exponentially close to the eigenvalue of ψ.

The maps $(s, \omega) \mapsto F_{s,\omega}, \psi \mapsto F_\psi$ give a mapping from $\mathcal{B} = \mathcal{B}_{eis} \cup \mathcal{B}_{\text{cusp}}$ to the vector space spanned by all co-closed forms on M_Y of eigenvalue $\leq T^2 + a \exp(-bY)$, for some absolute a, b. Call the image of this mapping $\mathcal{F}$. We argue now that $\mathcal{F}$ is linearly independent:

Indeed, in any relation

$$\sum a_j F_{s_j,\omega_j} + \sum_k b_k F_{\psi_k} = 0, \quad (0 \neq a_j \in \mathbf{C}, 0 \neq b_k \in \mathbf{C}). \tag{5.9.5.3}$$

we may suppose (by applying an orthogonal projection onto a suitable band of Laplacian eigenvalues) that all the s_j are "close to one another," i.e., there exists $T_0 \le T+1$ so that $|s_j - iT_0|$ and $|s_{\psi_k} - iT_0|$ are all exponentially small.

By § 5.9.2, the number of solutions to $f(s_j) = 0$ in the range $|T_0^2 - s_j^2| < a\exp(-bY)$ is absolutely bounded for Y large enough. As for the number of ψ_k that may appear in (5.9.5.3), this is (by the trivial upper bound in Weyl's law) polynomial in T_0. Therefore the number of terms in the putative linear relation is at worst polynomial in $T_{\max}$.

We shall derive a contradiction by showing that the "Gram matrix" of inner products for F_{s_j,ω_j} and F_{ψ_j} is nondegenerate (here j,k vary over those indices appearing in (5.9.5.3)). Clearly $\langle F_{s_j,\omega_j}, F_{\psi_k}\rangle$ is exponentially small and $\langle F_{\psi_j}, F_{\psi_k}\rangle$ differs from δ_{jk} by an exponentially small factor. These statements follow from the property (i) enunciated previously for both $F_{s,\omega}$ and F_ψ. As for the terms,

$$\langle F_{s_i,\omega_i}, F_{s_j,\omega_j}\rangle = [\omega_i, \omega_j] + \text{exponentially decaying} \tag{5.9.5.4}$$

by the Maass-Selberg relations; this is proved as (5.9.1.3).

But by the analysis of zeroes, assertion (iii) of Lemma 5.9.3, $[\omega_i, \omega_j]$ is itself exponentially small if $i \ne j$. Consequently, the Gram matrix is nonsingular by "diagonal dominance;" contradiction.

We have now exhibited a set of linearly independent co-closed 1-forms with eigenvalue $\le T^2+$ exponentially small. This establishes the first inequality in (5.9.4.1) (the precise value of the constants a, b in that equation are obtained by tracing through the estimates mentioned here).

5.9.6. Analysis of eigenvalues on M_Y near 0 by passage to combinatorial forms. We continue our proof Theorem 5.9.1 (a).

We will argue that any "very small" eigenvalue of the Laplacian on co-closed 1-forms on M_Y is in fact exactly zero. We prove in fact a stronger statement:

A. Any eigenvalue λ on co-closed forms on M_Y satisfying $\lambda \le AY^{-B}$ is actually zero, for suitable constants;
B. If $\omega_1, \dots, \omega_r$ form an orthonormal basis for $\Omega^+(0)$, then the functions F_{0,ω_i} constructed in the previous section[19] are a basis for harmonic forms on M_Y.

For this discussion, it does not, in fact, matter whether we discuss only co-closed 1-forms or all 1-forms, because of (5.9.4.3): any 1-form with eigenvalue polynomially close to zero must be co-closed. As always, however, we work with respect to absolute boundary conditions.

We will compare the analysis of j-forms and of *combinatorial* j-forms, that is to say, the cochain complex with respect to a fixed triangulation.

Fix a triangulation of $M_{\le Y}$ where each triangle has hyperbolic sides of length ≤ 1. This can be done with $O(\log Y)$ simplices, and for definiteness we do it in the following way: first fix $Y_0 \ge 1$ sufficiently large, then triangulate $M_{\le Y_0}$ for some fixed Y_0 (this can be done — any differentiable manifold may be triangulated). Now to triangulate the remaining "cusp region" $\mathcal{C}_{Y_0} - \mathcal{C}$, split it into regions of the type $M \le \mathrm{Ht}(x) \le 2M$ plus an end region $2^j M \le \mathrm{Ht}(x) \le Y$; we will assume that $2^j M \in [Y/2, Y/4]$. Each such region is diffeomorphic to the product $\mathbf{R}^2/L \times [0,1]$

[19] A priori, these functions were not guaranteed to be Laplacian eigenforms; however, the preceding statement (A) guarantees that they are in this case.

for a suitable lattice L, simply via the map $(x_1, x_2, y) \mapsto (x_1, x_2) \times \frac{y-M}{M}$ (with the obvious modification for the end cylinder); we fix a triangulation for the latter and pull it back.

We equip the resulting cochain complex with the "combinatorial" inner product: The characteristic function of distinct simplices form orthonormal bases. This being done, we define a "combinatorial" Laplacian on the cochain complex just as for the de Rham complex, i.e. $\Delta := d_c d_c^* + d^* d$, where d_c is the differential and d_c^* its adjoint with respect to the fixed inner product.

Suppose given an $N \times N$ symmetric integer matrix, all of whose eigenvalues μ satisfy the bound $|\mu| \leq A$, where A is real and larger than 1. Then also every nonzero eigenvalue satisfies the lower bound $|\mu| > A^{-N}$ if $\mu \neq 0$; this is so because all eigenvalues are algebraic integers, and in particular they can be grouped into subsets with integral products. On the other hand, every eigenvalue of the combinatorial Laplacian is bounded by the maximal number of simplexes "adjacent" to any given one: if we write out the combinatorial Laplacian with respect to the standard basis, the L^1-norm of each row is thus bounded. The same goes for $d_c^* d_c$ and $d_c d_c^*$. Therefore,

(5.9.6.1) Any nonzero eigenvalue λ of the combinatorial Laplacian,

$$\text{or of } d_c^* d_c, \text{ or of } d_c d_c^*, \text{ satisfies } |\lambda| \gg Y^{-m}$$

for some absolute constant m. This simple bound will play an important role.

Any j-form ω on $M_{\leq Y}$ induces (by integration) a combinatorial form, denoted ω_c, on the j-simplices of the triangulation. If $d\omega = 0$, then ω_c is closed, i.e. vanishes on the boundary of any combinatorial $(j+1)$-cycle.

Let us now consider $V(\epsilon)$, the space spanned by 1-forms on $M_{\leq Y}$ of eigenvalue $\leq \epsilon$. For topological reasons,

$$\dim V(\epsilon) \geq \dim V(0) = \dim H^1(M_{\leq Y}),$$

no matter how small ϵ. We shall show that $\dim V(\epsilon) = \dim H^1$ holds if $\epsilon < aY^{-b}$ for some absolute constants a, b. In particular, this shows that all eigenvalues less than aY^{-b} are identically zero.

Suppose to the contrary that $\dim V(\epsilon) > \dim H^1(M_{\leq Y})$. Consider the map from $V(\epsilon)$ to $\ker(d_c)/\mathrm{im}(d_c)$ given by first applying $\omega \mapsto \omega_c$, then taking the orthogonal projection onto $\ker(d_c)$, and finally projecting to the quotient. Since we're assuming that $\dim V(\epsilon) > \dim H^1(M_{\leq Y})$ this map must have a kernel. Let ω be a nonzero element of this kernel. The idea is to construct a combinatorial antiderivative for ω, then to turn it into an actual "approximate" antiderivative; the existence of this will contradict the fact that ω is (almost) harmonic.

There is an absolute $m > 0$ such that, for any $\omega \in V(\epsilon)$, we have:

$$\|d_c \omega_c\|_{L_c^2} \ll \epsilon^{1/2} Y^m \|\omega\|, \tag{5.9.6.2}$$

where L_c^2 denotes the combinatorial inner product and a is an absolute constant. Indeed, if K is any 2-simplex, we have $\langle d_c \omega_c, K \rangle = \langle \omega_c, \partial K \rangle = \int_{\partial K} \omega = \int_K d\omega$. But

$$\|d\omega\|_{L^\infty} \ll \epsilon^{1/2} Y^{m_1} \|\omega\|, \tag{5.9.6.3}$$

as follows from (say) the Sobolev inequality; the factor Y^{m_1} arises from the fact that the manifold $M_{\leq Y}$ has injectivity radius $\asymp Y^{-1}$. That shows (5.9.6.2).

Now (5.9.6.1) shows:

$$\mathrm{dist}(\omega_c, \ker(d_c)) \ll \epsilon^{1/2} Y^{m_2} \|\omega\|$$

for some absolute constant m_2; here the distance dist is taken with respect to the L^2-structure on the space of combinatorial forms. In fact, write $\omega_c = \omega_{c,k} + \omega_c'$ where $\omega_{c,k} \in \ker(d_c)$ and $\omega_c' \perp \ker(d_c)$. Then $\|d_c \omega_c'\| \ll \epsilon^{1/2} Y^m \|\omega\|$; in particular $\langle d_c^* d_c \omega_c', \omega_c' \rangle \ll (\epsilon^{1/2} Y^m \|\omega\|^2)$, and then we apply (5.9.6.1).

By our choice of ω, the orthogonal projection $\omega_{c,k}$ of ω_c to $\ker(d_c)$ belongs to image(d_c). So, there exists a combinatorial 0-form f_c such that:

$$\|\omega_c - d_c f_c\| \ll \epsilon^{1/2} Y^{m_2} \|\omega\|. \tag{5.9.6.4}$$

We now try to promote the combinatorial 0-form f_c to a function, which will be an approximate antiderivative for ω.

Let us now fix a base vertex of the triangulation, Q. For any other three-dimensional simplex σ fix a path of minimal combinatorial length γ_σ from Q to a vertex Q_σ of the simplex. For every point in σ, fix a linear path[20] γ_P from Q_σ to P. Now define $f : M_{\leq Y} \to \mathbf{C}$ via

$$f(P) = \int_{\gamma_\sigma + \gamma_P} \omega,$$

where $\gamma_\sigma + \gamma_P$ denotes concatenation of paths.

This defines f off a set of measure zero; f has discontinuities, i.e., doesn't extend to a continuous function on M, but they are small because of (5.9.6.3) and (5.9.6.4). Precisely, for any point P on M and a sufficiently small ball B about P, we have

$$\sup_{x,y \in B} |f(x) - f(y)| \ll \ \epsilon^{1/2} Y^{m_3} \|\omega\|$$

for suitable m_3.

On the interior of every simplex S, df is very close to ω: precisely,

$$\|df - \omega\|_{L^\infty(S^\circ)} \ll \epsilon^{1/2} Y^{m_4} \|\omega\|. \tag{5.9.6.5}$$

Again this follows from Stokes' formula and (5.9.6.3).

Finally, f almost satisfies absolute boundary conditions: if we denote by ∂_n the normal derivative at the boundary, we have

$$|\partial_n f|_{L^\infty} \leq \epsilon^{1/2} Y^{m_5} \|\omega\|. \tag{5.9.6.6}$$

This follows from (5.9.6.3) and the fact that ω itself satisfies absolute boundary conditions.

Now it is routine to smooth f to get a *smooth* function $\tilde{f}$ that satisfies (5.9.6.6) and (5.9.6.5), perhaps with slightly worse m_3, m_4 and implicit constants. [21]

[20] The notion of linear depends on an identification of the simplex with a standard simplex $\{x_i \in \mathbf{R} : x_i \geq 0, \sum x_i = 1\}$. For our purposes, the only requirements on these identifications is that their derivatives with respect to standard metrics should be polynomially bounded in Y; it is easy to see that this is achievable.

[21] We explain how to do this in some detail in a coordinate chart near the boundary. One then splits f into a part supported near and away from the boundary; the part away from the boundary can be handled by convolving f with a PU$)_2$-bi-invariant smooth kernel on $\mathrm{PGL}_2(\mathbf{C})$. Let $(x, y, z) \in S = \mathbf{R}/\mathbf{Z} \times \mathbf{R}/\mathbf{Z} \times [0, 1]$. Given a function $f : S \to \mathbf{C}$ that satisfies $f_z(x, y, 1) = 0$, we extend it to a function $f' : \mathbf{R}/\mathbf{Z} \times \mathbf{R}/\mathbf{Z} \times [0, 2] \to \mathbf{C}$ by forcing the symmetry $f(x, y, 2 - z) = f(x, y, z)$. Now let $\omega \in \mathbf{C}_c^\infty(\mathbf{R}^3)$ be such that $\int \omega = 1$, and let $\omega_\delta(\mathbf{x}) = \delta^{-3} \omega(\mathbf{x}/\delta)$. Now set $\tilde{f} = f' \star \omega$ (convolution on the abelian group $(\mathbf{R}/\mathbf{Z})^2 \times \mathbf{R}$). Note in particular that $d\tilde{f} = df \star \omega$.

Now

$$\langle \omega, \omega \rangle = \langle d\tilde{f}, \omega \rangle + \langle d\tilde{f} - \omega, \omega \rangle \tag{5.9.6.7}$$

$$= \langle \tilde{f}, d^* \omega \rangle + \text{boundary term} + \langle d\tilde{f} - \omega, \omega \rangle. \tag{5.9.6.8}$$

But $\langle \tilde{f}, d^*\omega \rangle$ vanishes because ω is co-closed; the boundary term is bounded by $\epsilon^{1/2} Y^{m_6} \|\omega\|^2$ because $\tilde{f}$ almost satisfies Neumann boundary conditions, i.e. (5.9.6.6). So

$$|\langle \omega, \omega \rangle| \ll \epsilon^{1/2} Y^{m_7} \|\omega\|^2$$

for suitable B. This is a contradiction whenever $\epsilon \leq AY^{-B}$ for suitable A, B, and completes the proof that

$$\dim V(\epsilon) = \dim V(0) \quad (\epsilon \ll Y^{-B}).$$

Remark 5.9.7. For later usage we compute the asymptotic behavior of the regulator for $M_{\leq Y}$; we'll show

$$\mathrm{reg}(H_1(M_{\leq Y})) \sim \mathrm{reg}(H_1(M))(\log Y)^{-h_{\mathrm{rel}}} \tag{5.9.7.1}$$

where h_{rel} is the number of relevant cusps of Y, and $\sim$ denotes that the ratio approaches 1 as $Y \to \infty$.

Let $\psi_1, \ldots, \psi_k$ be an orthogonal basis for coclosed harmonic cusp forms on M. (Thus, $k = \dim H^1_c(M, \mathbf{C})$). Let ν_i be the eigenfunctions associated to ψ_i under the map $\mathcal{B} \to \mathcal{F}$ of § 5.9.5. (Again, we emphasize that we only know that these are eigenfunctions because of what we proved in this section §5.9.6.)

Let $\omega_{k+1}, \ldots, \omega_{k+h_{\mathrm{rel}}}$ be a basis for $\Omega = \Omega^+(0)$; note that

$$1 - Y^{-4s} \Phi^-(-s) \Phi^+(s) \equiv 0$$

for $s = 0$, because $\Phi^-(s)$ and $\Phi^+(s)$ are inverses; thus we may form the eigenfunction $\nu_i := F_{0,\omega_i}$ on M_Y associated to $(0, \omega_i)$ for each $k+1 \leq i \leq k + h_{\mathrm{rel}}$. We set $\psi_i := E(0, \omega_i)$ for $k+1 \leq i \leq k + h_{\mathrm{rel}}$. Then ν_i is exponentially close to ψ_i. (As usual, this means: in L^2 norm on M_Y).

Again $\nu_1, \ldots, \nu_{k+h_{\mathrm{rel}}}$ are linearly independent and are harmonic. Since $\dim H^1(M_Y, \mathbf{C}) = \dim H^1(M, \mathbf{C}) = k + h_{\mathrm{rel}}$, therefore,

The ν_i, for $1 \leq i \leq k + h_{\mathrm{rel}}$, form a basis for harmonic forms.

Let $\gamma_1, \ldots, \gamma_{k+h_{\mathrm{rel}}}$ be a basis for $H_1(M_{\leq Y}, \mathbf{Z})$ modulo torsion. Then, to check (5.9.7.1), we note first of all that $\int_{\gamma_i} \nu_j \overset{Y \to \infty}{\to} \int_{\gamma_i} \psi_j$ for each i, j, i.e. the period matrix for M_Y approaches that for M. This requires a little more than simply the fact that ν_j is exponentially close to ψ_j which is, a priori, a statement only in L^2-norm. We omit the easy proof. Now (5.9.7.1) follows since

$$\det\left(\langle \nu_i, \nu_j \rangle\right)_{1 \leq i,j \leq k+h_{\mathrm{rel}}} \sim (\log Y)^{h_{\mathrm{rel}}} \det\left(\langle \psi_i, \psi_j \rangle\right)_{1 \leq i,j \leq k+h_{\mathrm{rel}}},$$

as follows from the fact that the ν_i, ψ_i are exponentially close, and the definition (5.3.1.1) of the inner product on Eisenstein harmonic forms for M.

5.9.8. Any eigenvalue of M_Y arises from a root of $f(s)$ or a cusp form. Again $T \leq T_{\max}$.

Let $\mathcal{F}$ be as constructed in § 5.9.5; the set of (near)-eigenforms associated to $\mathcal{B}_{cusp} \bigcup \mathcal{B}_{eis}$. Let $\mathcal{F}_0$ be the set of eigenforms of eigenvalue 0 belonging to $\mathcal{F}$, i.e., the harmonic forms.

Let

$$\epsilon_0 = AY^{-B} \tag{5.9.8.1}$$

be as in § 5.9.6, i.e., such that any coclosed eigen-1-form of eigenvalue $\lambda \leq \epsilon_0$ is known to satisfy $\lambda = 0$, and therefore to belong to the span of $\mathcal{F}_0$. As in the statement of the theorem, we write $\delta = a\exp(-bY)$; the constants b will be chosen sufficiently small to make various steps in the proof work.

We shall show that — for suitable b:

> (*) any co-closed eigen-1-form on M_Y with eigenvalue in $(\epsilon_0, (T-\delta)^2]$ that is orthogonal to all elements of $\mathcal{F} - \mathcal{F}_0$ is identically zero.

Since we already know that any co-closed 1-form with eigenvalue in $[0, \epsilon_0]$ belongs to the span of $\mathcal{F}_0$, this gives the second inequality of (5.9.4.1), and completes the proof of Theorem 5.9.1 part (b).

The idea of the proof: the Green identity (recalled below) shows that any 1-form f on M_Y that's a Laplacian eigenfunction cannot have constant term which is "purely" of the form $y^{-s}(adx_1 + bdx_2)$; its constant term must also[22] contain a piece $y^s(a'dx_1 + b'dx_2)$. But by subtracting a suitable Eisenstein series (rather: its restriction to M_Y) from f we can arrange that the constant term of f indeed looks like $y^{-s}(adx_1 + bdx_2)$. This leads to a new function M_Y whose constant term is very close to zero; we then seek to show it is very close to a restriction of a cusp form from M. This we do by spectrally expanding f on M. The trickiest point is to control the inner product of f with an Eisenstein series $E(\omega, t)$ when their eigenvalues are very close; for that, we use a pole-free region for Eisenstein series.

Let us suppose that $\eta = \eta_s$ is an co-closed eigenfunction of the 1-form Laplacian on $M_{\leq Y}$ with eigenvalue $-s^2 \in (\epsilon_0, (T-\delta)^2]$ and with $\|\eta\| = 1$, that is to say, $\int_{M_Y} \langle \eta, \eta \rangle = 1$. Note we are always assuming $T \leq T_{\max}$. Suppose that $\eta \perp \mathcal{F}$; we shall show $\eta = 0$.

The Green identity gives

$$(5.9.8.2)\quad \int_{M_{\leq Y}} \langle \omega_1, \Delta\omega_2 \rangle - \langle \Delta\omega_1, \omega_2 \rangle = B_1 + B_2,$$

$$B_1 = \int_{\partial M_Y} \langle d^*\omega_1, \omega_2(X) \rangle - \langle \omega_1(X), d^*(\omega_2) \rangle,$$

$$B_2 = \int_{\partial M_Y} \langle X.d\omega_1, \omega_2 \rangle - \langle \omega_1, X.d\omega_2 \rangle.$$

where X is a unit normal to the boundary – in our context it is simply $y\partial_y$ – and $\omega(X)$ means the evaluation of ω on X, while $X.d\omega$ is the result of contracting $d\omega$ with X.

At first, we do not use the fact that η satisfies absolute boundary conditions. Consider a fixed cusp of M, and write the constant term of η (see (5.2.1.1) for definition) as

$$(\eta)_N = \omega_s y^s + \omega_{-s} y^{-s} + \bar{\omega}_s y^s + \bar{\omega}_{-s} y^{-s}$$

where $\omega_s, \omega_{-s} \in \Omega^+ = \Omega^+(0), \bar{\omega}_s, \bar{\omega}_{-s} \in \Omega^-(0)$. Note that assumption that η is co-closed, i.e., that $d^*\eta \equiv 0$, implies that η_N does not contain terms of the form $y^t dy$.

[22]This corresponds to the fact that an "incoming" wave reflects and produces an "outgoing" wave. In other words there is a symplectic pairing on the possible asymptotics, with respect to which the possible realizable asymptotics form a Lagrangian subspace.

Define

$$\omega = \eta - (E(s, \omega_s) + E(-s, \bar{\omega}_s)))\,|_{M_Y},$$

a co-closed form on M_Y whose constant terms equals

$$\omega_N := (\bar{\omega}_{-s} + \omega_{-s} - \Phi^+(s)\omega_s - \Phi^-(-s)\bar{\omega}_s)y^{-s}.$$

In this way we have killed the y^s part of the constant term of η.

We apply the Green identity (5.9.8.2) with $\omega_1 = \omega_2 = \omega$ *and also* with Y replaced by $Y' := Y/2$. The left-hand side equals zero. By (5.2.1.2), we deduce that

$$|\omega(X)|_{L^\infty(\partial M_{Y/2})} \ll \exp(-bY),$$

on the other hand, $\|X.d\omega - s\omega_N\|_{L^\infty(\partial M_{Y/2})} \ll \exp(-bY)$ by the same (5.2.1.2). The Green identity implies that $(s-\bar{s})\|\omega_N\|^2_{L^2(\partial M_{Y/2})} \ll \exp(-bY)$; to be explicit, the norm here is simply $\|y^{-s}(adx_1 + bdx_2)\|_{L^2(\partial M_{Y/2})} \propto (|a|^2 + |b|^2)$, the constant of proportionality being the area of the cusp.

This estimate implies (by assumption, $-s^2 \leq \epsilon_0$, which gives a polynomial-in-Y lower bound on $s - \bar{s}$):

$$\|\omega_N\|^2 \ll \exp(-bY/2). \tag{5.9.8.3}$$

Making (5.9.8.3) explicit (by splitting into $dx_1 + idx_2$ and $dx_1 - idx_2$ parts, i.e. "holomorphic" and "antiholomorphic" parts):

$$\|\omega_{-s} - \Phi^-(-s)\bar{\omega}_s\|,\ \|\bar{\omega}_{-s} - \Phi^+(s)\omega_s\| \ll \exp(-bY/2). \tag{5.9.8.4}$$

In words, (5.9.8.4) says that the constant term of η resembles the consant term of $E(s, \omega_s) + E(-s, \bar{\omega}_s)$. We now want to check that s needs to be near a root of $f(s)$, which we will do by seeing what the boundary conditions tell us; we then want to argue that η itself differs from $E(s, \omega_s) + E(-s, \bar{\omega}_s)$ by (something very close to the) restriction of a cusp form from M, which we will do by spectral expansion.

We now use the absolute boundary conditions on η: they mean that $d\eta$ contracted with a boundary vector should be zero, which implies that $s(Y^s\omega_s + Y^s\bar{\omega}_s - Y^{-s}\omega_{-s} - Y^{-s}\bar{\omega}_{-s}) = 0$. Note that we have *exact* equality here: integrating $X.d\eta$ over the boundary picks up solely the constant term of η. (Thus, the boundary conditions gives one constraint for each Fourier coefficient, and this is the zeroth Fourier coefficient.)

Thus, considering "holomorphic" and "antiholomorphic" components,

$$\|Y^s\omega_s - Y^{-s}\omega_{-s}\| = \|Y^s\bar{\omega}_s - Y^{-s}\bar{\omega}_{-s}\| = 0. \tag{5.9.8.5}$$

Taken together, (5.9.8.4) and (5.9.8.5) imply that

$$\|Y^s\omega_s - Y^{-s}\Phi^-(-s)\bar{\omega}_s\|, \|Y^s\bar{\omega}_s - Y^{-s}\Phi^+(s)\omega_s\| \ll \exp(-bY/2),$$

and since $\Phi^\pm$ are unitary, we deduce that

$$\|\omega_s - Y^{-4s}\Phi^-(-s)\Phi^+(s)\omega_s\| \ll \exp(-bY/2). \tag{5.9.8.6}$$

This shows (more or less – details below) that s must be near zero of $f(s)$.

We now show estimates:

$$\|\omega_s\| \gg c_1 Y^{-c_2}, \|\omega\|_{L^2(M_Y)} \ll_N Y^{-N} \tag{5.9.8.7}$$

for suitable constants c_1, c_2 and for arbitrary N.

The proofs of both parts of (5.9.8.7) have the same flavor: we need to show that for any co-closed 1-form ξ (namely, either η or ω), then ξ is small if its constant term is. More precisely, we will prove

LEMMA 5.9.9. *For any $L > 1$, there are constants a, b, a' such that, for $\xi \in \{\eta, \omega\}$,*

$$\|\xi\|_{L^2(M_Y)} \ll aY^b \|\xi_N\| + a'Y^{-L}\|\eta\|_{L^2(M_Y)},$$

where the norm on ξ_N is defined in (5.9.9.5).

We prove this in § 5.9.9.1. Of course $\|\eta\|$ is supposed to be 1, but we write it as $\|\eta\|$ to preserve homogeneity of the equation.

The second assertion of (5.9.8.7) is directly the Lemma applied to $\xi = \omega$, together with (5.9.8.3); the first assertion of (5.9.8.7) by applying this Lemma to $\xi = \eta$ together with (5.9.8.4) and (5.9.8.5).

But (5.9.8.7) concludes the proof of (*) (on page 126, that is, the main assertion of the present section). In detail: The first estimate implies via Lemma 5.9.3 part (iv) that ω_s is close to a linear combination of solutions $\sum \omega_i'$ where (s_i', ω_i') is an exact solution of $Y^{-4s_i'}\Phi^-(-s_i')\Phi^+(s_i')\omega_i' = \omega_i'$ and all the s_i' are very close to s (in particular, $s_i' \neq 0$). The cited lemma gives certain bounds on the norms of the ω_i', which we use without comment in what follows. The second estimate of (5.9.8.7) implies that $\|\eta - E(s,\omega_s) - E(-s, \bar{\omega}_s)\|_{L^2(M_Y)}$ is $\ll Y^{-N}$, and so also[23] $\|\eta - \sum F_{s_i,\omega_i}\|_{L^2(M_Y)} \ll Y^{-N}$. In particular, if η were orthogonal to all $F_{s,\omega} \in \mathcal{F} - \mathcal{F}_0$ then $\eta = 0$, as claimed (since the matrix of inner products is nondegenerate – see discussion around (5.9.5.4)).

5.9.9.1. *Proof of Lemma 5.9.9.* We apologize for not giving a unified treatment of $\xi = \eta$ and $\xi = \omega$, but the two cases are slightly different because η satisfies boundary condtions and ω does not.

We regard ξ as a 1-form on M by extending it by zero; call the result $\tilde{\xi}$. Of course $\tilde{\xi}$ is not continuous, a point that will cause some technical trouble in a moment.

Outline: We shall expand $\tilde{\xi}$ in terms of a basis of 1-forms on M and take L^2-norms:

(5.9.9.1)
$$\tilde{\xi} = \sum_{\psi} |\langle \tilde{\xi}, \psi\rangle|^2 + \sum_{\nu} \int_{t=-\infty}^{\infty} |\langle \tilde{\xi}, E(\nu, it)\rangle|^2 \frac{dt}{2\pi} + \sum_f \int_0^{\infty} |\langle \tilde{\xi}, \frac{dE(f,it)}{\sqrt{1+t^2}}\rangle|^2 \frac{dt}{2\pi}$$

where the ν-sum is taken over an orthonormal basis for Ω^+, the f-sum is taken over an orthonormal basis for $C^\infty(0)$ (see § 5.2.2 for discussion) and the first ψ-sum is over *cuspidal* forms; we use the fact that arithmetic hyperbolic 3-manifolds have no discrete spectrum on forms, except for the locally constant 0- and 3-forms.

In fact, we later modify this idea slightly to deal with the poor convergence of the right-hand side, which behaves like the Fourier series of a non-differentiable function.

[23]Indeed, put $\bar{\omega}_i' = Y^{-2s}\Phi^+(s_i')\omega_i'$, so that $\bar{\omega}_s$ is exponentially close to $\sum \bar{\omega_i}'$ then also $E(s,\omega_s) + E(-s,\bar{\omega}_s)$ is exponentially close to $\sum E(s_i', \omega_i') + \sum E(-s_i', \bar{\omega}_i')$. To check the latter assertion one needs estimates on the derivative of the Eisenstein series in the s-variable; see discussion around (5.9.2.5) for this. In turn, each $E(s_i', \omega_i') + E(-s_i', \bar{\omega}_i')$ is exponentially close to $F_{s_i', \omega_i'}$, and no s_i' is zero.

Preliminaries: Note that in either case $\xi - \xi_N$ is "small": for $\xi = \eta$ the estimate (5.2.1.3) applies directly:

$$|\eta(x) - \eta_N(x)| \ll \exp(-b\ \mathrm{Ht}(x))\|\eta\|_{L^2}, \quad \mathrm{Ht}(x) \leq Y. \tag{5.9.9.2}$$

Note that the left-hand side only makes sense when x belongs to one of the cusps, i.e., $\mathrm{Ht}(x) \geq 1$. For $\xi = \omega$ one has

$$\begin{aligned} |\omega(x) - \omega_N(x)| &\leq |\eta - \eta_N| + |E(s,\omega) - E(s,\omega)_N| + |E(-s,\bar{\omega}) - E(-s,\bar{\omega})_N| \\ &\leq e^{-b\mathrm{Ht}(x)}\|\eta\|_{L^2}, \quad \mathrm{Ht}(x) \leq Y. \end{aligned} \tag{5.9.9.3}$$

where we used now (5.2.1.2) for the latter two terms, taking into account the Maass-Selberg relation (5.2.2.11), (5.9.2.2), and the fact that $\|\omega_s\|$ and $\|\bar{\omega}_s\|$ are bounded by constant multiples of $\|\eta\|_{L^2}$.

There are bounds of a similar nature for $d\omega - d\omega_N$ and $d\eta - d\eta_N$.

For any form, let $\|F\|_{\mathrm{bdy}}$ be the sum of the L^∞-norms of F, dF and d^*F along the boundary ∂M_Y. (5.2.1.3) and variants imply that for $\xi \in \{\eta, \omega\}$ we have, for suitable a, b, c,

$$\|\xi\|_{\mathrm{bdy}} \ll Y^c\|\xi\|_N + ae^{-bY}\|\eta\|, \tag{5.9.9.4}$$

where we define

$$\|\xi_N\|^2 = |a|^2 + |b|^2 + |a'|^2 + |b'|^2 \tag{5.9.9.5}$$

for $\xi_N = y^s(adx_1 + bdx_2) + y^{-s}(a'dx_1 + b'dx_2)$. (This norm is not "good" near $s = 0$, but we have in any case excluded this.)

The Green identity implies that if F is an eigenform of eigenvalue $-t^2 \leq T^2$ and $\xi \in \{\eta, \omega\}$:

$$\langle F, \xi\rangle_{M_Y} \ll \frac{\|F\|_{\mathrm{bdy}}\|\xi\|_{\mathrm{bdy}}}{|s|^2 - |t|^2}, \tag{5.9.9.6}$$

The spectral expansion: We now return to (5.9.9.1). To avoid the difficulties of poor convegence previously mentioned, we replaced $\tilde{\xi}$ by a smoothed version. We choose a "smoothing kernel": take a smooth self-adjoint section of $\Omega^1 \boxtimes (\Omega^1)^*$ on $\mathbf{H}^3 \times \mathbf{H}^3$, invariant under G_∞, and supported in $\{(P,Q) : d(P,Q) \leq C\}$, for some constant C. Then (the theory of point-pair invariants) there exist smooth functions $\hat{K}_1, \hat{K}_0$ such that $K \star \omega = \hat{K}_1(s)\omega$ whenever ω is a co-closed 1-form of eigenvalue $-s^2$, so that $d^*\omega = 0$, and $K \star \omega = \hat{K}_0(s)\omega$ whenever ω is of the form df, where f has eigenvalue $1 - s^2$. We can arrange matters such that $\hat{K}_\pm$ takes value in $[0, 2]$, $\hat{K}_1(s) = 1$, and so that the decay is rapid:

$$\hat{K}_{1,2}(u) \text{ decays faster than any positive power of } (T/u). \tag{5.9.9.7}$$

Now convolving with K and applying spectral expansion:

$$\begin{aligned} \langle\tilde{\xi} \star K, \tilde{\xi} \star K\rangle = &\sum_{\psi \text{ coclosed}} |K_1(s_\psi)|^2|\langle\tilde{\xi}, \psi\rangle|^2 + \sum_{\psi\perp\text{coclosed}} |K_0(s_\psi)|^2|\langle\tilde{\xi}, \psi\rangle|^2 \\ &+ \sum_\nu \int_{t=-\infty}^{\infty} \frac{dt}{2\pi}|\hat{K}_1(it)|^2\ |\langle\tilde{\xi}, E(\nu, it)\rangle|^2 \\ &+ \sum_f \int_t \frac{dt}{2\pi}|\hat{K}_0(it)|^2\ |\langle\tilde{\xi}, \frac{dE(f, it)}{\sqrt{1+t^2}}\rangle|^2 \end{aligned} \tag{5.9.9.8}$$

Here $-s_\psi^2$ is the Laplacian eigenvalue of ψ. The inclusion of the factors $|K(\cdots)|^2$ makes the right-hand side rapidly convergent.

We will now show that, for $\xi = \eta$ or $\xi = \omega$,

$$\|\tilde{\xi} \star K\|_{L^2} \ll aY^b \|\xi\|_{\text{bdy}} + a'Y^{-N}\|\eta\|_{L^2(M_Y)} \tag{5.9.9.9}$$

for arbitrary N, and a, b, a' possibly depending on N. This is all we need to finish the proof of the Lemma: Note that $\tilde{\xi} \star K$ agrees with ξ on $M_{\leq cY}$ (for a suitable constant $c \in (0,1)$ depending on the support of K). Now (5.9.9.2) and (5.9.9.3) imply that the L^2-norm of $\|\xi\|$ in the region $cY \leq y \leq Y$ is bounded by $a\exp(-bY)\|\eta\| + a'Y^{b'}\|\xi_N\|$. Therefore,

$$\begin{aligned}\|\xi\|_{L^2} &\ll \|\tilde{\xi} \star K\|_{L^2} + \int_{cY \leq \mathrm{Ht} \leq Y} |\xi|^2 \\ &\ll \|\tilde{\xi} \star K\|_{L^2} + a'Y^{b'}\|\xi_N\| + a\exp(-bY)\|\eta\| \\ &\ll a_3 Y^{b_3}\|\xi_N\| + a_4 Y^{-N}\|\eta\|\end{aligned}$$

where we used, at the last step, (5.9.9.9) and (5.9.9.4). Again, $a, b, a', \ldots$ are suitable constants, which may depend on N.

So, a proof of (5.9.9.8) will finish the proof of Lemma 5.9.9.

We discuss each term on the right-hand side of (5.9.9.8) in turn. In what follows, when we write "exponentially small" we mean bounded by $e^{-bY}\|\eta\|_{L^2}$; we shall use without comment the easily verified fact that $\frac{\|\omega\|_{L^2(M_Y)}}{\|\eta\|_{L^2}}$ is bounded by a polynomial in Y.

- Large eigenvalues: Either for $\xi = \eta$ or $\xi = \omega$, the total contribution to the right-hand side of (5.9.9.8) of terms with eigenvalue "large" (say $\geq Y^{1/10}$) is $\ll Y^{-N}\|\xi\|$ for any $N > 0$; this follows by the assumed decay (5.9.9.7) i.e. that $\hat{K}_{0,1}$ decays faster than any positive power of (T/u).

  In what follows, then, we may assume that we are only considering the inner product of $\tilde{\xi}$ with eigenfunctions of eigenvalue $\leq Y^{1/10}$. In particular, we may apply (5.2.1.2) to such eigenfunctions.
- Coclosed cuspidal ψ with eigenvalue in $[\epsilon_0, T^2]$:

  To analyze $\langle \eta, \psi \rangle$, when ψ is co-closed cuspidal, we recall that $\eta \perp F_\psi$ and F_ψ is exponentially close to ψ, at least whenever the eigenvalue of ψ lies in $[\epsilon_0, T^2]$. Thus such terms are an exponentially small multiple of $\|\eta\|$.

  To analyze $\langle \omega, \psi \rangle$ we simply note that ψ is perpendicular to $\eta - \omega = E(s, \omega) + E(-s, \bar{\omega})$ on M, and so (because ψ is cuspidal) almost perpendicular on M_Y: for ψ is exponentially small on $M - M_Y$.
- Coclosed cuspidal ψ with eigenvalue not in $[\epsilon_0, T^2]$:

  An application of Green's theorem (5.9.9.6) shows that $\langle \tilde{\xi}, \psi \rangle$ is an exponentially small multiple of $\|\xi\|_{\text{bdy}}$ for such ψ.

  Also, the number of such eigenvalues which are $\leq Y^{1/10}$ is at most a polynomial in Y.
- Cuspidal ψ that are perpendicular to co-closed:

  To analyze $\langle \eta, \psi \rangle$ when $\psi = df$ for some cuspidal eigenfunction f, we note that in fact $\langle \eta, df \rangle = \langle d^*\eta, f \rangle = 0$; the boundary term $\int_{\partial M_Y} *\eta \wedge f$ is identically zero because η satisfies absolute boundary conditions.

As before, ψ is exactly orthogonal to $\eta - \omega$, so almost orthogonal to it on M_Y, and so again $\langle \omega, \psi \rangle$ is bounded by an exponentially small multiple of $\|\eta\|$.

- Eisenstein terms:

  From (5.9.9.6) we get the estimate

$$|\langle \tilde{\xi}, E(\nu, t) \rangle_{M_Y}| \ll aY^b \frac{\|\xi\|_{\mathrm{bdy}}}{|t|^2 - |s|^2}, \tag{5.9.9.10}$$

  and there is a similar expression for the other Eisenstein term.

  This estimate is not good when t is very close to s (say, when $|s-t| \leq 1$). To handle that region, use Cauchy's integral formula and analytic continuation in the t-variable to compute $\langle \tilde{\xi}, E(\nu, t) \rangle$, this expression being antiholomorphic in t; this shows that the estimate

$$|\langle \tilde{\xi}, E(\nu, t) \rangle_{M_Y}| \ll aY^b \|\xi\|_{\mathrm{bdy}}, \tag{5.9.9.11}$$

  continues to hold when $|t - s| \leq 1$. Here we have used the fact that there exists a small disc around any $t \in i\mathbf{R}$ such that $E(\nu, t)$ remains holomorphic in that disc, together with bounds on $E(\nu, t)$ there; see discussion around (5.9.2.5).

  Now (5.9.9.10) and (5.9.9.11) suffice to bound the Eisenstein contribution to (5.9.9.8).

□

5.9.10. Modifications for functions, 2-forms, and 3-forms; conclusion of the proof. Since the proofs for (c) and (d) of the Theorem are largely the same as the proof of part (b), which we have now given in detail, we simply sketch the differences, mostly related to behavior of $\Psi(s)$ near 1.

5.9.10.1. *Informal discussion.* Recall that $\Psi(s)$ acts on an h-dimensional space, where h is the number of cusps. At $s = 1$, it has a pole whose residue is a projection onto a b_0-dimensional space, where b_0 is the number of components. The behavior of Ψ near 1 gives rise to two spaces of forms on M_Y of interest, one of dimension $h - b_0$ and one of dimension b_0. These two spaces are closely related to the kernel and image of $H^2(\partial M) \to H^3_c(M)$, which are $h - b_0$ and b_0 dimensional respectively.

(a) For f that belongs to the kernel of the residue of $\Psi(s)$ at $s = 1$, the function $s \mapsto E(s, f)$ is regular at $s = 1$. The 1-form $dE(s, f)$ is then *harmonic.* It (approximately) satisfies relative boundary conditions.

    These give rise (after a small modification) to an $h - b_0$ dimensional space of forms spanning the image of $H^0(\partial M_Y)$ inside $H^1(M_Y, \partial M_Y)$; or, more relevant to us, the 2-forms $*dE(s, f)$ give rise to an $h - b_0$-dimensional space of forms in $H^2(M_Y)$ that map isomorphically to $\ker(H^2(\partial M_Y) \to H^3_c(M_Y))$.

    This space was already used (see §5.3.1).

(b) The residue of Ψ at 1 means that one gets b_0 solutions to $\det(1 + Y^{-2s}\Psi(s)) = 0$ very close to $s = 1$.

    This gives rise (after a small modification, as before) to a b_0-dimensional space of functions on M_Y with very small eigenvalues (proportional to Y^{-2}), and satisfying relative (Dirichlet) boundary conditions. Roughly these eigenfunctions "want to be the constant function," but the constant function does not satisfy the correct boundary conditions.

These give the "extra eigenvalues" in part (d) of the Theorem.

5.9.10.2. *Part (c) of the theorem: co-closed* 0*-forms with absolute conditions.* One way in which the case of 0-forms is substantially easier: the "passage to combinatorial forms" is not necessary. For 1-forms, this was necessary only to handle eigenvalues on M_Y very close to zero; on the other hand, in the setting of 0-forms and absolute boundary conditions, the lowest nonzero eigenvalue is actually bounded away from 0.

More precisely, the analog of statement (*) from §5.9.8 now holds in the following form:

> (*) Any eigenfunction on M_Y with eigenvalue in $[0, (T_{\max} - \delta)^2]$
> which is orthogonal to all elements of the (analog of the sets) $\mathcal{F}$,
> is in fact zero.

In the current setting, $\mathcal{F}$ is the set of near-eigenforms constructed from cuspidal eigenfunctions on M, and also from Eisenstein series at parameters where $g(it) = 0$ for $t > 0$. [24]

This is proved exactly in the fashion of §5.9.8, with no significant modification: Let f be any eigenfunction on M_Y with eigenvalue $1 - s^2$, and we apply (5.9.8.2) with ω_1 replaced by f and ω_2 replaced by the Eisenstein series $E(s)$. There is no B_1-term, and thus the identity simply shows that the constant terms of f and the constant terms of $E(s)$ almost be proportional. One subtracts a suitable multiple of $E(s)$ from f and proceeds as before.

Note that in the prior analysis – e.g. prior to (5.9.8.3) – some of our estimates degnerated when s was close to zero, but this was just an artifact of using a poor basis; it is better to replace the roles of y^s, y^{-s} by the nondegenerate basis $e_s = y^s + y^{-s}, f_s = \frac{y^s - y^{-s}}{s}$, which eliminates this issue. This is related to the issue of why we do not include $s = 0$ as a root of $g(s) = 0$ in part (c) of the Theorem, i.e. why we only consider roots of $g(it)$ for $t > 0$.

5.9.10.3. *Part (d) of the Theorem: co-closed* 2*-forms with absolute conditions.* By applying the $*$ operator this is the same as computing eigenvalues on closed 1-forms with relative conditions, that is to say, away from the zero eigenvalue, the same as computing eigenvalues on *functions* with relative boundary conditions.

The analysis is now similar to that of the previous subsection, again now requiring consideration of $s \in [0, 1]$; but now there is a significant change related to $s \in [0, 1]$.

Namely, g' has roots for large Y in the region $s \in [0, 1]$, unlike $g(s)$ or $f(s)$. The reason for this is the singularity of $\Psi(s)$ at $s = 1$; this didn't show up for $g(s)$ because of the factor $\frac{1-s}{1+s}$ in front.

It will transpire that the number of such roots s is, for large Y, precisely the number of connected components of M, and they all lie very close to 1. The corresponding "monster eigenvalues" $1 - s^2$ are thus very close to zero.

Let R be the residue of $\Psi(s)$ at $s = 1$; we have computed it explicitly in § 5.2.2.5: it is a projection with nonzero eigenvalues parameterized by connected components of M; the eigenvalue corresponding to the connected component N is $\text{area}(\partial N)/\text{vol}(N)$.

[24] A priori, one needs to consider here not only $s \in i\mathbf{R}$ but also $s \in [0, 1]$; however, for sufficiently large Y there are no roots of g in $[0, 1]$, because Id dominates $\frac{1-s}{1+s} Y^{-2s} \Psi(s)$.

Now, the solutions to

$$\det(1+\Psi(s)Y^{-2s})=0$$

very close to $s=1$ are well-approximated, by a routine argument, by the solutions to

$$\det(1+\frac{R}{s-1}Y^{-2s})=0.$$

There is one solution to this for each nonzero eigenvalue λ of R, very close to $1-\frac{\lambda}{Y^2}$ (up to an error $o(1/Y^2)$). In particular the corresponding Laplacian eigenvalue is very close to $1-s^2 \approx 2\frac{\lambda}{Y^2}+o(1/Y^2)$.

We note for later reference, then, that the product of all nonzero "monster eigenvalues" is given by

$$\det{}'(2R/Y^2)(1+o(1)), \tag{5.9.10.1}$$

where again $\det'$ denotes the product of all nonzero eigenvalues.

This concludes our sketch of proof of parts (d), (e) of Theorem 5.9.1.

5.9.10.4. *Asymptotic behavior of regulators.* We now consider the relationship between the regulator of M_Y and that of M.

Clearly

$$\mathrm{reg}(H_0(M_Y)) \longrightarrow \mathrm{reg}(H_0(M)),$$

as $Y \to \infty$.

We have seen in (5.9.7.1) that

$$\mathrm{reg}(H_1(M_Y)) \sim \mathrm{reg}(H_1(M))(\log Y)^{-h_{\mathrm{rel}}}, \tag{5.9.10.2}$$

where h_{rel} is the number of relevant cusps.

As for the H^2-regulator, the analogue of (5.9.7.1) is the following, which is proved in a similar way (using the space of forms described in part (a) of §5.9.10.1):

$$\mathrm{reg}(H_2(M_Y)) \sim \mathrm{reg}(H_2(M))Y^{-2h'} \tag{5.9.10.3}$$

where $h'=\dim(H_2)-\dim(H_{2!})$, and $\sim$ denotes that the ratio approaches 1. Indeed, (5.3.1.3) was chosen in such a way as to make this relation simple.

On the other hand, we have seen that

$$\mathrm{reg}(H_2(M)) = \mathrm{reg}(H_{2!}(M)) \cdot \left(\left(\prod_{\mathcal{C}} 2\mathrm{area}(\mathcal{C}) \right) \prod_N \mathrm{vol}(N)^{-1} \cdot \det{}'(2R)^{-1} \right)^{1/2}$$

where the $\mathcal{C}$ product is taken over all cusps, the N-product is taken over all components N of M, and $\mathrm{vol}(N)$ denotes the volume of N. Finally, because $H_3(M_Y, \mathbf{C})$ is trivial,

$$\mathrm{reg}(H_3(M_Y)) \sim \mathrm{reg}(H_{3,\mathrm{BM}}(M)) \prod_N \mathrm{vol}(N)^{-1/2} \tag{5.9.10.4}$$

Consequently — by (5.3.3.3) — the ratio

$$\frac{\mathrm{reg}(M)}{\mathrm{reg}(M_Y)} \sim \left(\prod_{\mathcal{C}} 2\mathrm{area}(\mathcal{C}) \right)^{1/2} (\det{}'(2R))^{-1/2} (\log Y)^{h_{\mathrm{rel}}} Y^{-2h'},$$

where the product is taken over all connected components and taking the ratio for M and M' we arrive at:

$$\frac{\mathrm{reg}(M)}{\mathrm{reg}(M_Y)}\frac{\mathrm{reg}(M'_Y)}{\mathrm{reg}(M')} \sim \left(\frac{\det'(2R')Y^{2h'(M')}}{\det'(2R)Y^{2h'(M)}}\right)^{1/2}$$

$$(5.9.10.5) \qquad = \left(\frac{\det{}'(2R'/Y^2)}{\det{}'(2R/Y^2)}\right)^{1/2}.$$

since the terms $\prod \mathrm{area}(\mathcal{C})$ and $\log Y$ terms cancel and also $h'(M') - h(M) = -b_0(M) + b_0(M')$.

5.10. The proof of Theorem 5.8.3

Notation as in the statement of the Theorem.

In what follows we fix *two* truncation parameters $1 < Y' < Y$.

Let $K(t;x,y)$ be the the trace of the heat kernel on j-forms on M, and $k_t(x) = K(t;x,x)$. We suppress the j for typographical simplicity, but will refer to it when different js are treated differently. We define similarly K', k' for M', and K_Y, k_Y for M_Y, K'_Y, k'_Y for M'_Y. Recall our convention: in the cases of M_Y, M'_Y we compute with absolute boundary conditions, i.e. we work on the space of differential forms ω such that both ω and $d\omega$, when contracted with a normal vector, give zero. Set also $k_\infty(x) = \lim_{t\to\infty} k_t(x)$. Define similarly k'_∞ (for M') and $k_{\infty,Y}, k_{\infty,Y'}$ for M_Y and M'_Y respectively.

Set

$$\delta_t(x) = (k_t(x) - k_{Y,t}(x)), (x \in M_Y)$$

where we identify M_Y with a subset of M. We define δ'_t similarly. Set

$$(5.10.0.6) \quad I_j(Y,t) := (\mathrm{tr}^* \exp(-t\Delta_M) - \mathrm{tr}\exp(-t\Delta_{M_Y})) - \left(\mathrm{tr}^* \exp(-t\Delta_{M'}) - \mathrm{tr}\exp(-t\Delta_{M'_Y})\right),$$

where the Laplacians are taken on j-forms, and $I_j(Y,\infty) := \lim_{t\to\infty} I_j(Y,t)$; for the definition of regularized trace see Definition 5.3.5.

If we put $I_*(Y,t) = \frac{1}{2}\sum_j j(-1)^{j+1} I_j(Y,t)$, then that part of the expression of Theorem 5.8.3 involving analytic torsion equals

$$(5.10.0.7) \qquad -\frac{d}{ds}|_{s=0}\Gamma(s)^{-1}\int_0^\infty (I_*(Y,t) - I_*(Y,\infty))t^s\frac{dt}{t},$$

Note that, a priori, this should be understood in a fashion similar to that described after (5.3.5.3), that is to say, by dividing into intervals $[0,1]$ and $(1,\infty)$ separately.

In what follows, we fix j, and write simply $I(Y,t)$ for $I_j(Y,t)$; this makes the equations look much less cluttered. Then, using § 5.3.6, we may write

$$(5.10.0.8) \quad I(Y,t) = \int_{x\in M_{Y'}} \delta_t(x) - \int_{M'_{Y'}} \delta'_t(x) + \int_{x\in M_{[Y',Y]}} (k_t(x) - k'_t(x)) - (k_{t,Y}(x) - k'_{t,Y}(x)) + \int_{\mathcal{C}_Y} (k_t(x) - k'_t(x)).$$

and $I(Y,\infty)$ is obtained by substituting k_∞ for k_t at each point. Note, in the third integral on the left hand side, we have used the natural identification of $M_{[Y',Y]}$ with $M'_{[Y',Y]}$ to regard k' as a function on $M_{[Y',Y]}$. Now we have computed $\lim_{t\to\infty} \mathrm{tr}^* e^{-t\Delta_M}$ in (5.3.5.5); it equals

$$\begin{cases} b_0 = \dim H_0(M,\mathbf{C}), & j = 0,3 \\ \dim H_{j,!}(M,\mathbf{C}), & j = 1,2. \end{cases},$$

and similarly for M'. On the other hand, $\lim_{t\to\infty} \mathrm{tr}\ e^{-t\Delta_{M_Y}} = \dim H_j(M_Y,\mathbf{C})$ which equals $\dim H_j(M,\mathbf{C})$, and

$$\dim H_j(M,\mathbf{C}) - \dim H_{j,!}(M,\mathbf{C}) = \begin{cases} \frac{1}{2}\dim H_1(\partial M), & j=1, \\ \dim H_2(\partial M) - b_0(M), & j=2 \end{cases},$$

so that $I(Y,\infty) = 0$ for $j \neq 2,3$ — since the boundaries ∂M and $\partial M'$ are homotopy equivalent, and we get

(5.10.0.9)

$$\frac{d}{ds}|_{s=0}\Gamma(s)^{-1}\int_0^T I(Y,\infty)t^s\frac{dt}{t} = \begin{cases} (\log(T)+\gamma)(b_0(M)-b_0(M')) & (j=2) \\ (\log T+\gamma)(b_0(M)-b_0(M')), & (j=3). \\ 0, & j\in\{0,1\} \end{cases}$$

We claim that for any fixed $T > 0$

$$\lim_{Y\to\infty}\int_0^T I(Y,t)\frac{dt}{t} = 0. \tag{5.10.0.10}$$

Indeed, reasoning term by term in (5.10.0.8), and taking for definiteness $Y' = \sqrt{Y}$ so that the distance between "height Y" and "height $\sqrt{Y}$" is growing as $\log(Y)$,

(a) $\frac{\delta_t(x)}{t}, \frac{\delta'_t(x)}{t}$ both approach zero uniformly for $x \in M_{Y'}, Y > 2Y', t \in [0,T]$. This disposes of the first two terms.
(b) The terms $\frac{k_t(x)-k'_t(x)}{t}$ is independent of Y, Y'; uniformly for $t \in [0,T]$, it approaches zero as height$(x) \to \infty$. Since the measure of $M_{[Y',Y]}$ is bounded above, this disposes of the second term.
(c) As $Y' \to \infty$, $\frac{k_{t,Y}(x)-k'_{t,Y}(x)}{t}$ approaches zero uniformly for $x \in M_{[Y',Y]}$, $t \in [0,T]$.
(d) Similarly, for $x \in \mathcal{C}_Y$, $\frac{k_t(x)-k_t(x')}{t}$ approaches zero as $Y \to \infty$, uniformly for $t \in [0,T]$; this is just as in the case of the second term. This disposes of the final term.

These estimates are all "the local nature of the heat kernel." Indeed, (b) and (d) can be verified directly by using upper bounds for the heat kernel on hyperbolic 3-space, since k_t, k'_t are obtained by averaging this kernel. (a) and (c) can be verified with Brownian motion. For example, to analyze $k_{t,Y} - k'_{t,Y}$ for $x \in M_{[Y',Y]}$ we may analyze the difference between a Brownian motion on M_Y and M'_Y. The probability that a Brownian motion starting at x exits the cusp is bounded above by a function of type $\exp(-(\log Y')^2/t)$. This gives only an L^1-bound (in y) on the heat kernel $k_{t,Y}(x,y) - k'_{t,Y}(x,y)$, but then we obtain an L^∞-bound on $k_{t,Y}(x,x) - k'_{t,Y}(x,x)$ from the semigroup property and a trivial L^∞ estimate such as $k_{t,Y}(x,x) \ll (Y/t)^A$.

This proves (5.10.0.10). In combination we have shown that for any fixed T

(5.10.0.11)

$$\lim_{Y\to\infty} -\frac{d}{ds}|_{s=0}\Gamma(s)^{-1}\int_0^T (I_*(Y,t)-I_*(Y,\infty))\, t^s\frac{dt}{t} = \frac{1}{2}(\log T+\gamma)(b_0(M)-b_0(M')).$$

Now let us turn to the analysis of $t \in [T,\infty)$; we shall show (using crucially the analysis of small eigenvalues)

(5.10.0.12)

$$-\frac{d}{ds}|_{s=0}\Gamma(s)^{-1}\int_T^\infty (I_*(Y,t)-I_*(Y,\infty)t^s\frac{dt}{t} + \frac{1}{2}\left(\log\det{}'(Y^{-2}R) - \log\det{}'(Y^{-2}R')\right)$$
$$\longrightarrow -\frac{1}{2}(\log T+\gamma)(b_0(M)-b_0(M')), \text{ as } Y\to\infty$$

Recall that we are using $\det{}'$ to mean the product of nonzero eigenvalues, and R was defined after (5.3.3.3) Since we already saw (5.9.10.5) that

$$(\log \mathrm{RT}(M) - \log\mathrm{RT}(M_Y)) - (\log\mathrm{RT}(M') - \log\mathrm{RT}(M'_Y))$$
$$+\frac{1}{2}\left(\log\det{}'(Y^{-2}R) - \log\det{}'(Y^{-2}R')\right) \longrightarrow 0.$$

this, together with (5.10.0.9), will conclude the proof of the theorem; note that the analytic torsion quotient differs from (5.10.0.12) by a sign, owing to the sign difference between $\log\det\Delta$ and $\zeta'(0)$.

Now each term of $I_j(Y,t)$ can be written as a summation (or integral, in the case of continuous spectrum) of $e^{-\lambda t}$ over eigenvalues λ of j-forms on one of M, M_Y, M', M'_Y. We define I_j^{coclosed} simply be restricting these summations or integrals to be over co-closed forms, that is to say, the kernel of d^*. Then one can write each I_j in terms of I_j^{coclosed}; in fact, $I_j(Y,t)-I_j(Y,\infty) = (I_j^{\mathrm{coclosed}}(Y,t) - I_j^{\mathrm{coclosed}}(Y,\infty)) + (I_{j-1}(Y,t)^{\mathrm{coclosed}} - I_{j-1}(Y,\infty)^{\mathrm{coclosed}})$. In this way we reduce the analysis of (5.10.0.12) to the coclosed case.

Following our general policy of treating 1-forms in detail, we shall now show

$$\frac{d}{ds}|_{s=0}\Gamma(s)^{-1}\int_T^\infty \left(I_1^{\mathrm{coclosed}}(Y,t) - I_1^{\mathrm{coclosed}}(Y,\infty)\right)t^s\frac{dt}{t}\to 0 \tag{5.10.0.13}$$

(i.e., the contribution of coclosed 1-forms.) We shall then explain how the modifications of part (e) of Theorem 5.9.1 causes a slightly different analysis for $j\neq 1$, giving rise to the terms on the right of (5.10.0.12).

Write $N(x)$ for the number of eigenvalues on co-closed 1-forms on M_Y in $(0,x^2]$; enumerate these eigenvalues $0<\lambda_1\le\lambda_2\le\dots$ Define correspondingly $N_{\mathrm{cusp}}(x)$ for the number of such eigenvalues on co-closed cuspidal 1-forms on M. Note, in both cases, we are not counting eigenvalue 0.

Set $E(x)$ to be the "error" when we approximate $N(x)$ by means of Lemma 5.9.3:

$$E(x) = N(x) - \frac{\left(4xh_{\mathrm{rel}}\log(Y) - \sum_{i=1}^h \nu_i(x)\right)}{2\pi} - N_{\mathrm{cusp}}(x), \tag{5.10.0.14}$$

where h_{rel} is the size of the scattering matrix $\Phi^{\pm}$ and ν_i is as defined as before (cf. Lemma 5.9.3, (i)).

Write $\omega(s) = \det(\Phi^-(-s)\Phi^+(s))$. Since $\Phi^+(s)\Phi^-(s)$ is the identity, it follows that $\omega(s)\omega(-s)=1$; in particular, as we will use later, the function ω'/ω is *even* (symmetric under $s\mapsto -s$).

According to part (i) of the Lemma and Theorem 5.9.1, or more directly (5.9.4.1), $E(x)$ is bounded when in the range $x \le T_{\max} = T_{\max}(Y), Y \ge Y_0$; even more precisely $E(x)$ is "well-approximated" by $-\sum \left\{ \frac{4x \log Y - \nu_i(x)}{2\pi} \right\}$ — in the sense that

(5.10.0.15)
$$\int_X^{X+1} \left| E(x) + \sum_i \{ \frac{4x \log(Y) - \nu_i}{2\pi} \} \right| \le a \exp(-bY)$$

for some absolute constants a, b, whenever $X \le T_{\max}$. Here $\{\cdot\}$ denotes, as usual, fractional part.

Now let us analyze $\sum e^{-\lambda_i t}$, the sum being taken over all nonzero igenvalues of 1-forms on M_Y.

(5.10.0.16)
$$\begin{aligned}
\sum_{\lambda_i \ne 0} e^{-\lambda_i t} &= \int_0^\infty e^{-x^2 t} dN(x) \\
\underset{(a)}{=} & \int_0^\infty N(x) \cdot 2xte^{-x^2 t} dx \\
\underset{(b)}{=} & \sum \frac{-\nu_i(0)}{2\pi} + \frac{1}{2\pi} \int_0^\infty e^{-x^2 t} \left(4h_{\text{rel}} \log(Y) - \frac{\omega'}{\omega}(ix) + 2\pi \frac{dN_{\text{cusp}}}{dx} \right) dx \\
& + \int_0^\infty E(x) \cdot 2xte^{-x^2 t} \\
\underset{(c)}{=} & \frac{1}{2\pi} \int_{-\infty}^\infty e^{-x^2 t} \left(2h_{\text{rel}} \log(Y) - \frac{\omega'}{\omega}(ix) \right) + \sum_{\lambda \ne 0, \text{cusp eig.}} e^{-\lambda t} \\
& + \int_0^\infty E(x) \cdot 2xte^{-x^2 t}.
\end{aligned}$$

Note when we write, e.g. $\frac{dN_{\text{cusp}}}{dt} \cdot dt$, it should be understood as a distribution (it is, in fact, a measure).

Step (a) is integration by parts. At step (b), we first expanded $N(x)$ according to (5.10.0.14), then used integration by parts "in the reverse direction" and the observation that the derivative of $\sum \nu_i$ is $-\frac{\omega'}{\omega}$ by (5.9.2.1). In step (c) we used the fact that all $\nu_i(0) = 0$, this because $2\Phi^-(-s)\Phi^+(s)$ is the identity when $s = 0$ [25] and unfolded the integral from $[0, \infty)$ to $(-\infty, \infty)$, using the fact that ω'/ω is an even function.

The prior equation (5.10.0.16) says that the heat-trace on M_Y is closely related to the regularized heat-trace on M, up to an error term controlled by $E(x)$.

To proceed further we need to analyze the behavior of E, using:

LEMMA 5.10.1. *Suppose I is an open interval in $\mathbf{R}$ and m a monotone increasing piecewise differentiable function from I to $(-1/2, 1/2)$, satisfying $m' \in [A, B]$ where $B - A \ge 1$. Suppose that $\phi : I \to \mathbf{R}$ is smooth and $|\phi| + |\phi'| \le M$. Then*

$$\int \phi(x) \cdot m(x) dx \ll M \left(\frac{B - A}{A^2} \right).$$

[25] Were this not so, this term would be $\frac{-1}{4}\text{tr}(I - \Phi^-\Phi^+(0))$, which is related to a similar term in the trace formula.

PROOF. Translating I, we may suppose that $I = (a, b)$ and $m(0) = 0$, with $a < 0 < b$. Now $Ax \le m(x) \le Bx$ for $x > 0$; therefore, $b \in [\frac{1}{2B}, \frac{1}{2A}]$. Similarly, $-a$ belongs to the same interval.

Also

$$\begin{aligned}\int_I \phi(x)m(x) &= \int_I Ax\phi(0)dx + \int_I Ax(\phi(x)-\phi(0)) + \int_I (m(x)-Ax)\phi(x)dx \\ &= A\phi(0)(b^2-a^2)/2 + O(A.M.\int_I |x|^2) + O((B-A)M\int_I |x|) \\ &= O((B-A)M/A^2).\end{aligned}$$

where we used the fact that $\int_I |x| \ll \frac{1}{A^2}$. $\square$

Return to (5.10.0.13). In view of (5.10.0.16) and its analogue on the manifold M'_Y (write E' for the analogue of E on M'_Y), together with (5.3.5.4) describing the regularized trace, we may express $\int_T^\infty \left(I_1^{\text{coclosed}}(Y,t) - I_1^{\text{coclosed}}(Y,\infty)\right)\frac{dt}{t}$ as

$$\int_T^\infty dt\left(\int_0^\infty E(x) - E'(x)\right)2xe^{-x^2t}dx \tag{5.10.1.1}$$

$$= 2\int_0^\infty (E(x)-E'(x))e^{-x^2T}\frac{dx}{x} \tag{5.10.1.2}$$

indeed this is absolutely convergent, so the interchange in order is justified.

Fix $0 < \epsilon < 1$. We estimate (5.10.1.2) by splitting the integral into ranges of x:

$$[0, \frac{1}{100\log Y}) \cup [\frac{1}{100\log Y}, \epsilon) \cup [\epsilon, T_{\max}) \cup [T_{\max}, \infty).$$

Recall that $T_{\max}$ was a parameter defined in (5.9.2.4), and our analysis of small eigenvalues works "up to eigenvalue $T_{\max}^2$."

- The integral $\int_{T_{\max}}^\infty$ is handled by trivial estimates. Indeed, $E(x)$ and $E'(x)$ are easily bounded by a polynomial in x all of whose coefficients are at worst polynomials in $\log Y$. But, given the explicit definition (5.9.2.4), the size of $\int_{T_{\max}}^\infty x^N e^{-x^2T}$ decays faster than any polynomial in $\log Y$.
- The integral $\int_\epsilon^{T_{\max}}$ is seen to be $O_{\epsilon,T}((\log Y)^{-1/2})$:

  We approximate E and E' by means of (5.10.0.15). Thus the integral in question becomes a sum of integrals of the form

  $$\int_\epsilon^{T_{\max}} e^{-x^2T}\frac{dx}{x}\sum_i\{\frac{4x\log(Y)-\nu_i}{2\pi}\},$$

  up to an exponentially small error. We handle the integral for each i separately in what follows.

  Now split the range of integration (ϵ, T) as a union $\bigcup_i (a_j, a_{j+1})$ so that $\{\frac{4x\log(Y)-\nu_i}{2\pi}\}$ goes from $-1/2$ to $1/2$ as x increase from a_i to a_{i+1}. The number of such intervals is $O(T_{\max}\cdot\log Y)$. (At the endpoints, one can move ϵ and $T_{\max}$ by an amount $\approx 1/\log Y$ to ensure this is true; the integral over these small "edge intervals" has in any case size $O_\epsilon((\log Y)^{-1}.)$

  We now apply Lemma 5.10.1 to each $\int_{a_i}^{a_{i+1}}$. By virtue of (5.9.2.3), we may take $A = 4\log Y - a(\log\log Y)^b$ and $B = 4\log Y + a(\log\log Y)^b$ for appropriate a, b, and also M to be the maximum value of $e^{-x^2T}/x + (e^{-x^2T}/x)'$ in the range $(\epsilon, T_{\max})$, which is bounded by $O_{\epsilon,T}(1)$.

- To handle $\int_0^{\frac{1}{100 \log Y}}$ note that

$$|E(x) - E'(x)| \leq \sum_i |\nu_i| + |\nu_i'| \tag{5.10.1.3}$$

for $x \leq \frac{1}{100 \log Y}$; this follows from the definition (5.10.0.14) of E, E' together with the fact that there are no nonzero cuspidal eigenvalues x^2 on M, or any eigenvalues x^2 on M_Y, for this range of x. Since $|\nu_i(x)|/x$ is bounded for $x \in (0, \frac{1}{100 \log Y})$, independent of Y, it follows that the integral $\int_0^{\frac{1}{100 \log Y}}$ is bounded by $O((\log Y)^{-1/2})$.
- It remains to consider $\int_{\frac{1}{100 \log Y}}^{\epsilon}$.

  By using (5.10.0.15) it suffices to bound instead

$$\int_{\frac{1}{100 \log Y}}^{\epsilon} \frac{dx}{x} \left| \sum_i \{\frac{4x \log(Y) - \nu_i}{2\pi}\} - \sum_i \{\frac{4x \log(Y) - \nu_i'}{2\pi}\} \right|,$$

the difference between the two integrals going to zero as $Y \to \infty$.

  To bound this it will suffice to bound, instead, the corresponding difference of integer parts

$$\int_{\frac{1}{100 \log Y}}^{\epsilon} \frac{dx}{x} \left| \sum_i \left[\frac{4x \log(Y) - \nu_i}{2\pi}\right] - \sum_i \left[\frac{4x \log(Y) - \nu_i'}{2\pi}\right] \right|,$$

because the difference is bounded, up to constants, by the sum of quantities $|\nu_i(x)/x|$; and the integral of $|\nu_i(x)/x|$ over the region in question is bounded by $O(\epsilon)$ – after all, $|\nu_i(x)/x|$ is again absolutely bounded for $x \in [\frac{1}{100 \log Y}, \epsilon)$, independent of Y.

  Now the functions $4x \log(Y) - \nu_i$ or $4x \log(Y) - \nu_i'$, considered just in the region $[0, \epsilon)$, are all monotonic, and they all cross $2\pi\mathbf{Z}, 4\pi\mathbf{Z}$ etc. "at nearby points." More precisely, if we write $a_{i,n}$ for the solution to $4x \log(Y) - \nu_i(x) = 2\pi n$, then $|a_{i,n} - a_{j,n}| \ll \frac{n}{(\log Y)^2}$; and a similar result holds comparing $a_{i,n}$ and the analogously defined $a_{j,n}'$ for ν_j'. Each such value of n up to $O(\log Y \cdot \epsilon)$, then, contributes at most $\frac{n}{(\log Y)^2} \cdot \frac{\log Y}{n} = O(\frac{1}{\log Y})$ to the above integral. In this way, the integral above is bounded above by $O(\epsilon)$.

These bounds show that

$$\limsup_{Y \to \infty} \left| \int_T^{\infty} (I_1^{\text{coclosed}}(Y, t) - I_1^{\text{coclosed}}(Y, \infty)) \frac{dt}{t} \right| = O(\epsilon)$$

but ϵ is arbitrary; so the limit is zero. We have concluded the proof of (5.10.0.13).

To check (5.10.0.12) we now outline the changes in the prior analysis in the other case (i.e. co-closed j-forms for $j \neq 1$).

5.10.1.1. *The analysis for I_j for $j \neq 1$.*

- The analysis of 0-forms, i.e. I_0^{coclosed}, is the same, indeed simpler.

  The role $\Phi^-(-it)\Phi^+(it)$ is replaced by $\Psi(it)$. The main difference: the lowest eigenvalue on the continuous spectrum of M is now not zero, and also the nonzero eigenvalue spectrum of M_Y is bounded away from zero uniformly in Y. This avoids many complications in the above proof.

- The analysis of 2-forms, i.e. I_2^{coclosed}, is slightly differently owing to the different formulation of (d) of Theorem 5.9.1.

  Indeed, Theorem 5.9.1 (d) and the discussion of § 5.9.10 especially §5.9.10.3 describes certain co-closed eigenvalues on 2-forms for M_Y, M'_Y that are very close to zero. Denote these "monster eigenvalues" by λ_i for M_Y, and μ_j forM'_Y.

  The same analysis that we have just presented now shows that

(5.10.1.4)

$$\int_T^\infty \frac{dt}{t}(I_2^{\text{coclosed}}(Y,t) - I_2^{\text{coclosed}}(Y,\infty) = -\int_T^\infty \frac{dt}{t}\left(\sum_i e^{-\lambda_i t} - \sum_j e^{-\mu_j t}\right) + o(1),$$

  Recalling that $\int_T^\infty e^{-\lambda t}\frac{dt}{t} = \int_\infty^{T\lambda} \frac{e^{-u}}{u} du \sim -\log(T\lambda) - \gamma + o(1)$, for small λ, and using the analysis of § 5.9.10 (see (5.9.10.1)) we obtain that the left-hand side (5.10.1.4) equals

$$o(1) + \left(\log\det{}'(2Y^{-2}R) - \log\det{}'(2Y^{-2}R')\right) + (b_0(M) - b_0(M'))(\log T + \gamma)$$

  as $Y \to \infty$. This accounts exactly for the extra terms in (5.10.0.12).
- Coclosed 3-forms are simply multiples of the volume and all have eigenvalue 0.

This finishes the proof of (5.10.0.12) and so also finishes the proof of Theorem 5.8.3.

CHAPTER 6

Comparisons between Jacquet–Langlands pairs

Our main goal in this chapter is to compare the torsion homology of two manifolds in a Jacquet–Langlands pair Y, Y'.

We begin with notation (§6.1), recollections on the classical Jacquet–Langlands correspondence (§6.2), and on the notion of newform (§6.3).

After some background, we state and prove the crudest form of a comparison of torsion homology (§6.4, Theorem 6.4.1) and we then interpret "volume factors" in that Theorem as being related to congruence homology, thus giving a slightly refined statement (§6.6, Theorem 6.6.3). These statements are quite crude – they control only certain *ratios* of orders of groups, and we devote the remainder of the Chapter to trying to reinterpret them as relations between the orders of spaces of newforms.

In § 6.7 we introduce the notion of *essential homology* and *dual-essential homology.* These are two variants of homology which (in different ways) "cut out" congruence homology. We observe that Ihara's lemma becomes particularly clean when phrased in terms of essential homology (Theorem 6.7.6).

In § 6.8 we start on the core business of the chapter: the matter of trying to interpret Theorem 6.6.3 as an equality between orders of newforms. To do so, we need to identify the alternating ratios of orders from Theorem 6.6.3 with orders of newforms. In § 6.8 we consider several special cases (Theorem 6.8.2, 6.8.6, 6.8.8) where one can obtain somewhat satisfactory results.

In the final section § 6.9, we sketch some generalizations of § 6.8, using the spectral sequence of Chapter 4. The strength of our results is limited because of our poor knowledge about homology of S-arithmetic groups; nonetheless, we see that certain mysterious factors from § 6.8 are naturally related to the order of K_2.

6.1. Notation

We begin with the notion of a *Jacquet–Langlands pair*: given sets Σ, S, T, S', T' of finite places satisfying:

$$\Sigma = S \coprod T = S' \coprod T'$$

we take $\mathbb{G}, \mathbb{G}'$ to be inner forms of $\mathrm{PGL}(2)/F$ that are ramified at the set of finite primes S and S' respectively and ramified at all real infinite places of F. Let $Y(K_\Sigma)$ be the "level Σ" arithmetic manifold associated to $\mathbb{G}$ and $Y'(K_\Sigma)$ the analogous construction for $\mathbb{G}'$ (see § 2.2.2 for details).

We refer to a pair of such manifolds as a *Jacquet–Langlands pair.*

We recall that $Y(\Sigma)$ and $Y'(\Sigma)$ have the same number of connected components (see (3.3.4)). We refer to this common number as b_0 or sometimes as $\#Y = \#Y'$.

6.2. The classical Jacquet Langlands correspondence

As mentioned in the introduction to this Chapter, the comparison of torsion orders rests on the classical Jacquet–Langlands correspondence. Accordingly we recall a corollary to it, for the convenience of the reader. We continue with the notation of § 6.1.

For every j, let $\Omega^j(Y(K_\Sigma))^{\mathrm{new}}$ denote the space of *new cuspidal* differential j-forms on $Y(K_\Sigma)$. By this we shall mean the orthogonal complement, on cuspidal j-forms, of the image of all degeneracy maps $Y(K_R) \to Y(K_\Sigma)$, $S \subset R \subset \Sigma$. If $Y(K_\Sigma)$ is noncompact (i.e., S is empty) *then we restrict only to cuspidal j-forms.* Then the Jacquet–Langlands correspondence implies that

> The Laplacian Δ_j acting on the two spaces of forms $\Omega^j(Y(K_\Sigma))^{\mathrm{new}}$ and $\Omega^j(Y'(K_\Sigma))^{\mathrm{new}}$ are *isospectral*; more precisely, there is, for every $\lambda \in \mathbf{R}$, an isomorphism of λ-eigenspaces:
>
> $$\Omega^j(Y(K_\Sigma))^{\mathrm{new}}_\lambda \xrightarrow{\sim} \Omega^j(Y'(K_\Sigma))^{\mathrm{new}}_\lambda$$
>
> that is equivariant for the action of all Hecke operators outside Σ.

For the Jacquet–Langlands correspondence in its most precise formulation, we refer to [**43**, Chapter 16]; for a partial translation (in a slightly different context) into a language closer to the above setting, see [**70**].

Specializing to the case $\lambda = 0$, we obtain the Hecke-equivariant isomorphism

$$H^j(Y(K_\Sigma), \mathbf{C})^{\mathrm{new}} \xrightarrow{\sim} H^j(Y'(K_\Sigma), \mathbf{C})^{\mathrm{new}}.$$

We recall the notion of new in the (co)homological setting below.

6.3. Newforms, new homology, new torsion, new regulator

Recall (§3.10) that the $\mathfrak{q}$-*new* space of the first homology $H_1(Y(K_\Sigma), \mathbf{Z})$ is the quotient of all homology classes coming from level $\Sigma - \mathfrak{q}$.

So $H_1(Y(K_\Sigma), \mathbf{C})^{\mathrm{new}}$ will be, by definition, the cokernel of the map

$$\bigoplus_{v \in \Sigma - S} H_1(\Sigma - \{v\}, \mathbf{C})^2 \xrightarrow{\Psi^\vee} H_1(\Sigma, \mathbf{C})$$

which (since we are working in characteristic zero) is naturally isomorphic to the kernel of the push-forward map $H_1(\Sigma, \mathbf{C}) \xrightarrow{\Psi} \bigoplus H_1(\Sigma - \{v\}, \mathbf{C})^2$. However, when we work over $\mathbf{Z}$, the cokernel definition will be the correct one.

The classical Jacquet–Langlands correspondence (that we have just discussed) proves the existence of an isomorphism

$$H_1(Y(K_\Sigma), \mathbf{C})^{\mathrm{new}} \xrightarrow{\sim} H_1(Y'(K_\Sigma), \mathbf{C})^{\mathrm{new}},$$

which is equivariant for the action of all prime-to-Σ Hecke operators.

In the next section we shall formulate certain corresponding theorems for torsion, but only at a numerical level. To motivate the definitions that follow, note (see §3.10.4) the existence of a sequence, where all maps are degeneracy maps:

(6.3.0.5)
$$H_1(\Sigma - S, \mathbf{C})^{2^{|S|}} \longrightarrow \cdots \longrightarrow \bigoplus_{\{v,w\} \subset \Sigma - S} H_1(\Sigma - \{v, w\}, \mathbf{C})^4 \longrightarrow \bigoplus_{v \in \Sigma - S} H_1(\Sigma - \{v\}, \mathbf{C})^2 \longrightarrow H_1(\Sigma, \mathbf{C}).$$

It has homology only at the last term, and this homology is isomorphic to $H_1(\Sigma, \mathbf{C})^{\text{new}}$. Therefore, $H_1(\Sigma, \mathbf{C})^{\text{new}}$ can be decomposed, as a virtual module for the (prime-to-Σ) Hecke algebra, as an alternating sum of the left-hand vector spaces.

Motivated by this, we now define:

DEFINITION 6.3.1. *If S denotes the set of finite places of F which ramify in D, we define the* new regulator*, the* new essential regulator*,* new analytic torsion*, the* new naive torsion*, and the* new volume *as follows:*

$$\begin{aligned}
\operatorname{reg}^{\text{new}}(\Sigma) &= \prod_{S\subset R\subset\Sigma} \operatorname{reg}(Y(K_R))^{(-2)^{|\Sigma\setminus R|}} \\
\operatorname{reg}_E^{\text{new}}(H_1(\Sigma)) &= \frac{\operatorname{reg}^{\text{new}}(\Sigma)}{\operatorname{vol}^{\text{new}}(\Sigma),} \\
\tau_{\text{an}}^{\text{new}}(\Sigma) &= \prod_{S\subset R\subset\Sigma} \tau_{\text{an}}(Y(K_R))^{(-2)^{|\Sigma\setminus R|}} \\
h_{\text{tors}}^{\text{new}}(\Sigma) &= \prod_{S\subset R\subset\Sigma} |H_{1,\text{tors}}(Y(K_R), \mathbf{Z})|^{(-2)^{|\Sigma\setminus R|}} \\
\operatorname{vol}^{\text{new}}(\Sigma) &= \prod_{S\subset R\subset\Sigma} \operatorname{vol}(Y(K(R))^{(-2)^{|\Sigma\setminus R|}}
\end{aligned}$$

If we wish to emphasize the dependence on Y in this definition (with Σ explicit) we write $\operatorname{reg}^{\text{new}}(Y)$, etc. etc. Observe that h^{new} is not, *a priori*, the order of any group; indeed much of the later part of this Chapter is concerned with attempting to relate h^{new} and the size of a newform space.

The above definitions are equally valid in the split or non-split case; however, the definition of regulator varies slightly between the two cases (compare § 3.1.2 and § 5.3.2).

The quantity $\operatorname{reg}_E^{\text{new}}$ is meant to signify the "essential" part of the regulator, i.e. what remains after removing the volume part. One computes from the definition that:

$$\begin{aligned}
\operatorname{reg}_E^{\text{new}} &= \prod_{S\subset R\subset\Sigma} \left(\frac{\operatorname{reg}(H_1(R, \mathbf{Z}))}{\operatorname{reg}(H_2(R, \mathbf{Z})}\right)^{(-2)^{|\Sigma\setminus R|}}, \quad \mathbb{G} \text{ nonsplit,} \\
&= \prod_{S\subset R\subset\Sigma} \left(\frac{\operatorname{reg}(H_1(R, \mathbf{Z}))}{\operatorname{reg}(H_{2!}(R, \mathbf{Z})}\right)^{(-2)^{|\Sigma\setminus R|}}, \quad \mathbb{G} \text{ split.}
\end{aligned}$$

6.4. Torsion Jacquet–Langlands, crudest form

We continue with the previous notation; in particular, Y, Y' are a Jacquet–Langlands pair.

THEOREM 6.4.1. *The quantity*

$$\frac{h_{\text{tors}}^{\text{new}}(Y)}{\operatorname{vol}^{\text{new}}(Y)\operatorname{reg}_E^{\text{new}}(Y)} \tag{6.4.1.1}$$

is the same with Y replaced by Y'.

A major goal of the Chapter is to refine this statement by trying to interpret this numerical value in terms of quantities that relate more directly to Galois representations.

PROOF. (of Theorem 6.4.1).

We first give the proof in the compact case, i.e., both S, S' are nonempty. First note that there is an equality of analytic torsions

$$\tau_{\mathrm{an}}^{\mathrm{new}}(Y) = \tau_{\mathrm{an}}^{\mathrm{new}}(Y'), \tag{6.4.1.2}$$

Because of the sequence (6.3.0.5), it is sufficient to show that the spectrum of the Laplacian Δ_j acting on *new* forms on $Y(K_\Sigma)$ and $Y'(K_\Sigma)$ coincide. But that is a direct consequence of the Jacquet–Langlands correspondence as recalled in § 6.2.

Since we are in the compact case (that is to say, both S, S' are nonempty) (6.4.1.2) $\Longrightarrow$ (6.4.1.1): this follows directly from the Cheeger-Müller theorem, in the form given in § 3.1.2.

Otherwise, we may suppose that $S = \emptyset$ and $S' \neq \emptyset$, so that Y is noncompact but Y' is compact. If we suppose $S' = \{\mathfrak{p}, \mathfrak{q}\}$, then the statement has already been shown in Corollary 5.8.2. (In fact, all the primary theorems of the remainder of the chapter will be proved under this assumption.)

For $S = \emptyset$, arbitrary S', the proof is essentially the same as for the quoted Corollary: indeed, that corollary follows in a straightforward way from Theorem 5.8.3, and that deduction can be carried through in the same fashion for arbitrary S'.

□

6.5. Comparison of regulators and level-lowering congruences: a conjecture

As a consequence of Theorem 6.4.1 we see, in particular, that if Y, Y' are a Jacquet–Langlands pair (see §6.1) then

$$\frac{\mathrm{reg}_E^{\mathrm{new}}(Y)}{\mathrm{reg}_E^{\mathrm{new}}(Y')} \in \mathbf{Q}^\times, \tag{6.5.0.3}$$

where *new* regulators are defined as a certain alternating quotient of regulators.

It is possible to give another proof of (6.5.0.3), *independent of the Cheeger–Müller theorem.* The idea is to use Waldspurger's theorem on toric periods [**72, 71**] in order to evaluate the regulators; in particular, this shows that *both* $\mathrm{reg}(Y)$ *and* $\mathrm{reg}(Y')$ *are, up to algebraic numbers and powers of* π*, equal to a certain ratio of special values of* L*-functions.* For a precise statement, which is not needed here, see [**16**, Theorem 5.1].

In principle, the advantage of such an independent proof of (6.5.0.3) is that it could lead to a precise understanding of the ratio, and, therefore, of the effect on torsion. However, we have nothing unconditional in this direction at present, and correspondingly we do not present in entirety this alternate proof. Rather, we will examine a special case and, on the basis of this, present a principle — (6.5.2.1) below — that seems to govern the relationship between the regulators between the two manifolds in a Jacquet–Langlands pair.

6.5.1. The case of elliptic curves. Consider, as a warm-up example, the case of an indefinite division algebra D over $\mathbf{Q}$, ramified at a set of places S.

Let f be an *integrally normalized* form on the corresponding adelic quotient X_S, a certain Shimura curve. (The notion of *integrally normalized* is defined in [**55**] using algebraic geometry; it is likely that corresponding results hold when D is definite, for the "naive" integral normalization.) Let f^{JL} be the Jacquet–Langlands transfer of f to the split group GL_2: it may be considered as a classical holomorphic modular form for a suitable congruence subgroup and level.

In his thesis, Prasanna established the following principle:

(6.5.1.1) Level-lowering congruences for f^{JL} at S reduces $\langle f, f\rangle$.

A heuristic way to remember this principle is the following: Suppose f^{JL} is of squarefree level N, and that we realize the Jacquet–Langlands correspondence from $X_0(N)$ to X_S in some explicit way — for example, via Θ-correspondence. This correspondence is insensitive to arithmetic features, so $\Theta(f^{JL})$ will not be arithmetically normalized. A level lowering congruence for f^{JL} means that it is congruent (mod ℓ) to a newform g of level N/p, for some p. Heuristically, one might expect $\Theta(f^{JL}) \cong \Theta(g)$; but $\Theta(g)$ is zero for parity reasons. Therefore, $\Theta(f^{JL})$ is "divisible by ℓ"; what this means is that to construct the arithmetically normalized form one needs to divide by an additional factor of ℓ, depressing $\langle f, f\rangle$.

6.5.2. From Q to imaginary quadratic fields. Now we analyze the case of an imaginary quadratic field, or more generally a field with just one complex place, as considered elsewhere in this paper.

Let $Y(\Sigma), Y'(\Sigma)$ be a Jacquet–Langlands pair (see § 6.1). These therefore correspond to groups $\mathbb{G}$ and $\mathbb{G}'$ ramified at sets of places S, S' respectively. We assume we are not in the trivial case $\mathbb{G} = \mathbb{G}'$; in particular, $S \cup S'$ is nonempty.

We assume that there is an equality

$$\dim H^1(Y(\Sigma), \mathbf{Q}) = \dim H^1(Y'(\Sigma), \mathbf{Q}) = 1$$

and the corresponding motive is an elliptic curve C. We also denote by π the corresponding automorphic representation for $\mathbb{G}$ and π' the automorphic representation for $\mathbb{G}'$. Thus, C is a "modular" elliptic curve over F whose conductor has valuation 1 at all primes in Σ; we put modular in quotes since there is no uniformization of C by $Y(\Sigma)$ in this setting.

Our belief (stated approximately) is that:

> Level-lowering congruences for C at places of $S' - S$ (respectively $S - S'$, respectively outside $S \cup S'$) reduce (respectively increase, respectively have no effect on)

(6.5.2.1)
$$\frac{\mathrm{reg}(Y)}{\mathrm{reg}(Y')}.$$

Explicitly, we say that C admits a level lowering congruence modulo ℓ at a prime $\mathfrak{q}$ if the ℓ-torsion $C[\ell]$ is actually unramified at $\mathfrak{q}$. If we working over $\mathbf{Q}$, this would actually mean there exists a classical modular form at level $\Sigma/\mathfrak{q}$ that is congruent to the form for C, modulo ℓ; in our present case, this need not happen (as can be seen from the numerical computations of [**16**]). In this instance, in any case, we expect that the regulator ratio of (6.5.2.1) would contain a factor of ℓ either in its numerator or denominator, as appropriate.

A little thought with the definition of $\mathrm{reg}(Y)$ shows that this is indeed analogous to (6.5.1.1): the smaller that $\mathrm{reg}(Y')$ is, the smaller the L^2-norm of a harmonic representative for a generator of $H^1(Y', \mathbf{Z})$. We will not formulate a precise conjecture along the lines of (6.5.2.1): at this stage proving it seems out of reach except in special cases (e.g. base change forms) but we hope it will be clear from the following discussion what such a conjecture would look like.

We have numerical evidence for this principle (see §1.1.4 and Chapter 8 of [**16**]); and our our later results comparing Jacquet–Langlands pairs (especially Theorem 6.8.8) also give evidence for (6.5.2.1). Finally, the following (very conditional) Theorem, then, provides some evidence for the belief (6.5.2.1).

THEOREM 6.5.3. *Denote by C_χ the quadratic twist of the elliptic curve C by the quadratic character χ. Suppose the validity of the conjecture of Birch and Swinnerton-Dyer for each C_χ (see* (6.5.5.1) *for the precise form used). Let ℓ be a prime larger than* 3 *such that the Galois representation of G_F on ℓ-torsion is irreducible. Suppose moreover that*

i. *Cycles $\gamma_{\mathscr{D}}$ associated to $\mathbb{G}$-admissible data $\mathscr{D}$ (these are defined in §6.5.4) generate the first homology $H_1(Y(\Sigma), \mathbf{Z}/\ell\mathbf{Z})$ and cycles $\gamma_{\mathscr{D}}$ associated to $\mathbb{G}$-admissible data $\mathscr{D}'$ generate $H_1(Y'(\Sigma), \mathbf{Z}/\ell\mathbf{Z})$.*
ii. *Fix a finite set of places T and, for each $w \in T$, a quadratic character χ_w of $F_w^\times$; then there exist infinitely many quadratic characters χ such that $\chi|F_w = \chi_w$ for all w, ℓ does not divide the order of $Ш(C_\chi)$, and C_χ has rank* 0.

Then[1]

$$\left(\frac{\mathrm{reg}(H_1(Y, \mathbf{Z}))}{\mathrm{reg}(H_1(Y', \mathbf{Z}))}\right)^2 \sim \frac{\prod_{v \in S} v(j)}{\prod_{v \in S'} v(j)} \tag{6.5.3.1}$$

where $\sim$ denotes that the powers of ℓ dividing both sides is the same; j is the j-invariant of C, and S (resp. S') is the set of places which are ramified for D (resp. D').

In order to make clear the relation to (6.5.2.1), note that prime divisors of $v(j)$ (for $v \in S$) measure "level lowering congruences" at $v \in S$: by Tate's theory, for such v, the ℓ-torsion $C[\ell]$ over F_v is isomorphic as a Galois module – up to a possible twist by an unramified character – to the quotient

$$\langle q_v^{1/\ell}, \mu_\ell \rangle / q_v^{\mathbf{Z}}.$$

In particular, if $v(q_v) = v(j)$ is divisible by ℓ and ℓ is co-prime to the residue characteristic of F_v, the mod ℓ-representation is unramified at v if the residue characteristic of v is different from ℓ, and "peu ramifiée" (in the sense of Serre [**67**]) if the residue characteristic of v is ℓ. This is, conjecturally, exactly the condition that detects whether ℓ should be a level lowering prime. So the right-hand side indeed compares level lowering congruences on Y and Y'.

A word on the suppositions. Item (ii) is essentially Question B(S) of [**55**]. We have absolutely no evidence for (i) but we can see no obvious obstruction to it, and it seems likely on random grounds absent an obstruction. In the course of proof we verify that the relevant cycles span *rational* homology.

[1]This ratio of H_1-regulators differs by a volume factor from $\mathrm{reg}(Y)/\mathrm{reg}(Y')$, by (3.1.2.2). This arithmetic significance of the volume factors is discussed, and completely explained in terms of congruence homology, in § 6.6.

In short, the idea is this: To evaluate $\mathrm{reg}(H_1(Y))$ and $\mathrm{reg}(H_1(Y'))$, we integrate harmonic forms on Y (resp. Y') against cycles $\gamma_{\mathscr{D}}$. By a formula of Waldspurger, these evaluations are L-values; when we use Birch Swinnerton-Dyer, we will find that the Tamagawa factors c_v, as in (6.5.5.1), are slightly different on both sides, and this difference corresponds to the right-hand side of (6.5.3.1).

6.5.4. We now give the exact definition of the cycles $\gamma_{\mathscr{D}}$ that appear in the statement of the theorem:

Recall that we write $F_\infty = F \otimes \mathbf{R}$. Fix an isomorphism

$$\mathbb{G}(F_\infty) \xrightarrow{\sim} \mathrm{PGL}_2(\mathbf{C}) \times \mathrm{SO}_3^r$$

such that K_∞ is carried to $\mathrm{PU}_2 \times \mathrm{SO}_3^r$, where PU_2 is the image in $\mathrm{PGL}_2(\mathbf{C})$ of the stabilizer of the standard Hermitian form $|z_1|^2 + |z_2|^2$, and let H be the subgroup

$$H = (\text{diagonal in } \mathrm{PGL}_2(\mathbf{C}) \times \mathrm{SO}_2^r)$$

(for an arbitrary choice of SO_2s inside SO_3).

A "$\mathbb{G}$-admissible datum" $\mathscr{D}$ consisting of a triple

$$(\text{maximal subfield } E \subset D, \text{ element } g \in \mathbb{G}(\mathbb{A}), \text{ quadratic characters } \chi_1, \chi_2 \text{ of } \mathbb{A}_F^\times/F^\times)$$

satisfying the following conditions:

(a) E is the quadratic field associated, by class field theory, to $\chi_1\chi_2$.
(b) Let $\mathbf{T} \subset \mathbb{G}$ be the centralizer of E; it's a torus and $\mathbf{T}(F) \simeq E^\times/F^\times$. We require that $g^{-1}\mathbf{T}(\mathbb{A})^\circ g \subset H \times K$, where $\mathbf{T}(\mathbb{A})^\circ$ denotes the maximal compact subgroup.
(c) At any place v, one of the following possibilities occur:
 (ci) D_v is split, π_v is unramified, and at most one of χ_1 or χ_2 is ramified;
 (cii) D_v is ramified and π_v is one-dimensional, and $\chi_{1,v} \neq \chi_{2,v}$ are both unramified.
 (ciii) D_v is split and π_v is Steinberg and $\chi_{1,v} = \chi_{2,v}$ are both unramified.
 (For v archimedean, regard all characters as unramified; thus for v real, we understand ourselves to be in case (cii), declaring all characters to be unramified; for v complex, we regard ourselves to be in case (ciii), understanding Steinberg" as "the tempered cohomological representation of $\mathrm{PGL}_2(\mathbf{C})$.)

We often write in what follows $E_v := (E \otimes F_v)^\times$.

For each such datum $\mathscr{D}$, define an element $\gamma_{\mathscr{D}} \in H_1(Y, \mathbf{Z})$ as follows:

Put $K_{\mathscr{D}} = gK_\infty K g^{-1}$. Also, let $\psi_{\mathscr{D}}$ be the character of $\mathbb{A}_E^\times/E^*$ obtained as $\mathrm{N} \circ \chi_1 = \mathrm{N} \circ \chi_2$. The quotient $Y_T := \mathbf{T}(F)\backslash\mathbf{T}(\mathbb{A})/K_{\mathscr{D}} \cap \ker(\psi)$ is a compact 1-manifold (possibly disconnected). The map

$$\mathrm{emb} : t \mapsto tg, \quad \mathbf{T}(F)\backslash\mathbf{T}(\mathbb{A}) \to \mathbb{G}(F)\backslash\mathbb{G}(\mathbb{A}) \tag{6.5.4.1}$$

descends to a map $\mathrm{emb} : Y_T \to Y(K)$; its image is a finite collection of closed arcs $\{G_i\}_{1\le i\le I}$ on Y. Let $[G_i] \in H_1(Y(K), \mathbf{Z})$ be the homology class of G_i. Moreover, ψ factors through to a function from the finite set of G_i to ± 1; we may speak of $\psi(G_i)$ as the constant value taken by ψ on G_i. Accordingly, put

$$\gamma_{\mathscr{D}} = \sum_1^I \psi(G_i).[G_i] \in H_1(Y(K), \mathbf{Z}).$$

Note that this will actually be trivial if $K_{\mathscr{D}}$ is not contained in the kernel of ψ.

6.5.5. Note that C has multiplicative reduction at $v \in \Sigma$. For any quadratic character χ of $\mathbb{A}_F^\times/F^\times$, let C_χ be the quadratic twist of C by χ, and $Ш(\chi)$ its Tate-Shafarevich group. The BSD conjecture predicts that

$$L(C_\chi, \frac{1}{2}) = \frac{2}{\sqrt{\Delta_F}} \left(\prod_v c_v(C_\chi) \right) \frac{Ш(C_\chi)\Omega(C_\chi)}{C_\chi(F)^2_{\mathrm{tors}}} \tag{6.5.5.1}$$

where $\Omega(C_\chi)$ is the complex period of C, and c_v is the number of components of the special fiber of the Néron model of C_χ at v.

LEMMA 6.5.6. *There exist infinitely many $\mathbb{G}$-admissible data $\mathscr{D}$, definition as above, such that $L(\frac{1}{2}, \pi \times \chi_1)$ and $L(\frac{1}{2}, \pi \times \chi_2)$ are both nonzero.*

Before the proof, some recollections about root numbers: For representations σ of $\mathrm{PGL}_2(k)$, k a local field, we denote by $\varepsilon(\sigma) \in \pm 1$ the root number (in this setting, these are independent of the choice of additive character.) Recall that if σ is a principal series induced from the character β, then $\varepsilon(\sigma) = \beta(-1)$; also if σ is the Steinberg representation (resp. its twist by an unramified quadratic character), then $\varepsilon(\sigma) = -1$ (resp. 1).

PROOF. It suffices to show that there are infinitely many quadratic characters (χ_1, χ_2) that satisfy condition (c) of the criterion (recalled below), and, simultaneously, $L(\frac{1}{2}, \pi \times \chi_i) \neq 0$.

(ci) D_v is split, π_v is unramified, and at most one of χ_1 or χ_2 is ramified;
(cii) D_v is ramified and π_v is one-dimensional, and $\chi_{1,v} \neq \chi_{2,v}$ are both unramified.
(ciii) D_v is split and π_v is Steinberg and $\chi_{1,v} = \chi_{2,v}$ are both unramified.

Once this is done then we may choose E, g to satisfy (a), (b) too.

We will use the following simple fact: If T is a finite set of places, Π is a cuspidal automorphic representation of PGL_2, and there is $v \notin T$ so that Π_v is Steinberg (up to an unramified twist), there exists a quadratic character ω such that $\varepsilon(\frac{1}{2}, \Pi \times \omega) = -\varepsilon(\Pi)$, and moreover $\omega = 1$ at all places in T and ω is unramified at v.

Indeed enlarge T to contain all archimedean and ramified places of Π, with the exception of v, and take ω such that ω is unramified nontrivial at v and ω is trivial at all places in T. Then

$$\frac{\varepsilon(\Pi \times \omega)}{\varepsilon(\Pi)} = -\prod_{w \notin T \cup \{v\}} \omega(-1) = -\prod_{w \in T \cup \{v\}} \omega(-1) = -1,$$

so that either Π or $\Pi \times \omega$ has global root number 1.

Next, it has been proven by Waldspurger [**73**, Theorem 4, page 288] that, if the global root number of a PGL_2-automorphic cuspidal representation π is trivial, then one can find a quadratic characters θ *trivial at any fixed finite set of places* with $L(\frac{1}{2}, \pi \times \theta) \neq 0$.

Let T be the set of places where either D_v or π_v is ramified. The set T contains a finite Steinberg place by assumption. By what we noted above, applied to the Jacquet-Langlands transfer of π together with T with this Steinberg place removed, there exists χ_1 unramified at T so that $\varepsilon(\pi \times \chi_1) = 1$. By Waldspurger's result (but applied to $\pi \times \chi_1$) we may actually choose χ_1, still unramified in T, such that $L(\frac{1}{2}, \pi \times \chi_1) \neq 0$.

Now let T' be the set of places where D_v, π_v or $\chi_{1,v}$ are ramified. Clearly, there exists at least one χ_2 so that (χ_1, χ_2) satisfy the conditions of (c) – after all, χ_2 is unconstrained outside T', and so it amounts to a finite set of local constraints.

Note that (ci)–(ciii) imply that, for every place v,

$$\chi_1\chi_2(-1)\varepsilon(\frac{1}{2}, \pi_v \times \chi_1)\varepsilon(\frac{1}{2}, \pi_v \times \chi_2) = \begin{cases} 1, & D_v \text{ split} \\ -1, & D_v \text{ ramified.} \end{cases} \tag{6.5.6.1}$$

at all places v. Note this is also valid for v archimedean: if v is complex, it is clear since $\chi_{1,v} = \chi_{2,v}$, and if v is real, then D_v is ramified and we have $\varepsilon(\frac{1}{2}, \pi_v \times \chi_{1,v}) = \varepsilon(\frac{1}{2}, \pi_v \times \chi_{2,v})$.

Taking the product of (6.5.6.1) over all places v, and noting that $\varepsilon(\frac{1}{2}, \pi \times \chi_1) = 1$, we see that the global root number $\varepsilon(\pi \times \chi_2) = 1$ also. Now applying Waldspurger's result again, now with T replaced by T', we may find such χ_2' with $L(\frac{1}{2}, \pi \times \chi_2') \neq 0$ and $\chi_{2,v}' = \chi_{2,v}$ for each $v \in T'$. The pair (χ_1, χ_2') still satisfies (c): For $v \in T'$ it is the same as (χ_1, χ_2), and for $v \notin T'$ all of D_v, π_v and $\chi_{1,v}$ are all unramified so we are in case (ci). □

Note also that the conditions in the definition of "$\mathbb{G}$-admissible datum" automatically force that, for every v, there exists a linear functional $\ell : \pi_v \to \mathbf{C}$ which transforms under $\mathbf{T}(F_v)$ by the character ψ. Indeed, it is known that if D_v is split and π_v is principal series such a functional always exists; so it is enough to consider the case when either D_v is nonsplit or when π_v is a Steinberg representation. In that case, the claim follows from (c-iii) and the results of Saito and Tunnell [**61, 69**]: such a functional exists if and only if (6.5.6.1) holds.

6.5.7. Proof of Theorem 6.5.3. We denote by D the quaternion algebra corresponding to $\mathbb{G}$, and similarly D' for $\mathbb{G}'$.

Let ω (resp ω') be a harmonic form on Y resp. Y' corresponding to C and let π resp. π' be the corresponding automorphic representation. Let R be the generator of the subgroup $\int_\gamma \omega \subset \mathbf{R}$, where γ ranges over $H_1(Y(\Sigma), \mathbf{Z})$. Define similarly R'.

The results of Waldspurger [**72, 71**] show that if $\mathscr{D}$ is $\mathbb{G}$-admissible, then:

$$\frac{|\int_{\gamma_{\mathscr{D}}} \omega|^2}{\langle \omega, \omega \rangle} \propto \Delta_{\chi_1\chi_2}^{1/2} L(1/2, \pi \times \chi_1) L(1/2, \pi \times \chi_2) \tag{6.5.7.1}$$

and a similar formula on $\mathbb{G}'$ *with the same constant of proportionality* – that is to say, if $\mathscr{D}$ is $\mathbb{G}$ admissible, and $\mathscr{D}'$ is $\mathbb{G}'$-admissible, then the ratio LHS/RHS for $(\mathbb{G}, \mathscr{D})$ equals the corresponding ratio for $(\mathbb{G}', \mathscr{D}')$. Here we wrote Δ_χ for the discriminant of the quadratic field extension of F attached to the character χ.

For each $\mathbb{G}$-admissible datum $\mathscr{D}$ for $Y(\Sigma)$ we may write $\int_{\gamma_{\mathscr{D}}} \omega = m_{\mathscr{D}} R$, where $m_{\mathscr{D}} \in \mathbf{Z}$; similarly, for each $\mathbb{G}'$-admissible datum $\mathscr{D}$ for $Y'(\Sigma)$ we may write $\int_{\gamma_{\mathscr{D}}} \omega' = m'_{\mathscr{D}} R'$. Our assumption (i) implies that the greatest common divisor of $\mathfrak{m}_{\mathscr{D}}$, where $\mathscr{D}$ ranges over $\mathbb{G}$-admissible data, is relatively prime to ℓ; similarly for m'.

It follows from (6.5.7.1) that

$$\left(\frac{\mathrm{reg}(H_1(Y))}{\mathrm{reg}(H_1(Y'))} \right)^2 \sim \frac{g}{g'},$$

where

$$g = \gcd_{(\chi_1, \chi_2) \in \mathcal{X}} \Delta_{\chi_1\chi_2}^{1/2} L(1/2, \pi \times \chi_1) L(1/2, \pi \times \chi_2),$$

where the greatest common divisor is taken over the set $\mathcal{X}$ of all (χ_1, χ_2) that give rise to a $\mathbb{G}$-admissible datum; and g' is defined similarly, with the gcd taken over a corresponding set $\mathcal{X}'$ for $\mathbb{G}'$; again $\sim$ denotes that the powers of ℓ dividing both sides coincide.

We shall now apply the conjecture of Birch and Swinnerton-Dyer (6.5.5.1):

$$L(1/2, \pi \times \chi) = 2 \frac{(\prod_v c_v(C_\chi)) \,\text{Ш}(C_\chi)}{\# C_\chi(F)^2_{\text{tors}}} \frac{\Omega_{C_\chi}}{\Delta_F^{1/2}}$$

We have $\Omega(C_\chi) = \Omega(C) \cdot \Delta_\chi^{-1/2}$ and $\Delta_{\chi_1}\Delta_{\chi_2} = \Delta_{\chi_1\chi_2}$; both of these use the assumption (c) on data $\mathscr{D}$. Because we assume that the Galois action on mod ℓ torsion is irreducible, we see that the ℓ-part of $C_\chi(F)_{\text{tors}}$ is trivial for any χ. Also, by [**60**], the product of $c_v(C_\chi)$ over all $v \notin \Sigma$ is a power of 2.

Therefore,

$$\frac{g}{g'} \sim \frac{\gcd_{(\chi,\chi') \in \mathcal{X}} \text{Ш}(C_\chi)\text{Ш}(C_{\chi'}) \prod_{v \in \Sigma} c_v(C_\chi) c_v(C_{\chi'})}{\gcd_{(\chi,\chi') \in \mathcal{X}'} \text{Ш}(C_\chi)\text{Ш}(C_{\chi'}) \prod_{v \in \Sigma} c_v(C_\chi) c_v(C_{\chi'})}, \tag{6.5.7.2}$$

where, again, $\sim$ means that the powers of ℓ dividing both sides coincide.

For $v \in \Sigma$, put $A_v := \gcd_{(\chi_1,\chi_2) \in \mathcal{X}} c_v(C_{\chi_1}) c_v(C_{\chi_2})$. Now $c_v(C_\chi)$ is wholly determined by the v-component χ_v for $v \in \Sigma$. So we may choose (χ_1, χ_2) in such a way that $c_v(C_{\chi_1}) c_v(C_{\chi_2}) \sim A_v$ for every $v \in \Sigma$.

According to assumption (iii) of the Theorem we may find χ_1' so that $\chi_1' = \chi_1$ at all places $v \in \Sigma$ (in particular, χ_1' is unramified everywhere in Σ) and also $\text{Ш}(C_{\chi_1'})$ is prime-to-ℓ.

According to assumption (iii) of the Theorem, again, we may find χ_2' so that $\chi_2' = \chi_2$ at all places $v \in \Sigma$ and at all places where χ_1 is ramified, and also also $\text{Ш}(C_{\chi_2'})$ is prime-to-ℓ.

Now $(\chi_1', \chi_2') \in \mathcal{X}$ still. Indeed, $(\chi_1', \chi_2') = (\chi_1, \chi_2)$ at all places of Σ, by construction, and at all other places at least one is unramified. Therefore, the numerator of (6.5.7.2) – call it g_1 – has the property that $\frac{g_1}{\prod_v A_v}$ is relatively prime to ℓ.

Reasoning similarly for $\mathcal{X}'$ we find:

$$g \sim \prod_{v \in \Sigma} \frac{A_v}{A_v'},$$

We are reduced to the local problem of computing A_v for $v \in \Sigma$. An examination of the definition of $\mathcal{X}$ and $\mathcal{X}'$ shows that: $A_v \sim 1$ *unless* D_v is ramified, and $A_v' \sim 1$ *unless* D_v' is ramified. In those cases, C has multiplicative reduction at v, and, as we have observed

$$A_v \sim c_v(C) = |v(j)|.$$

This concludes the proof. □

6.6. Torsion Jacquet–Langlands, crude form: matching volume and congruence homology

We now return to the general case and show that *the "volume" factors in the Theorem 6.4.1 are accounted for by congruence homology.*

Write $\Sigma = S \cup T$, and $R = S \cup V$, where $\emptyset \subseteq V \subseteq T$. We now prove an elementary lemma. Recall that w_F is the number of roots of unity, and that w_F divides 4 unless $F = \mathbf{Q}(\sqrt{-3})$.

LEMMA 6.6.1. *The groups $H_{1,\mathrm{cong}}(R, \mathbf{Z})$ and $H^1_{\mathrm{cong}}(R, \mathbf{Q}/\mathbf{Z})$ have order*

$$\left(\cdot \prod_S (N\mathfrak{q}+1) \cdot \prod_V (N\mathfrak{q}-1) \right)^{\#Y}$$

up to powers of $\ell | w_F$, where $\#Y$ is the number of connected components of $Y(\Sigma)$.

PROOF. Recall that we have described the connected components of $Y(K_\Sigma)$ in Remark 3.3.4:

$$Y(K_\Sigma) = \coprod_A Y_0(\Sigma, \mathfrak{a}) = \coprod_A \Gamma_0(\Sigma, \mathfrak{a}) \backslash \mathbf{H}^3,$$

Thus, (by definition, as in § 3.7.0.2) it suffices to compute the product of the orders of the congruence groups $H_{1,\mathrm{cong}}$ of $Y_0(\Sigma, \mathfrak{a})$ for all $\mathfrak{a} \in A$.

By strong approximation this amounts to computing the size of the maximal abelian quotients of K^1_v for all places v which come from abelian quotients of K_v (that is, the inverse image of the map $K^{1,\mathrm{ab}}_v \rightarrow K^{\mathrm{ab}}_v$). The independence of the intermediate choice of K follows from the fact that K is "p-convienient" (see Definition 3.7.3). In particular, (following Lemma 3.7.4, as well as equation 3.7.3.1) this computation is independent of the choice of connected component, and the answer, up to the exceptional factors above, is

$$\left(\prod_v \left| \text{inverse image of } K^{\mathrm{ab}}_v \text{ in } K^{1,\mathrm{ab}}_v \right| \right)^{\#Y}$$

For $v \in S$, the level structure K_v arises from the maximal order in a quaternion algebra over $\mathscr{O}_v$, and when $v \in V$, K_v is of "$\Gamma_0(v)$-type" in $\mathrm{GL}_2(\mathscr{O}_v)$. If we compute with the norm one element groups K^1_v, we arrive at the answer above. In general, we have to ensure that the image of the determinant (which has 2-power order) acts trivially on these quotients (equivalently, the abelian quotients of K^1_v lift to abelian quotients of K_v). This follows by an explicit verification: in the quaternion algebra case, it follows from the fact that (in the notation of § 3.2.2) that the map $K^1_v \rightarrow l^1$ lifts to a map $K_v \rightarrow l^\times$, and in the $\Gamma_0(v)$-case because $K^1_v \rightarrow k^\times$ lifts to a map $K_v \rightarrow (k^\times)^2$. □

Let us make the following definition (cf. Definition 6.3.1):

DEFINITION 6.6.2. *If S denotes the set of finite places of F which ramify in D, we define the* new essential numerical torsion *as follows:*

$$\begin{aligned} h^{E,\mathrm{new}}_{\mathrm{tors}}(\Sigma) &= \prod_{S \subset R \subset \Sigma} \frac{|H_{1,\mathrm{tors}}(R, \mathbf{Z})|^{(-2)^{|\Sigma \backslash R|}}}{|H_{1,\mathrm{cong}}(R, \mathbf{Z})|^{(-2)^{|\Sigma \backslash R|}}} \\ h^{\mathrm{new}}_{\mathrm{cong}}(\Sigma) &= \prod_{S \subset R \subset \Sigma} |H_{1,\mathrm{cong}}(R, \mathbf{Z})|^{(-2)^{|\Sigma \backslash R|}} \end{aligned}$$

As in the case of h^{new}, the number $h^{E,\mathrm{new}}$ is not, a priori, the order of any group, and again one of our concerns later in this Chapter will be to try to interpret it as such.

Clearly, $h^{\text{new}}_{\text{tors}}(\Sigma) = h^{E,\text{new}}_{\text{tors}}(\Sigma) \cdot h^{\text{new}}_{\text{cong}}(\Sigma)$. We have the following refined version of Theorem 6.4.1.

THEOREM 6.6.3. *The quantity* $\dfrac{h^{E,\text{new}}_{\text{tors}}(Y)}{\text{reg}^{\text{new}}_{E}(Y)}$ *is the same with* Y *replaced by* Y'.

PROOF. In light of Theorem 6.4.1, it suffices to prove that the ratio $\dfrac{h^{\text{new}}_{\text{cong}}(Y)}{\text{vol}^{\text{new}}(Y)}$ is the same when Y is replaced by Y'. This expression can be written as an alternating product, each of whose terms looks like

$$\frac{|H_{1,\text{cong}}(R, \mathbf{Z})|}{\text{vol}(Y(K(R))}$$

to the power of $(-2)^{|\Sigma \setminus R|} = (-2)^{|V|}$. Recall that $\Sigma = S \cup T$, whereas $R = S \cup V$. By combining Borel's volume formula (Theorem 3.5.4) with Lemma 6.6.1, this term (without the exponent) can also be written as the $\#Y$th power of:

$$\left(\frac{\mu \cdot \zeta_F(2) \cdot |d_F|^{3/2}}{2^m(4\pi^2)^{[F:\mathbf{Q}]-1}}\right)^{-1} \cdot \prod_S \frac{(N\mathfrak{q}+1)}{(N\mathfrak{q}-1)} \prod_T \frac{(N\mathfrak{q}-1)}{(N\mathfrak{q}+1)},$$

(recall that $\mu \in \mathbf{Z}$ depends only on F). Let us count to what power each term is counted in the alternating product.

(1) The volume (related) term $\dfrac{\nu_N(\pi^2)^{[F:\mathbf{Q}]-1}}{\mu\zeta_F(2)|d_F|^{3/2}}$ occurs to exponent

$$1 - 2\binom{d}{1} + 2^2\binom{d}{2} - 2^3\binom{d}{3} + \ldots = (1-2)^d = (-1)^d.$$

(2) The factor $\dfrac{(N\mathfrak{q}+1)}{(N\mathfrak{q}-1)}$ for $\mathfrak{q} \in S$ occurs to the same exponent, for the same reason.

(3) The factor $\dfrac{(N\mathfrak{q}+1)}{(N\mathfrak{q}-1)}$ for $\mathfrak{q} \in \Sigma \setminus S$, which occurs to exponent -1 in the expression above and so to exponent $-(-2)^{|V|}$ in the alternating product, but only appears when $\mathfrak{q} \in V$, occurs to exponent to exponent

$$-1 + 2\binom{d-1}{1} - 2^2\binom{d-1}{2} + \ldots = -(1-2)^{d-1} = (-1)^d.$$

Both the volume term and the term $\dfrac{(N\mathfrak{q}+1)}{(N\mathfrak{q}-1)}$ for $\mathfrak{q} \in \Sigma$ do not depend on whether we are considering Y or Y'. Thus it suffices to show that in the product above they occur to the same exponent. Yet we have just calculated that they occur to exponent $(-1)^d\#Y$ and $(-1)^{d'}\#Y'$ respectively, where $d = |\Sigma \setminus S|$ and $d' = |\Sigma \setminus S'|$. We have already noted (cf. §3.3.3) that $\#Y = \#Y'$. On the other hand, since $|S| \equiv |S'| \mod 2$, it follows that $d \equiv d' \mod 2$, and so $(-1)^d = (-1)^{d'}$; the result follows. □

6.7. Essential homology and the torsion quotient

In light of Theorem 6.6.3 it is desirable to give an interpretation to the number $h^{E,\text{new}}_{\text{tors}}(Y)$ as the order of a certain group (representing "newforms.") In view of Definition 6.6.2 there are two issues: firstly, interpreting quantities like $\frac{|H_{1,\text{tors}}(\Sigma,\mathbf{Z})|}{|H_{1,\text{cong}}(\Sigma,\mathbf{Z})|}$ as group orders; and secondly, similarly interpreting the alternating product over

$R \subset \Sigma$. In this section we discuss the first issue. In other words, we will introduce notions of "essential" homology, that is to say, variant versions of homology which excise the congruence homology. There will be different ways of doing so, adapted to different contexts.

We have defined previously (§3.7) a congruence quotient of homology. By means of the linking pairing (§3.4.1 and §5.1.2) on $H_{1,\mathrm{tors}}(\Sigma, \mathbf{Z}_p)$, we may thereby also define a *congruence subgroup* of homology:

DEFINITION 6.7.1. *Suppose $p > 2$ is not an orbifold prime. Let $H_{1,\mathrm{cong}*}$ be the orthogonal complement, under the linking form, to the kernel of $H_{1,\mathrm{tors}}(\Sigma, \mathbf{Z}_p) \to H_{1,\mathrm{cong}}(\Sigma, \mathbf{Z}_p)$.*

Recall that the linking form is defined in §5.1.2 in the noncompact case for non-orbifold primes p and for level structures $K = K_\Sigma$, although this definition relies on a peculiarity of these level structures – that the cohomology of the cusps is vanishing (Lemma 5.4.4).

Note that the order of $H_{1,\mathrm{cong}}$ and $H_{1,\mathrm{cong}*}$ need not be the same; they differ by the p-part of *liftable* congruence homology (see § 3.7.1.2):

(6.7.1.1)
$$\frac{|H_{1,\mathrm{cong}}|}{|H_{1,\mathrm{cong}*}|} = h_{\mathrm{lif}}(\Sigma) \text{ (equality of } p\text{-parts.)}$$

DEFINITION 6.7.2. *Suppose $p > 2$ is not an orbifold prime.*
We define the essential *and* dual-essential *homology;*

$$H_1^E(\Sigma, \mathbf{Z}_p) = \ker(H_1 \to H_{1,\mathrm{cong}});$$

$$H_1^{E^*}(\Sigma, \mathbf{Z}_p) = H_1/H_{1,\mathrm{cong}*}.$$

We make a similar definition with $\mathbf{Z}_p$ replaced by any ring in which 2 and orbifold primes are invertible.

The first definition is quite general; as we observed above, the second relies, in the split case, on the vanishing of boundary homology of the cusps (Lemma 5.4.4) and thus is well-defined *only* when $p > 2$ and is not an orbifold prime. Although, e.g. $H_1^{E^*}(\Sigma, \mathbf{Z})$ is not defined for this reason, we will allow ourselves nonetheless to write

(6.7.2.1)
$$|H_1^{E^*}(\Sigma, \mathbf{Z})_{\mathrm{tors}}|$$

for the product of orders of $|H_1^{E^*}(\Sigma, \mathbf{Z}_p)_{\mathrm{tors}}|$ over all non-orbifold primes; equivalently, for the order of $H_1^{E^*}(\Sigma, \mathbf{Z}')_{\mathrm{tors}}$, where $\mathbf{Z}'$ is obtained by inverting all orbifold primes. This will be a convenient abbreviation since, in the rest of this chapter, we will usually work only "up to orbifold primes."

REMARK 6.7.3. Note that if the cohomology $H_1(\Sigma, \mathbf{Z}_p)$ is pure torsion, these notions are dual, i.e. there is a perfect pairing $H_1^E(\Sigma, \mathbf{Z}_p) \times H_1^{E^*}(\Sigma, \mathbf{Z}_p) \to \mathbf{Q}_p/\mathbf{Z}_p$.

REMARK 6.7.4. Even if the congruence homology is nontrivial, the essential homology may or may not differ, as an abstract group, from the actual homology; this is again related to the existence of liftable congruence homology (§ 3.7.1.2).

REMARK 6.7.5. The level lowering maps $\Phi^\vee$ and $\Psi^\vee$ induce maps $\Phi^\vee_E$ and $\Psi^\vee_E$ on essential homology, and similarly for dual-essential. (For example, consider the map $\Psi^\vee : H_1(\Sigma/\mathfrak{q}, \mathbf{Z})^2 \to H_1(\Sigma, \mathbf{Z})$. On torsion it is adjoint, under the linking form, to $\Psi : H_{1,\mathrm{tors}}(\Sigma, \mathbf{Z}) \to H_{1,\mathrm{tors}}(\Sigma/\mathfrak{q}, \mathbf{Z})^2$. It follows from that the $\Psi^\vee$ carries $H_{1,\mathrm{cong}*}(\Sigma/\mathfrak{q})$ to $H_{1,\mathrm{cong}*}(\Sigma)$; in particular, it induces a map $\Psi^\vee_E$ on dual-essential homology.) By virtue of this remark, one can define correspondingly essential new homology, dual-essential new homology, and so on.

These notions have different utility in different contexts. In particular, Ihara's lemma achieves a very elegant formulation in terms of essential homology, whereas dual statements seem better adapted to dual-essential homology:

THEOREM 6.7.6. *Suppose $p \nmid w_H$, the number of roots of unity in the Hilbert class field, and assume that the other conditions for Ihara's lemma (§ 4.1) are valid. Then the level-lowering map on essential homology*

$$\Psi : H_1^E(\Sigma, \mathbf{Z}_p) \longrightarrow H_1^E(\Sigma/\mathfrak{q}, \mathbf{Z}_p)^2$$

is surjective.

Under the same conditions, assume further that $H_1(\Sigma, \mathbf{Q}_p)^{\mathfrak{q}-\mathrm{new}} = 0$. Then the level-raising map on dual-essential homology

$$\Psi^\vee : H_1^{E^*}(\Sigma/\mathfrak{q}, \mathbf{Z}_p)^2 \to H_1^{E^*}(\Sigma, \mathbf{Z}_p)$$

is injective. More generally the same result holds localized at $\mathfrak{m}$ a maximal ideal of $\mathbf{T}_\Sigma$, assuming that $H_1(\Sigma, \mathbf{Z}_p)_\mathfrak{m}$ is torsion.

We note that $w_H = 2$ if F has a real place; otherwise, the only primes dividing w_H are 2 and 3.

6.7.7. The proof of the first statement in Theorem 6.7.6. Ihara's lemma implies that any element $h \in H_1^E(\Sigma/\mathfrak{q}, \mathbf{Z}_p)^2$ lifts to $h' \in H_1(\Sigma, \mathbf{Z}_p)$. The image of h' in $H_1^{\mathrm{cong}}(\Sigma, \mathbf{Z}_p)$ under $H_1^{\mathrm{cong}}(\Sigma, \mathbf{Z}_p) \to H_1^{\mathrm{cong}}(\Sigma/\mathfrak{q}, \mathbf{Z}_p)^2$ is trivial. Now (the homological version of) (3.7.6.2) and the following Lemma imply that we can modify h' so that it actually lies in $H_1^E(\Sigma, \mathbf{Z}_p)$:

LEMMA 6.7.8. *Suppose that $p \nmid w_H$. Under the assumptions of Ihara's Lemma, the image of the level raising map on cohomology:*

$$\Phi : H^1(\Sigma/\mathfrak{q}, \mathbf{Q}_p/\mathbf{Z}_p)^2 \to H^1(\Sigma, \mathbf{Q}_p/\mathbf{Z}_p)$$

has trivial intersection with the $\mathfrak{q}$-congruence cohomology (see §3.7.6 for definition). Similarly, the kernel of the level lowering map

$$\Psi : H_1(\Sigma, \mathbf{Z}_p) \to H_1(\Sigma/\mathfrak{q}, \mathbf{Z}_p)^2$$

on homology surjects onto the congruence homology at $\mathfrak{q}$.

PROOF. Let $\mathfrak{r}$ be an auxiliary prime (different from $\mathfrak{q}$, the primes dividing p, and not contained in Σ) with the property that $p \nmid N(\mathfrak{r}) - 1$ and that $\mathfrak{r}$ is principal. Such a prime exists by the Cebotarev density theorem and the fact that $p \nmid w_H$. Let $\Sigma\mathfrak{r} = \Sigma \cup \{\mathfrak{r}\}$. The prime $\mathfrak{r}$ has the property that $H^1(\Sigma\mathfrak{r}, \mathbf{Q}_p/\mathbf{Z}_p)$ does not have any (p-torsion) $\mathfrak{r}$-congruence cohomology, since (§ 3.7.6) the $\mathfrak{r}$-congruence homology has order dividing $N(\mathfrak{r}) - 1$ (as a consequence of our assumptions on $\mathfrak{r}$).

We have the following commutative diagram:

$$\begin{array}{ccc} H^1(\Sigma/\mathfrak{q}, \mathbf{Q}_p/\mathbf{Z}_p)^4 & \xrightarrow{\Phi_{\mathfrak{q}} \oplus \Phi_{\mathfrak{q}}} & H^1(\Sigma, \mathbf{Q}_p/\mathbf{Z}_p)^2 \\ \Big\downarrow{\scriptstyle \Phi_{\mathfrak{r}} \oplus \Phi_{\mathfrak{r}}} & & \Big\downarrow{\scriptstyle \Phi_{\mathfrak{r}}} \\ H^1(\Sigma\mathfrak{r}/\mathfrak{q}, \mathbf{Q}_p/\mathbf{Z}_p)^2 & \xrightarrow{\Phi_{\mathfrak{q}}} & H^1(\Sigma\mathfrak{r}, \mathbf{Q}_p/\mathbf{Z}_p) \end{array}$$

Here the map $\Phi_{\mathfrak{q}} \oplus \Phi_{\mathfrak{q}}$ is simply the map $\Phi_{\mathfrak{q}}$ applied to the first and second, and third and fourth copy of $H^1(\Sigma/\mathfrak{q})$, whereas the map $\Phi_{\mathfrak{r}}^2$ is defined on the first and third, and second and fourth factors respectively.

We first prove that this is commutative. First note that conjugation by $\begin{pmatrix} \pi_{\mathfrak{q}} & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} \pi_{\mathfrak{r}} & 0 \\ 0 & 1 \end{pmatrix}$ commute. It follows that the two nontrivial degeneracy maps $d_{\mathfrak{q}}$ and $d_{\mathfrak{r}}$ commute (the other degeneracys map are the obvious inclusions which certainly commute with everything.) It follows that the composition of maps on the upper corner is:

$$(\alpha, \beta, \gamma, \delta) \mapsto (\alpha + d_{\mathfrak{q}}\beta, \gamma + d_{\mathfrak{q}}\delta) \mapsto \alpha + d_{\mathfrak{q}}\beta + d_{\mathfrak{r}}\gamma + d_{\mathfrak{r}}d_{\mathfrak{q}}\delta,$$

whereas the lower corner is

$$(\alpha, \beta, \gamma, \delta) \mapsto (\alpha + d_{\mathfrak{r}}\gamma, \beta + d_{\mathfrak{r}}\delta) \mapsto \alpha + d_{\mathfrak{r}}\gamma + d_{\mathfrak{q}}\beta + d_{\mathfrak{r}}d_{\mathfrak{q}}\delta.$$

Assume, now, that $\Phi_{\mathfrak{q}}(\alpha, \beta)$ is congruence cohomology at $\mathfrak{q}$; to prove the lemma it suffices to show that it vanishes. We may choose (α', β') such that $\Phi_{\mathfrak{q}}(\alpha', \beta') = -[\mathfrak{r}]\Phi_{\mathfrak{q}}(\alpha, \beta)$: since $\mathfrak{r}$ is principal, $\alpha' = -\alpha$ and $\beta' = -\beta$ will do. Then

$$\Phi_{\mathfrak{q}} \oplus \Phi_{\mathfrak{q}}(\alpha, \beta, \alpha', \beta') = (\Phi_{\mathfrak{q}}(\alpha, \beta), -[\mathfrak{r}]\Phi_{\mathfrak{q}}(\alpha, \beta))$$

lies in the kernel of $\Phi_{\mathfrak{r}}$. By commutivity,

$$(\Phi_{\mathfrak{r}}(\alpha, \alpha'), \Phi_{\mathfrak{r}}(\beta, \beta'))$$

lies in the kernel of $\Phi_{\mathfrak{q}}$.

By Ihara's lemma, the kernel of $\Phi_{\mathfrak{q}}$ consists of classes of the form $([\mathfrak{q}]x, -x)$ where $x \in H^1_{\mathrm{cong}}(\Sigma/\mathfrak{q}, \mathbf{Q}_p/\mathbf{Z}_p)$.

In view of our assumptions on $\mathfrak{r}$, both of the degeneracy maps

$$H^1_{\mathrm{cong}}(\Sigma/\mathfrak{q}, \mathbf{Q}_p/\mathbf{Z}_p) \rightarrow H^1_{\mathrm{cong}}(\Sigma\mathfrak{r}/\mathfrak{q}, \mathbf{Q}_p/\mathbf{Z}_p)$$

are isomorphisms. In particular, there exists $a \in H^1_{\mathrm{cong}}(\Sigma/\mathfrak{q}, \mathbf{Q}_p/\mathbf{Z}_p)$ such that

$$\Phi_{\mathfrak{r}}(\alpha, \alpha') = \Phi_{\mathfrak{r}}(a, 0)$$

and similarly there is b such that $\Phi_{\mathfrak{r}}(\beta, \beta') = \Phi_{\mathfrak{r}}(b, 0)$.

Hence $(\alpha - a, \alpha')$ and $(\beta - b, \beta')$ lie in the kernel of $\Phi_{\mathfrak{r}}$ – that is to say, $\alpha, \beta \in H^1_{\mathrm{cong}}(\Sigma/\mathfrak{q})$. Then the image of (α, β) under $\Phi_{\mathfrak{q}}$ cannot be $\mathfrak{q}$-congruence by (3.7.6.2).

The claim about homology follows from an identical (dualized) argument. $\square$

6.7.9. The proof of the second statement of Theorem 6.7.6. Note that the second part of Theorem 6.7.6 is dual to the first statement when $H_1(\Sigma, \mathbf{Q}) = 0$. Let us now show that it continues to hold when $H_1(\Sigma, \mathbf{C})^{\mathfrak{q}-\text{new}} = 0$. The argument will be the same whether we first tensor with $\mathbf{T}_{\Sigma,\mathfrak{m}}$ or not, hence, we omit $\mathfrak{m}$ from the notation.

First of all, to show that $\Psi^\vee : H_1^{E^*}(\Sigma/\mathfrak{q}, \mathbf{Z}_p)^2 \to H_1^{E^*}(\Sigma, \mathbf{Z}_p)$ is injective, it is enough to verify the corresponding statement on the torsion subgroups (since it is clear on torsion-free quotients without any assumptions.)

On the other hand, the duality on $H_{1,\text{tors}}$ shows that the injectivity of $\Psi^\vee_{\text{tors}}$ is equivalent to the *surjectivity* of

$$\Psi : H_{1,E,\text{tors}}(\Sigma, \mathbf{Z}_p) \to H_{1,E,\text{tors}}(\Sigma/\mathfrak{q}, \mathbf{Z}_p)^2.$$

But we have shown in the first part of the Theorem that $\Psi : H_{1,E}(\Sigma, \mathbf{Z}_p) \to H_{1,E}(\Sigma/\mathfrak{q}, \mathbf{Z}_p)^2$ is surjective; since it is injective on torsion-free parts (this is where the assumption $H_1(\Sigma, \mathbf{Q}_p)^{\text{new}} = 0$ is used), it must also be surjective on torsion.

6.8. Torsion Jacquet–Langlands, refined form: spaces of newforms.

In this section, we outline to what extent we can interpret the *a priori* rational number $h^{E,\text{new}}_{\text{tors}}$ as the order of a space of newforms. Recall we have defined the notion of newforms in §3.10; moreover, that definition makes sense if we replace "homology" by "essential homology" or "dual-essential homology," and thus we also have notions of "essential newforms" and "dual-essential newforms." Although the notion of essential newforms seems more natural, it turns out that dual-essential newforms will be better for our purposes.

We will focus on the case $S = \emptyset$, $S' = \{\mathfrak{p}, \mathfrak{q}\}$. Note in that case the assumption (4.1.3.1) for Ihara's lemma is known to be true for $\mathbb{G}$. In particular, we will prove the "Theorem A" and "Theorem A†" and 'Theorem B" from the introduction to the manuscript. We do not aim for generality – rather illustrating some situations where one can obtain interesting results. The final section §6.9 briefly outlines the less satisfactory situation in the general case.

Again, we continue with the same notation as set earlier in the Chapter, so that Y, Y' are a Jacquet–Langlands pair.

6.8.1. The case where $H^1(Y(\Sigma), \mathbf{C}) = 0$. Suppose that $H^1(Y(\Sigma), \mathbf{C}) = 0$. Then we have an equality, from Theorem 6.6.3,

$$h^{E,\text{new}}_{\text{tors}}(Y) = h^{E,\text{new}}_{\text{tors}}(Y').$$

Note that $H^1(Y(R), \mathbf{C}) = 0$ for all $S \subset R \subset \Sigma$, and hence there is an equality:

$$h^{E,\text{new}}_{\text{tors}}(Y) = \prod_{S \subset R \subset \Sigma} |H_1^E(Y(R), \mathbf{Z})|^{(-2)^{\Sigma \setminus R}}.$$

To what extent does this differ from the order of the group $H_1^{E^*}(\Sigma, \mathbf{Z})^{\text{new}}$ of dual-essential newforms?

THEOREM 6.8.2. *Suppose $S = \emptyset, S' = \{\mathfrak{p}, \mathfrak{q}\}$. If $H_1(\Sigma, \mathbf{C}) = 0$, then there is an equality*

$$|H_1^{E^*}(Y(\Sigma), \mathbf{Z})^{\text{new}}| = \chi \cdot |H_1^{E^*}(Y'(\Sigma), \mathbf{Z})|.$$

for an integer $\chi \in \mathbf{Z}$, and away from primes ≤ 3 — i.e. the ratio is a rational number whose numerator and denominator are divisible only by 2 and 3.

Recall (see around (6.7.2.1)) that the order of $H_1^{E^*}(\Sigma, \mathbf{Z})$ makes sense away from orbifold primes.

Note that even under this quite simple assumption, we still obtain a factor χ which need not be trivial. We will see (§6.8.3) that we expect χ to be related to K_2; this idea is also supported by numerical evidence and certain computations with Galois deformation rings that have not been included in this document; they can be found in [**16**].

PROOF. All the statements that follow are to be understood to be valid only up to the primes 2 and 3.

We have $h_{\text{tors}}^{E,\text{new}}(Y) = h_{\text{tors}}^{E,\text{new}}(Y')$, and, in this case, $h_{\text{tors}}^{E,\text{new}}(Y')$ is simply the order of $H_1^{E^*}(Y'(\Sigma), \mathbf{Z})$.

It remains to analyze the dual-essential new homology for Y. That group is, by definition, the cokernel of the level-raising map:

$$H_1^{E^*}(\Sigma/\mathfrak{p})^2 \oplus H_1^{E^*}(\Sigma/\mathfrak{q})^2 \to H_1^{E^*}(\Sigma).$$

Dualizing (see Remark 6.7.3– all groups involved are torsion) this is dual to the kernel of the right-most map in

$$H_1^{E}(\Sigma/\mathfrak{pq})^4 \leftarrow H_1^{E}(\Sigma/\mathfrak{p})^2 \oplus H_1^{E}(\Sigma/\mathfrak{q})^2 \leftarrow H_1^{E}(\Sigma). \tag{6.8.2.1}$$

Away from primes dividing w_H, the left-most map is surjective (Theorem 6.7.6) and the alternating ratio of orders of the sizes of the groups equals $h_{\text{tors}}^{E,\text{new}}(\Sigma)$. The result follows, where χ equals the order of the homology of the sequence at the middle term. □

6.8.3. The nature of χ. We prove Theorem A† from the introduction, in particular, yielding a relationship between χ from the prior Theorem 6.8.2 and K_2.

As in the statement of Theorem A† we suppose that the class number is odd (so that all the involved manifolds have a single connected component), that $\Sigma = \{\mathfrak{p}, \mathfrak{q}\}$ (so that we are working "at level 1" on the quaternionic side), and, as before, that $H_1(\Sigma)$ is pure torsion. We fix a prime $\ell > 3$, and will compute ℓ-parts. Set g to be the ℓ-part of $\gcd(N\mathfrak{p} - 1, N\mathfrak{q} - 1)$.

We will show that:

$$\text{the } \ell\text{-part of } \#K_2(\mathscr{O}_F) \text{ divides } \chi \cdot g \tag{6.8.3.1}$$

which implies Theorem A†.

The integer χ is the order of the homology at the middle of (6.8.2.1).

Now consider the square, where all homology is taken with coefficients in $\mathbf{Z}_\ell$, and we impose Atkin-Lehner signs (i.e., all homology at level $\Sigma/\mathfrak{p}$ should be in the $-$ eigenspace for $w_\mathfrak{q}$, all homology at level $\Sigma/\mathfrak{q}$ should be in the $-$ eigenspace for $w_\mathfrak{p}$, and at level Σ we impose both $-$ conditions):

$$\begin{array}{ccccc}
H^E_{1,\mathrm{tors}}(\Sigma/\mathfrak{pq}) & \longleftarrow & H^E_{1,\mathrm{tors}}(\Sigma/\mathfrak{p})^- \oplus H^E_{1,\mathrm{tors}}(\Sigma/\mathfrak{q})^- & \longleftarrow & H^E_{1,\mathrm{tors}}(\Sigma)^{--}. \\
\downarrow & & \downarrow & & \downarrow \\
H_{1,\mathrm{tors}}(\Sigma/\mathfrak{pq}) & \longleftarrow & H_{1,\mathrm{tors}}(\Sigma/\mathfrak{p})^- \oplus H_{1,\mathrm{tors}}(\Sigma/\mathfrak{q})^- & \longleftarrow & H_{1,\mathrm{tors}}(\Sigma)^{--}. \\
\downarrow & & \downarrow f & & \downarrow \\
H_{1,\mathrm{cong}}(\Sigma/\mathfrak{pq}) & \longleftarrow & H_{1,\mathrm{cong}}(\Sigma/\mathfrak{p})^- \oplus H_{1,\mathrm{cong}}(\Sigma/\mathfrak{q})^- & \longleftarrow & H_{1,\mathrm{cong}}(\Sigma)^{--}.
\end{array}$$

Each vertical row is a short exact sequence, by definition. Also, $\Sigma = \{\mathfrak{p}, \mathfrak{q}\}$, so that $\Sigma/\mathfrak{pq}$ is the trivial level structure, but we have continued to use Σ for compatibility with our discussion elsewhere. Also note that $H_{1,\mathrm{cong}}(\Sigma)^{--}$ is trivial, by computations similar to that of §6.7.2.

When we speak about "homology" of this square in what follows, we always mean the homology of the horizontal rows. For example, the homology at the middle term of the bottom row is $\left(k_{\mathfrak{p}}^{\times} \oplus k_{\mathfrak{q}}^{\times}\right) \otimes \mathbf{Z}_\ell$ (recall that there is only one connected component). Now let M denote the homology group at the middle of the upper row, and let $\tilde{M}$ denote the homology group in the center of the square; then we have a surjection:

$$M \twoheadrightarrow \ker\left(\tilde{M} \stackrel{f}{\to} (k_{\mathfrak{p}}^{\times} \oplus k_{\mathfrak{q}}^{\times}) \otimes \mathbf{Z}_\ell\right). \tag{6.8.3.2}$$

But $\tilde{M}$ arose already, in the computations of the homology of S-arithmetic groups (§4.4, e.g. (4.4.1.1)); the spectral sequence of that section shows that we have an exact sequence

$$\text{a quotient of } \mathbf{Z}_\ell/g \to H_2(Y(K_\Sigma[\frac{1}{\mathfrak{pq}}], \mathbf{Z}_\ell) \twoheadrightarrow \tilde{M},$$

We have also proven (same proof as Theorem 4.5.1 (i), see (4.5.7.11)) that there is a surjection

$$H_2(Y(K_\Sigma[\frac{1}{\mathfrak{pq}}]), \mathbf{Z}_\ell) \twoheadrightarrow K_2(\mathscr{O}_F[\frac{1}{\mathfrak{pq}}]) \otimes \mathbf{Z}_\ell. \tag{6.8.3.3}$$

Finally, the groups $\tilde{M}, H_2(\dots), K_2(\mathscr{O}_F[\frac{1}{\mathfrak{pq}}]$ all have maps to $(k_{\mathfrak{p}}^{\times} \oplus k_{\mathfrak{q}}^{\times}) \otimes \mathbf{Z}_\ell$ (the map f from (6.8.3.2), the pull-back of f to $H_2(\dots)$, and the tame symbols at $\mathfrak{p}, \mathfrak{q}$, respectively), and these three maps are all *compatible*. The kernel of the tame symbols on $K_2(\mathscr{O}_F[\frac{1}{\mathfrak{pq}}])$ is $K_2(\mathscr{O}_F)$. That implies that

$$\text{the } \ell\text{-part of } \#K_2(\mathscr{O}_F) \text{ divides } |M| \cdot g.$$

But M is a direct summand of the group whose order defines χ, so we are done proving (6.8.3.1).

6.8.4. No newforms. We continue under the assumption of $S = \emptyset, S' = \{\mathfrak{p}, \mathfrak{q}\}$.

Now let us suppose that $H_1(\Sigma, \mathbf{C})$ is non-zero, but that $H_1(\Sigma, \mathbf{C})$ is completely accounted for by the image of the space of newforms of level $\Sigma/\mathfrak{q}$, that is to say,

$H_1(\Sigma, \mathbf{C}) \simeq H_1(\Sigma/\mathfrak{q}, \mathbf{C})^2$. In this case, we see that $\mathrm{reg}^{\mathrm{new}}(Y') = 1$. and Theorem 6.6.3 manifests itself as an equality

$$\frac{h^{E,\mathrm{new}}_{\mathrm{tors}}(Y)}{\mathrm{reg}^{\mathrm{new}}_E(Y)} = h^{E,\mathrm{new}}_{\mathrm{tors}}(Y'), \tag{6.8.4.1}$$

We may also compute $\mathrm{reg}^{\mathrm{new}}_E(Y)$ as well, namely, by Theorem 5.5.5,

$$\mathrm{reg}^{\mathrm{new}}_E(Y) = \frac{|H_{1,\mathrm{lif}}(\Sigma/\mathfrak{q};\mathfrak{q})|^2}{\Delta}, \tag{6.8.4.2}$$

where Δ is the order of the cokernel of the map

$$\begin{pmatrix} (N(\mathfrak{q})+1) & T_{\mathfrak{q}} \\ T_{\mathfrak{q}} & (N(\mathfrak{q})+1) \end{pmatrix}$$

on $H_1^{\mathrm{tf}}(\Sigma/\mathfrak{q}, \mathbf{Z})^2$ to itself. Therefore,

$$h^{E,\mathrm{new}}_{\mathrm{tors}}(Y)\Delta = h^{E,\mathrm{new}}_{\mathrm{tors}}(Y') \cdot |H_{1,\mathrm{lif}}(\Sigma/\mathfrak{q};\mathfrak{q})|^2 \tag{6.8.4.3}$$

In particular, the factor Δ manifests itself as torsion on the non-split quaternion algebra – for example, if ℓ divides $|\Delta|$ but not $|H_{1,\mathrm{lif}}|$, then ℓ must divide the order of a torsion H_1 group for Y'.

A rather easy consequence is "Theorem B" (Theorem 1.1.1) from the introduction:

THEOREM 6.8.5. *Suppose that $S = \emptyset, S' = \{\mathfrak{p}, \mathfrak{q}\}$. Suppose that $l > 3$ is a prime such that:*

1. $H_1(Y(\Sigma/\mathfrak{p}), \mathbf{Z}_l) = 0$*,*
2. $H_1(Y(\Sigma/\mathfrak{q}), \mathbf{Z}_l)$ *and* $H_1(Y(\Sigma), \mathbf{Z}_l)$ *is torsion free,*
3. $H_1(Y'(\Sigma), \mathbf{C}) = 0$*.*

Then l can divide $H_1(Y'(\Sigma), \mathbf{Z})$ if and only if l divides $\frac{\Delta}{(N(\mathfrak{p})-1)^{\#Y}}$, where $\#Y$ is the number of connected components.

Note that this notation is not precisely compatible with that used in Theorem B, but the content is the same (taking account that Theorem B is stated in the context of odd class number, so that $\#Y = 1$).

PROOF. The assumptions imply that there is no mod-ℓ torsion at level $\Sigma, \Sigma/\mathfrak{p}, \Sigma/\mathfrak{q}$ or $\Sigma/\mathfrak{p}\mathfrak{q}$. Therefore

$$h^{\mathrm{new}}_{\mathrm{tors}}(Y) = 1.$$

Next we must account for congruence homology. Assumption (1) implies that there is no congruence cohomology of characteristic l at level $\Sigma/\mathfrak{p}$, and thus ℓ is prime to $H_{1,\mathrm{cong}}(\Sigma/\mathfrak{p})$ and $H_{1,\mathrm{cong}}(\Sigma/\mathfrak{p}\mathfrak{q}, \mathbf{Z})$. That means that the l-part of $H_{1,\mathrm{cong}}(\Sigma/\mathfrak{q})$ and $H_{1,\mathrm{cong}}(\Sigma)$ arises entirely from $\mathfrak{p}$-congruence classes; in particular, they have order equal to (the l-part of) $(N(\mathfrak{p})-1)^{\#Y}$, where $\#Y$ is the number of components of Y. From that we deduce that

$$h^{\mathrm{new}}_{\mathrm{tors},E}(Y) = (N(\mathfrak{p})-1)^{\#Y} \text{ (equality of } \ell\text{-parts)}.$$

As for the quantity $H_{1,\mathrm{lif}}(\Sigma/\mathfrak{q};\mathfrak{q})$ which shows up above – our assumption (2) implies that it equals simply $(N\mathfrak{p}-1)^{\#Y}$. So (6.8.4.3) shows that the size of $H_1^E(Y'(\Sigma), \mathbf{Z}_\ell)$ (or $H_1^{E^*}$; they both have the same order here) equals the ℓ-part of $\frac{\Delta}{(N\mathfrak{p}-1)^{\#Y}}$. ∎

Now we pass to a more difficult case, to illustrate the type of results that we can still obtain about newforms:

THEOREM 6.8.6. *Suppose* $S = \emptyset, S' = \{\mathfrak{p}, \mathfrak{q}\}$. *Suppose that* $\ell > 3$ *is a prime such that:*

(i) *The natural map* $H_1(\Sigma, \mathbf{Q}_\ell) \rightarrow H_1(\Sigma/\mathfrak{q}, \mathbf{Q}_\ell)^2$ *is an isomorphism, and* $H_1(\Sigma/\mathfrak{p}, \mathbf{Q}_\ell) = 0$*;*

(ii) *There are no mod* ℓ *congruences between torsion forms at level* $\Sigma/\mathfrak{pq}$ *and characteristic zero forms at level* $\Sigma/\mathfrak{q}$*, i.e. there is no maximal ideal for the Hecke algebra* $\mathbf{T}_{\Sigma/\mathfrak{q}} \otimes \mathbf{Z}_\ell$ *that has support both in* $H_{1,\mathrm{tors}}(\Sigma/\mathfrak{pq}, \mathbf{Z}_\ell)$ *and* $H_{1,\mathrm{tf}}(\Sigma/\mathfrak{q}, \mathbf{Z}_\ell)$*.*

Then, up to ℓ*-units,*

$$|H_1^{E^*}(Y(\Sigma), \mathbf{Z}_\ell)^{\mathrm{new}}| = |H_1^{E^*}(Y'(\Sigma), \mathbf{Z}_\ell)| \cdot \chi.$$

for an integer $\chi \in \mathbf{Z}$*.*

Again, we believe that the integer χ is related to K_2, as was the case in §6.8.3.

PROOF. In what follows equalities should be regarded as equalities of ℓ-parts. The proof is a series of diagram chases. Examine the diagram (recall the definition of $H_{1,\mathrm{cong}*}$ from § 6.7)

$$\begin{array}{ccccc}
H_{1,\mathrm{cong}*}(\Sigma/\mathfrak{pq})^4 & \longleftarrow & H_{1,\mathrm{cong}*}(\Sigma/\mathfrak{p})^2 \oplus H_{1,\mathrm{cong}*}(\Sigma/\mathfrak{q})^2 & \longleftarrow & H_{1,\mathrm{cong}*}(\Sigma). \\
\downarrow & & \downarrow & & \downarrow \\
H_{1,\mathrm{tors}}(\Sigma/\mathfrak{pq})^4 & \longleftarrow & H_{1,\mathrm{tors}}(\Sigma/\mathfrak{p})^2 \oplus H_{1,\mathrm{tors}}(\Sigma/\mathfrak{q})^2 & \longleftarrow & H_{1,\mathrm{tors}}(\Sigma). \\
\downarrow & & \downarrow & & \downarrow \\
H^{E^*}_{1,\mathrm{tors}}(\Sigma/\mathfrak{pq})^4 & \longleftarrow & H^{E^*}_{1,\mathrm{tors}}(\Sigma/\mathfrak{p})^2 \oplus H^{E^*}_{1,\mathrm{tors}}(\Sigma/\mathfrak{q})^2 & \longleftarrow & H^{E*}_{1,\mathrm{tors}}(\Sigma).
\end{array}$$

The horizontal arrows are level-lowering maps. The vertical rows are exact by definition. Inspecting this diagram, and using (6.7.1.1), we deduce the equality (6.8.6.1)

$$h^{E,\mathrm{new}}_{\mathrm{tors}}(Y) = \frac{|H_1^{E^*}(\Sigma, \mathbf{Z}_\ell)^{\mathrm{tors}}| \left(\ldots \Sigma/\mathfrak{pq} \cdots\right)^4}{\left(\cdots \Sigma/\mathfrak{p} \cdots\right)^2 \left(\ldots \Sigma/\mathfrak{q} \cdots\right)^2} \cdot |H_{1,\mathrm{lif}}(\Sigma, \mathbf{Z}_\ell)|^{-1} \cdot |H_{1,\mathrm{lif}}(\Sigma/\mathfrak{q}, \mathbf{Z}_\ell)|^2.$$

Now consider the diagram with exact vertical rows:

$$\begin{array}{ccccc}
H^{E^*}_{1,\mathrm{tors}}(\Sigma/\mathfrak{pq})^4 & \longrightarrow & H^{E^*}_{1,\mathrm{tors}}(\Sigma/\mathfrak{p})^2 \oplus H^{E^*}_{1,\mathrm{tors}}(\Sigma/\mathfrak{q})^2 & \longrightarrow & H^{E^*}_{1,\mathrm{tors}}(\Sigma). \\
\downarrow & & \downarrow & & \downarrow \\
H_1^{E^*}(\Sigma/\mathfrak{pq})^4 & \xrightarrow{A} & H_1^{E^*}(\Sigma/\mathfrak{p})^2 \oplus H_1^{E^*}(\Sigma/\mathfrak{q})^2 & \longrightarrow & H_1^{E^*}(\Sigma). \\
\downarrow & & \downarrow & & \downarrow \\
H_{1,\mathrm{tf}}(\Sigma/\mathfrak{pq})^4 & \longrightarrow & H_{1,\mathrm{tf}}(\Sigma/\mathfrak{p})^2 \oplus H_{1,\mathrm{tf}}(\Sigma/\mathfrak{q})^2 & \xrightarrow{B} & H_{1,\mathrm{tf}}(\Sigma).
\end{array}$$

Taking into account Lemma 4.2.1 to analyze the cokernel of B, and noting that Theorem 6.7.6 and assumption (ii) of the theorem statement prove that A is injective away from primes dividing w_H, the diagram shows that

$$|H_1^{E^*}(Y(\Sigma), \mathbf{Z}_\ell)^{\mathrm{new}}| = \chi \cdot \frac{\Delta}{|H_{1,\mathrm{lif}}(\Sigma/\mathfrak{q};\mathfrak{q})|} \cdot \frac{H_1^{E^*}(\Sigma, \mathbf{Z}_\ell)^{\mathrm{tors}} \left(\ldots \Sigma/\mathfrak{pq} \cdots\right)^4}{\left(\cdots \Sigma/\mathfrak{p} \cdots\right)^2 \left(\ldots \Sigma/\mathfrak{q} \cdots\right)^2}, \tag{6.8.6.2}$$

where $\chi \in \mathbf{Z}$. As before, Δ is the determinant of $(T_{\mathfrak{q}}^2 - (1+N(\mathfrak{q}))^2)$ on $H_1(\Sigma/\mathfrak{q}, \mathbf{C})$, and the equality is an equality of ℓ-parts.

Combining (6.8.6.1) and (6.8.6.2), and using the fact from (6.8.4.2) that $\mathrm{reg}_E^{\mathrm{new}}(Y) = h_{\mathrm{lif}}(\Sigma/\mathfrak{q};\mathfrak{q})^2 \Delta^{-1}$,

$$\begin{aligned} |H_1^{E^*}(Y(\Sigma), \mathbf{Z}_\ell)^{\mathrm{new}}| &= \chi \cdot h_{\mathrm{tors}}^{E,\mathrm{new}}(Y) \cdot \Delta \cdot \frac{h_{\mathrm{lif}}(\Sigma) h_{\mathrm{lif}}(\Sigma/\mathfrak{q};\mathfrak{q})^{-1}}{h_{\mathrm{lif}}(\Sigma/\mathfrak{q})^2} \\ &= \chi \cdot \frac{h_{\mathrm{tors}}^{E,\mathrm{new}}(Y)}{\mathrm{reg}_E^{\mathrm{new}}(Y)} \cdot \frac{h_{\mathrm{lif}}(\Sigma) h_{\mathrm{lif}}(\Sigma/\mathfrak{q};\mathfrak{q})}{h_{\mathrm{lif}}(\Sigma/\mathfrak{q})^2}. \end{aligned} \tag{6.8.6.3}$$

We now need to analyze liftable congruence homology. Consider the commutative diagram

$$\begin{array}{ccccc} H_1(\Sigma/\mathfrak{q}, \mathbf{Z}_\ell)^2_{\mathrm{tors}} & \xleftarrow{\Psi_{\mathrm{tors}}} & H_1(\Sigma, \mathbf{Z}_\ell)_{\mathrm{tors}} & \longleftarrow & \ker(\Psi_{\mathrm{tors}}) \\ \downarrow & & \downarrow & & \downarrow \\ H_1(\Sigma/\mathfrak{q}, \mathbf{Z}_\ell)^2_{\mathrm{cong}} & \xleftarrow{\mathrm{id} \oplus [\mathfrak{q}]} & H_1(\Sigma, \mathbf{Z}_\ell)_{\mathrm{cong}} & \longleftarrow & H_1(\Sigma, \mathbf{Z}_\ell)_{\mathfrak{q}-\mathrm{cong}} \\ \downarrow & & \downarrow & & \\ H_{1,\mathrm{lif}}(\Sigma/\mathfrak{q})^2 & \xleftarrow{\iota} & H_{1,\mathrm{lif}}(\Sigma) & & \end{array}$$

By a diagram chase using Lemma 6.7.8 – note that $\ker(\Psi_{\mathrm{tors}}) = \ker(\Psi)$, by assumption – the induced map $\iota : H_{1,\mathrm{lif}}(\Sigma) \to H_{1,\mathrm{lif}}(\Sigma/\mathfrak{q})^2$ is injective. Write L for image of $H_1(\Sigma/\mathfrak{q}, \mathbf{Z}_\ell)_{\mathrm{tors}}$ inside $H_1(\Sigma/\mathfrak{q}, \mathbf{Z}_\ell)_{\mathrm{cong}}$. Then one easily sees that the image of ι has size equal to the index of $L \cap [\mathfrak{q}]L$ in $H_1(\Sigma/q, \mathbf{Z}_\ell)$. So,

$$h_{1,\mathrm{lif}}(\Sigma) = \frac{|H_1(\Sigma/\mathfrak{q}, \mathbf{Z}_\ell)_{\mathrm{cong}}|}{|L \cap [\mathfrak{q}]L|}.$$

On the other hand, by definition,

$$h_{\mathrm{lif}}(\Sigma/\mathfrak{q};\mathfrak{q}) = \frac{|H_1(\Sigma/\mathfrak{q}, \mathbf{Z}_\ell)_{\mathrm{cong}}|}{|L + [\mathfrak{q}]L|} \text{ and } h_{\mathrm{lif}}(\Sigma/\mathfrak{q}) = \frac{|H_1(\Sigma/\mathfrak{q}, \mathbf{Z}_\ell)_{\mathrm{cong}}|}{|L|}.$$

Putting these together, we see that $h_{\mathrm{lif}}(\Sigma) \cdot h_{\mathrm{lif}}(\Sigma/q;\mathfrak{q}) = h_{\mathrm{lif}}(\Sigma/\mathfrak{q})^2$; and, returning to (6.8.6.3) and recalling (6.8.4.1), we deduce

$$\begin{aligned} |H_1^{E^*}(Y(\Sigma), \mathbf{Z}_\ell)^{\mathrm{new}}| &= \chi \cdot h_{\mathrm{tors}}^{E,\mathrm{new}}(Y') \\ &= \chi \cdot |H_1^{E*}(Y'(\Sigma), \mathbf{Z}_\ell)|. \end{aligned}$$

Note that the change in essential homology exactly accounts for the change in regulator, when accounted for properly. □

6.8.7. An example with level-lowering and newforms. We continue to suppose that $S' = \{\mathfrak{p}, \mathfrak{q}\}$ and $S = \emptyset$.

Now suppose that $H_1(\Sigma, \mathbf{C}) \neq 0$, but that we have $H_1(\Sigma/\mathfrak{q}, \mathbf{C}) = 0$ for all $\mathfrak{q} \in \Sigma$. Recall that Theorem 6.6.3 in this case says that

$$\frac{h^{E,\mathrm{new}}_{\mathrm{tors}}(Y)}{h^{E,\mathrm{new}}_{\mathrm{tors}}(Y')} = \frac{\mathrm{reg}_E(Y)}{\mathrm{reg}_E(Y')}.$$

We deduce from this the following: Suppose that $H_1(Y(\Sigma), \mathbf{Z})$ and $H_1(Y'(\Sigma), \mathbf{Z})$ are p-torsion free (or, more generally, that their p-torsion subgroups have the same order). Let us also assume that p is co-prime to the order of the congruence homology. Then

> p divides $\mathrm{reg}_E(Y')/\mathrm{reg}_E(Y)$ only if there exists a level lowering prime $\mathfrak{m}$ of characteristic p for $H_1(Y(\Sigma), \mathbf{Z})$.

Indeed, if p is to divide $\mathrm{reg}_E(Y')/\mathrm{reg}_E(Y)$, there must be p-torsion for $Y(\Sigma/\mathfrak{p})$ or $Y(\Sigma/\mathfrak{q})$; Ihara's lemma shows that this must divide a level lowering prime, since there is no p-torsion at level $Y(\Sigma)$. This provides theoretical evidence towards the conjecture formulated as Principle 6.5.2.1.

More generally, one can prove, similarly to the prior results, that:

THEOREM 6.8.8. *Suppose $S = \emptyset, |S'| = 2$. Suppose that $H_1(\Sigma/\mathfrak{q}, \mathbf{C}) = 0$ for all $\mathfrak{q} \in S'$. Then there is an equality*

$$\frac{|H_1^{E^*}(Y(\Sigma), \mathbf{Z})^{\mathrm{new},\mathrm{tors}}|}{|H_1^{E^*}(Y'(\Sigma), \mathbf{Z})^{\mathrm{tors}}|} \cdot \frac{1}{|\mathscr{L}(\Sigma)|} = \chi \cdot \frac{\mathrm{reg}_E(Y)}{\mathrm{reg}_E(Y')}, \tag{6.8.8.1}$$

where p divides $\mathscr{L}(\Sigma)$ if and only if there exists a prime $\mathfrak{m}$ of the Hecke algebra, of characteristic dividing p, which occurs in the the support of an eigenform of characteristic zero for $Y(\Sigma)$, but also occurs in the support of the integral cohomology at some strictly lower level. Moreover, $\chi \in \mathbf{Z}$.

Our expectation is that the two terms on the left-hand side of (6.8.8.1) match termwise with those on the right-hand side, that is to say:

1. The spaces $H_1^{E^*}(\Sigma, \mathbf{Z})^{\mathrm{new}}$ are the same for Y and Y', possibly up to factors arising from K_2-classes which correspond to χ,
2. The ratio of regulators corresponds exactly to level lowering primes $\mathfrak{m}$, which are accounted for by the factor $\mathscr{L}(\Sigma)$,

and the theorem above shows that the "product" of these two statements holds, which can be considered as a consistency check.

6.9. The general case

In the previous section, we have given a variety of cases – assuming that $S = \emptyset$ and $|S'| = 2$ – where our results lead to direct results about the orders of spaces of newforms.

In this section, we discuss what can be said in a general setting. Since the results here are more conditional, because of our lack of knowledge about the higher cohomology of S-arithmetic groups, we content ourselves with a brief summary of what can be proven by similar methods, and do not give details.

Consider the following ratio:

$$\frac{|H_1^{E}(Y(\Sigma), \mathbf{Z})^{\mathrm{new},\mathrm{tors}}|}{|H_1^{E}(Y'(\Sigma), \mathbf{Z})^{\mathrm{new},\mathrm{tors}}|} \cdot \frac{\mathrm{reg}_E^{\mathrm{new}}(Y')}{\mathrm{reg}_E^{\mathrm{new}}(Y)}. \tag{6.9.0.2}$$

What is our expectation for this quantity?

We expect that spaces of newforms should be equal when localized at some maximal ideal $\mathfrak{m}$, with the exception of a correction factor related to K-theoretic classes.

Similarly, we also expect (see the discussion in § 6.5.1) that the new regulators should differ exactly at level lowering primes; that is, p dividing the residue characteristic of primes $\mathfrak{m}$ which are supported on the space of newforms which lift to characteristic zero, but are *also* supported — possibly at the level of torsion — at lower level. (Note that, if neither Y nor Y' are split, there can be level lowering primes contributing to both the numerator and denominator of this quantity.)

On the basis of this discussion, we expect that this quantity (6.9.0.2) is always equal to:

$$\frac{\chi_D(Y)}{\chi_D(Y')} \cdot \frac{|\mathscr{L}(Y)|}{|\mathscr{L}(Y')|}, \tag{6.9.0.3}$$

where $\mathscr{L}(Y)$ and $\mathscr{L}(Y')$ are finite $\mathbf{T}$-modules recording the level lowering congruences described above, and χ_D is related to K-theory. (In particular, if Y is split, then $\mathscr{L}(Y')$ should be trivial and so $|\mathscr{L}(Y)| \in \mathbf{Z}$ is divisible exactly by level lowering primes.)

What can we prove? Indeed, an analysis along the lines of the prior sections does show a result of the type (6.9.0.3), where

- $\chi(\Sigma)$ is defined as a certain alternating product of orders of cohomology of S-arithmetic groups, in the range where according to Conjecture 4.4.4 all the groups involved are Eisenstein.
- $\widetilde{\mathscr{L}}(Y)$ and $\widetilde{\mathscr{L}}(Y')$ are virtual $\mathbf{T}$-modules related to Y and Y' with the following property:

  $\widetilde{\mathscr{L}}(Y)$ is finite and has support at an ideal $\mathfrak{m}$ of $\mathbf{T}$ only if one of the following occurs:
  (1) $\mathfrak{m}$ occurs in the support of $H_1(\Sigma, \mathbf{Z})^{\mathrm{tf}}$, but also occurs in the support of $H_1(T, \mathbf{Z})$ for some $S \subset T \subsetneq \Sigma$ of smaller level,
  (2) $d > 2$, $\mathfrak{m}$ has characteristic p, and p divides $N(\mathfrak{p}) - 1$ or $N(\mathfrak{p}) + 1$ for some $p \in \Sigma \setminus S \cap S'$.

Some remarks:

(1) If $d > 2$, we lose control of factors dividing $N(\mathfrak{p}) - 1$ or $N(\mathfrak{p}) + 1$. This comes from the difficulty of understanding the interaction of the H_3-row of the spectral sequence in the non-split case (and similarly, the contribution to the H_2 row from the fundamental class of the cusps).
(2) The level lowering primes $\mathfrak{m}$ we expect to occur in $\mathscr{L}(Y)$ should actually have support in $H_1(\Sigma, \mathbf{Z})^{\mathrm{tf,new}}$; that is, occur as characteristic zero *newforms*. Moreover, although we know that $\mathfrak{m}$ contributing to $\mathscr{L}(Y)$ above have the listed property, there is no converse; similarly, we can't control (for big d) whether such primes contribute to the numerator or denominator.

It would be good to improve our understanding on any of these points! It would of course be good to show that the terms of (6.9.0.2) and (6.9.0.3) match up “term by term”, and not merely that their product is equal.

Bibliography

[1] Milton Abramowitz and Irene A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.

[2] Avner Ash. Galois representations attached to mod p cohomology of $GL(n, \mathbf{Z})$. *Duke Math. J.*, 65(2):235–255, 1992.

[3] Karim Belabas and Herbert Gangl. Generators and relations for $K_2\mathscr{O}_F$. *K-Theory*, 31(3):195–231, 2004.

[4] Tobias Berger. On the Eisenstein ideal for imaginary quadratic fields. *Compos. Math.*, 145(3):603–632, 2009.

[5] Nicolas Bergeron, M. H. Sengün, and Akshay Venkatesh. Torsion homology growth and cycle complexity on arithmetic manifolds. `http://arxiv.org/abs/1401.6989`.

[6] Nicolas Bergeron and Akshay Venkatesh. Asymptotic growth of torsion homology for arithmetic groups. to appear in J. Inst. Math. Jussieu, 2012.

[7] A. Borel. Commensurability classes and volumes of hyperbolic 3-manifolds. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 8(1):1–33, 1981.

[8] A. Borel and H. Jacquet. Automorphic forms and automorphic representations. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 189–207. Amer. Math. Soc., Providence, R.I., 1979. With a supplement "On the notion of an automorphic representation" by R. P. Langlands.

[9] Armand Borel and Nolan R. Wallach. *Continuous cohomology, discrete subgroups, and representations of reductive groups*, volume 94 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, N.J., 1980.

[10] Kevin Buzzard. Integral models of certain Shimura curves. *Duke Math. J.*, 87(3):591–612, 1997.

[11] Kevin Buzzard and Toby Gee. The conjectural connections between automorphic representations and galois representations. Proceedings of the LMS Durham Symposium 2011 (to appear), 2011.

[12] Frank Calegari and Nathan M. Dunfield. Automorphic forms and rational homology 3-spheres. *Geom. Topol.*, 10:295–329 (electronic), 2006.

[13] Frank Calegari and Matthew Emerton. On the ramification of Hecke algebras at Eisenstein primes. *Invent. Math.*, 160(1):97–144, 2005.

[14] Frank Calegari and David Geraghty. Modularity lifting beyond Taylor–Wiles. preprint, 2012.

[15] Frank Calegari and Barry Mazur. Nearly ordinary Galois deformations over arbitrary number fields. *J. Inst. Math. Jussieu*, 8(1):99–177, 2009.

[16] Frank Calegari and Akshay Venkatesh. A torsion jacquet-langlands correspondence, manuscript. `http://arxiv.org/abs/1212.3847`.

[17] Jeff Cheeger. Analytic torsion and the heat equation. *Ann. of Math. (2)*, 109(2):259–322, 1979.

[18] Laurent Clozel. La conjecture se serre dans le cas complexe. unpublished manuscript, Séminaire Henniart–Mézard, 1999, 2007.

[19] Laurent Clozel. La conjecture de serre dans le cas pair. unpublished manuscript, 2007.

[20] Chandan Singh Dalawat. Some aspects of the functor k_2 of fields. preprint.

[21] R. M. Damerell. *L*-functions of elliptic curves with complex multiplication. II. *Acta Arith.*, 19:311–317, 1971.

[22] Henri Darmon, Fred Diamond, and Richard Taylor. Fermat's last theorem. In *Elliptic curves, modular forms & Fermat's last theorem (Hong Kong, 1993)*, pages 2–140. Int. Press, Cambridge, MA, 1997.
[23] Johan L. Dupont and Chih Han Sah. Scissors congruences. II. *J. Pure Appl. Algebra*, 25(2):159–195, 1982.
[24] J. Elstrodt, F. Grunewald, and J. Mennicke. *Groups acting on hyperbolic space*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998. Harmonic analysis and number theory.
[25] Dirk Feldhusen. *Nenner der Eisensteinkohomologie der* GL(2) *über imaginär quadratischen Zahlkörpern*. Bonner Mathematische Schriften [Bonn Mathematical Publications], 330. Universität Bonn Mathematisches Institut, Bonn, 2000. Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 2000.
[26] L. M. Figueiredo. Serre's conjecture for imaginary quadratic fields. *Compositio Math.*, 118(1):103–122, 1999.
[27] Tobias Finis, Fritz Grunewald, and Paulo Tirao. The cohomology of lattices in $SL(2,\mathbb{C})$. *Experiment. Math.*, 19(1):29–63, 2010.
[28] D. Flath. Decomposition of representations into tensor products. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 179–183. Amer. Math. Soc., Providence, R.I., 1979.
[29] Eberhard Freitag. *Hilbert modular forms*. Springer-Verlag, Berlin, 1990.
[30] Joshua S. Friedman. Regularized determinants of the Laplacian for cofinite Kleinian groups with finite-dimensional unitary representations. *Comm. Math. Phys.*, 275(3):659–684, 2007.
[31] Benedict H. Gross. On the Satake isomorphism. In *Galois representations in arithmetic algebraic geometry (Durham, 1996)*, volume 254 of *London Math. Soc. Lecture Note Ser.*, pages 223–237. Cambridge Univ. Press, Cambridge, 1998.
[32] F. Grunewald, H. Helling, and J. Mennicke. SL_2 over complex quadratic number fields. I. *Algebra i Logika*, 17(5):512–580, 622, 1978.
[33] Fritz Grunewald. Notes on Bianchi manifolds. unpublished notes, 1972.
[34] G. Harder. `http://www.math.uni-bonn.de/people/harder/Manuscripts/buch/`.
[35] G. Harder. Eisenstein cohomology of arithmetic groups. The case GL_2. *Invent. Math.*, 89(1):37–118, 1987.
[36] Michael Harris, Kai-Wen Lan, Richard Taylor, and Jack Thorne. Galois representations for regular algebraic cusp forms over CM-fields. preprint.
[37] Michael Harris and Richard Taylor. *The geometry and cohomology of some simple Shimura varieties*, volume 151 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001. With an appendix by Vladimir G. Berkovich.
[38] Allen Hatcher. Bianchi orbifolds of small discriminant. unpublished notes, 1983.
[39] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.
[40] Kevin Hutchinson. A new approach to Matsumoto's theorem. *K-Theory*, 4(2):181–200, 1990.
[41] M. N. Huxley. Scattering matrices for congruence subgroups. In *Modular forms (Durham, 1983)*, Ellis Horwood Ser. Math. Appl.: Statist. Oper. Res., pages 141–156. Horwood, Chichester, 1984.
[42] Henryk Iwaniec. *Spectral methods of automorphic forms*, volume 53 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2002.
[43] H. Jacquet and R. P. Langlands. *Automorphic forms on $GL(2)$*. Lecture Notes in Mathematics, Vol. 114. Springer-Verlag, Berlin, 1970.
[44] Krzysztof Klosin. Ihara's lemma for imaginary quadratic fields. *J. Number Theory*, 128(8):2251–2262, 2008.
[45] Harald König. *Eisenstein-Kohomologie von* $\mathrm{Sl}_2(\mathbf{Z}[i])$. Bonner Mathematische Schriften [Bonn Mathematical Publications], 222. Universität Bonn Mathematisches Institut, Bonn, 1991. Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 1991.
[46] John Lott and Mel Rothenberg. Analytic torsion for group actions. *J. Differential Geom.*, 34(2):431–481, 1991.
[47] Wolfgang Lück. Analytic and topological torsion for manifolds with boundary and symmetry. *J. Differential Geom.*, 37(2):263–322, 1993.
[48] Colin Maclachlan and Alan W. Reid. The arithmetic of hyperbolic 3-manifolds. 219:xiv+463, 2003.

[49] Barry Mazur. An introduction to the deformation theory of Galois representations. In *Modular forms and Fermat's last theorem (Boston, MA, 1995)*, pages 243–311. Springer, New York, 1997.
[50] A. Mazzoleni. A new proof of a theorem of Suslin. *K-Theory*, 35(3-4):199–211 (2006), 2005.
[51] C. Mœglin and J.-L. Waldspurger. *Spectral decomposition and Eisenstein series*, volume 113 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1995. Une paraphrase de l'Écriture [A paraphrase of Scripture].
[52] Kenneth N. Moss. Homology of $SL(n, \mathbf{Z}[1/p])$. *Duke Math. J.*, 47(4):803–818, 1980.
[53] Werner Müller. Analytic torsion and R-torsion of Riemannian manifolds. *Adv. in Math.*, 28(3):233–305, 1978.
[54] Werner Müller. Eta invariants and manifolds with boundary. *J. Differential Geom.*, 40(2):311–377, 1994.
[55] Kartik Prasanna. Integrality of a ratio of Petersson norms and level-lowering congruences. *Ann. of Math. (2)*, 163(3):901–967, 2006.
[56] Kenneth A. Ribet. Mod p Hecke operators and congruences between modular forms. *Invent. Math.*, 71(1):193–205, 1983.
[57] Kenneth A. Ribet. Congruence relations between modular forms. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983)*, pages 503–514, Warsaw, 1984. PWN.
[58] Kenneth A. Ribet. On modular representations of $Gal(\overline{\mathbf{Q}}/\mathbf{Q})$ arising from modular forms. *Invent. Math.*, 100(2):431–476, 1990.
[59] J. Rohlfs. On the cuspidal cohomology of the Bianchi modular groups. *Math. Z.*, 188(2):253–269, 1985.
[60] Karl Rubin. Fudge factors in the Birch and Swinnerton-Dyer conjecture. In *Ranks of elliptic curves and random matrix theory*, volume 341 of *London Math. Soc. Lecture Note Ser.*, pages 233–236. Cambridge Univ. Press, Cambridge, 2007.
[61] Hiroshi Saito. On Tunnell's formula for characters of GL(2). *Compositio Math.*, 85(1):99–108, 1993.
[62] Peter Sarnak. Nonvanishing of L-functions on germ$R(s) = 1$. In *Contributions to automorphic forms, geometry, and number theory*, pages 719–732. Johns Hopkins Univ. Press, Baltimore, MD, 2004.
[63] Peter Scholze. On torsion in the cohomology of locally symmetric varieties. preprint.
[64] Mehmet Şengün. On the integral cohomology of Bianchi groups. *Exp. Math.*, 20(4):487–505, 2011.
[65] Jean-Pierre Serre. Le problème des groupes de congruence pour SL2. *Ann. of Math. (2)*, 92:489–527, 1970.
[66] Jean-Pierre Serre. *Arbres, amalgames, SL_2*. Société Mathématique de France, Paris, 1977. Avec un sommaire anglais, Rédigé avec la collaboration de Hyman Bass, Astérisque, No. 46.
[67] Jean-Pierre Serre. Sur les représentations modulaires de degré 2 de $Gal(\overline{\mathbf{Q}}/\mathbf{Q})$. *Duke Math. J.*, 54(1):179–230, 1987.
[68] Rebecca Torrey. On Serre's conjecture over imaginary quadratic fields. *J. Number Theory*, 132(4):637–656, 2012.
[69] Jerrold B. Tunnell. Local ϵ-factors and characters of GL(2). *Amer. J. Math.*, 105(6):1277–1307, 1983.
[70] Marie-France Vignéras. Variétés riemanniennes isospectrales et non isométriques. *Ann. of Math. (2)*, 112(1):21–32, 1980.
[71] J.-L. Waldspurger. Quelques propriétés arithmétiques de certaines formes automorphes sur $GL(2)$. *Compositio Math.*, 54(2):121–171, 1985.
[72] J.-L. Waldspurger. Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie. *Compositio Math.*, 54(2):173–242, 1985.
[73] J.-L. Waldspurger. Correspondances de Shimura et quaternions. *Forum Math.*, 3(3):219–307, 1991.
[74] Andrew Wiles. Modular elliptic curves and Fermat's last theorem. *Ann. of Math. (2)*, 141(3):443–551, 1995.
[75] R. Zimmert. Zur SL_2 der ganzen Zahlen eines imaginär-quadratischen Zahlkörpers. *Invent. Math.*, 19:73–81, 1973.

Index