A MINICOURSE ON MICROLOCAL ANALYSIS FOR WAVE PROPAGATION

ANDRÁS VASY

1. Introduction

This minicourse describes a microlocal framework for the linear analysis that has been useful for the global understanding of wave propagation phenomena. While it is useful to have some background in microlocal analysis since relatively sophisticated frameworks and notions are discussed and it may be easier for the reader to start with simpler cases, we in fact cover the subject from scratch.

The next section, Section 2 covers the differential operator aspects of totally characteristic, or b-, operators, in particular pointing out the differences from the local theory, or from the global theory on compact manifolds without boundary.

The basics of microlocal analysis in Section 3 are covered roughly following Melrose's lecture notes [31] (which introduced the author himself to the subject!), but in a generalized version. These generalizations concern both introducing the scattering algebra and allowing variable order operators. The former was first described systematically from a geometric perspective in [30] from the beginning; see [41, 37] for earlier descriptions in $\mathbb{R}^n$; here it serves to have a pseudodifferential algebra with Fredholm properties on a non-compact space right from the outset to orient the reader. The latter was discussed by Duistermaat and Unterberger [43, 7], though not quite in the setting we are interested in. We remark that apart from solving elliptic PDEs, the elliptic scattering algebra results shown in this section also allow one to deal with the local geodesic X-ray transform [42], and were the key ingredient in showing boundary rigidity in a fixed conformal class [40].

Section 4 gives a thorough description of propagation phenomena, starting with generalizations of Hörmander's propagation of singularities theorem [28] to the scattering algebra (where it is due to Melrose in [30]) variable order settings, and including complex absorption and radial points. These are then used to discuss the limiting absorption principle in scattering theory as well as the Klein-Gordon equation on Minkowski-like spaces. The approach of this minicourse, given the

---

Date: Draft dated October 22, 2016.

The author was supported in part by National Science Foundation grants DMS-1068742 and DMS-1361432. He is very grateful to the students at Stanford University in the Math 256B course he taught in Winter 2014, for which these notes were originally written, for their patience and comments. He is also grateful to the Université de Grenoble for its hospitality during the summer school in 2014 on mathematical aspects of general relativity, as well as to the participants; these notes again provided the backbone of the material covered. In addition, he is grateful to the hospitality of the Université de Nice in summer 2015 where he gave a lecture series based on the more advanced parts of these notes and where these notes were put in their final form. Thanks are also due to Peter Hintz for a careful proofreading of the final version. But most of all the author is grateful to Richard Melrose, from whom he learnt microlocal analysis, and whose notes are followed, in a generalized form, in Section 3.

limited time, is that the proofs are given in the basic microlocal setting; extension to the b-operators (presented in the recent literature in detail) is then straightforward, and we do not give the detailed proofs in that setting here. Note that we also use large parameter, or semiclassical, estimates, which are discussed briefly in Section 5. While their incorporation in the earlier section would have been straightforward, they result in further notational overhead, so it seemed better to only sketch the basic properties and leave the details to the reader, who may also refer to Zworski's recent treatment of the topic [58].

Section 5 shows, following [48, 47], how the tools developed can be applied to understand the analytic continuation of the resolvent of the Laplacian on conformally compact, asymptotically hyperbolic, spaces. While we do not discuss it here, another application of these microlocal tools is the work of Dyatlov and Zworski [11] to dynamical zeta functions for Anosov flows; we refer to [49] to a concise overview of some recent applications and to detailed references.

Finally Section 6 introduces the b-pseudodifferential operator algebra of Melrose [34] by local reduction to cylindrical models in Euclidean space. This actually does not give quite as much of the full 'small calculus' as the analogous localization for the standard algebra on compact manifolds without boundary and for the scattering algebra, but it is sufficient for us. This completes the tools necessary for the linear analysis of waves on Lorentzian scattering spaces, which are generalizations of asymptotically Minkowski spaces, [48, 2, 26, 12], as well as Kerr-de Sitter space, which was described in [48, 26], apart from a treatment of the trapped set in the latter case (for which we provide references at the end of the section). Indeed, it was the Kerr-de Sitter project [48] that started the work on the non-elliptic microlocal framework presented here.

In fact, these techniques extend to non-linear problems. The extension to semilinear wave equations was done in [26], to quasilinear problems without trapping in [24], and to quasilinear problems with trapping in [22]. The quasilinear works introduce b-pseudodifferential operators with coefficients with Sobolev regularity (rather than smoothness) in the spirit of Beals and Reed [3], and to deal with trapping, tame linear estimates; it turns out that these extensions of the 'smooth' microlocal framework described in these notes is not hard. Given these linear estimates, the non-linear techniques are rather straightforward by now; an easy version of Nash-Moser iteration due to Saint Raymond [38] is used in [22] to deal with the losses of derivatives due to the trapping. These quasilinear results were then used in the recent proof of the stability of slowly-rotating Kerr-de Sitter black holes by Hintz and Vasy [23].

The author hopes that these detailed lecture notes will provide a more accessible path, available in a single place, to the microlocal material that has proved so useful in our understanding of so many problems!

2. The overview

The ultimate goal of this minicourse is to understand the analysis of not necessarily elliptic totally characteristic, or b-(pseudo)differential operators. These are operators on manifolds M with boundaries or corners, though they can also arise on complete manifolds without boundary after one appropriately compactifies or bordifies them (i.e. makes them into a manifold with boundary or corners, not necessarily compact), as we discuss below. We study both elliptic and hyperbolic operators, with the Laplacian or the d'Alembertian of a b-metric (i.e. a metric corresponding to this structure) giving main examples. An important aspect of b-analysis is, however, that the principal symbol, on which ellipticity, etc., is based, does not completely capture the problem: there is also an analytic family of operators at the boundary ∂M, depending on the dual of the normal variable to the boundary, the poles of whose inverse determine both Fredholm properties and asymptotic expansions.

To be more concrete, consider the vector space of vector fields $\mathcal{V}_{\mathrm{b}}(M)$ tangent to ∂M, M an n-dimensional manifold with boundary. Recall that $V \in \mathcal{V}_{\mathrm{b}}(M)$ is equivalent to the statement that for all $f \in \mathcal{C}^\infty(M)$ with $f|_{\partial M} = 0$ one has $Vf|_{\partial M} = 0$; this in turn is equivalent to the same statement for a single f with a non-degenerate differential at ∂M (such a non-degenerate function, when non-negative, is called a *boundary definining function*). This space $\mathcal{V}_{\mathrm{b}}(M)$ is a left $\mathcal{C}^\infty(M)$ module, and is a Lie algebra: if f is a function that vanishes at ∂M, $V, W \in \mathcal{V}_{\mathrm{b}}(M)$, then VWf, WVf vanish at ∂M, thus so does

$$[V, W]f = (VW - WV)f.$$

Further, in a local coordinate chart mapping an open set O in M to an open set U in $[0, \infty)_x \times \mathbb{R}^{n-1}_y$, with ∂M mapped to $x = 0$ (thus x is a local boundary defining function), any smooth vector field has the form

$$V = b_0 \partial_x + \sum_{j=1}^{n-1} b_j \partial_{y_j},\ b_j \in \mathcal{C}^\infty(M),$$

so the tangency of V to ∂M means $Vx|_{x=0} = 0$, i.e. $b_0|_{x=0} = 0$, so $b_0 = xa_0$, $a_0 \in \mathcal{C}^\infty(M)$. Correspondingly,

$$V = a_0(x\partial_x) + \sum_j a_j \partial_{y_j},\ a_j \in \mathcal{C}^\infty(M), \tag{2.1}$$

locally; conversely, any vector field with such local coordinate expressions is in $\mathcal{V}_{\mathrm{b}}(M)$.

Our main interest is in differential operators generated by such $V \in \mathcal{V}_{\mathrm{b}}(M)$. Namely we let $\mathrm{Diff}^m_{\mathrm{b}}(M)$ consist of finite sums of up to m-fold products of elements of $\mathcal{V}_{\mathrm{b}}(M)$. With the usual Fourier analysis convention that $D = \frac{1}{i}\partial$, typical examples, in local coordinates, are operators such as

$$(xD_x)^2 + \sum_j D^2_{y_j}$$

which is elliptic, corresponding to a Riemannian space with a cylindrical end, as well as, after appropriate conjugation and division by a factor, to conic spaces, and

$$(xD_x - \sum_j y_j D_{y_j})^2 - \sum_j D^2_{y_j},$$

which is hyperbolic, and corresponds to a neighborhood of the static patch in a de Sitter space.

Typically the spaces of interest are not presented as manifolds with boundary; we need to bordify them for this purpose. Thus, an exact cylindrical end metric is traditionally written as

$$g = dr^2 + h(y, dy),\ r \to +\infty,$$

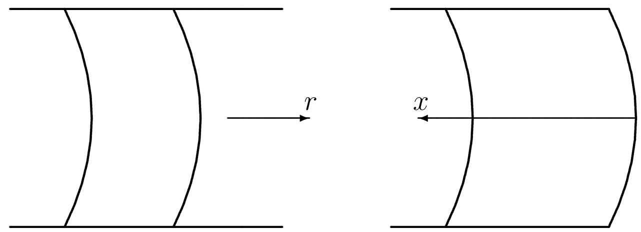

FIGURE 1. Left: a cylindrical end, with $r \to +\infty$. Right: the bordification, with $x = 0$ corresponding to $r = \infty$.

with Laplacian

$$\Delta_g = D_r^2 + \Delta_h.$$

Letting $x = e^{-r}$ (see Figure 1) means $r \to +\infty$ corresponds to $x \to 0$, and $dr = -\frac{dx}{x}$, $D_r = -xD_x$ shows that such a metric and Laplacian indeed has the stated form if one bordifies the end by gluing $x = 0$ to it, i.e. replacing $(r_0, +\infty)_r \times Y$ by $[0, e^{-r_0})_x \times Y$, with the identification

$$(r, y) \mapsto (e^{-r}, y)$$

of

$$(r_0, +\infty)_r \times Y \to (0, e^{-r_0})_x \times Y;$$

combining this with the compact core of the original manifold one obtains the manifold with boundary M. (More precisely, if the original manifold is $\tilde{M}$, the manifold M is the disjoint union of $\tilde{M}$ and $[0, r_0^{-1}) \times Y$ with the identification of the cylindrical end in $\tilde{M}$ with $(0, r_0^{-1}) \times Y$ as above, with a base for the topology of M given by the images under the equivalence relation of open sets in either $\tilde{M}$ or $[0, r_0^{-1}) \times Y$, and with coordinate charts given by those in $\tilde{M}$ as well as $[0, r_0^{-1})$ times coordinate charts in Y.) Note that a smooth function $f \in \mathcal{C}^\infty(M)$ has a Taylor series expansion at ∂M in terms of powers of x; this amounts to powers of e^{-r} in the original coordinates.

While we postpone detailed discussion, it turns out that wave, or more general Klein-Gordon, equations on asymptotically de Sitter spaces, or even Kerr-de Sitter spaces, describing rotating black holes in a background with a positive cosmological constant (reflecting our current understanding of the universe) also give rise to b-problems that the tools we develop can be used to analyze. A byproduct of this discussion is a new perspective to analyze asymptotically hyperbolic spaces, and indeed it sheds light on the role so-called even metrics play on these. Before continuing, we first comment on a different class of spaces, which do not seem to be b-spaces at the outset, where b-analysis in fact turns out to be useful.

On a manifold with boundary M, there is a conformally related class of vector fields, called scattering vector fields $\mathcal{V}_{\mathrm{sc}}(M)$. These arise by letting ρ be a boundary defining function of M (i.e. $\rho \geq 0$, ρ vanishes exactly on ∂M, and $d\rho$ is non-degenerate at ∂M; x was a local boundary defining function above), and letting

$$\mathcal{V}_{\mathrm{sc}}(M) = \rho\mathcal{V}_{\mathrm{b}}(M).$$

In local coordinates, such vector fields thus have the form

$$V = a_0(x^2\partial_x) + \sum_j a_j(x\partial_{y_j}),\ a_j \in \mathcal{C}^\infty(M),$$

locally; conversely, any vector field with such local coordinate expressions is in $\mathcal{V}_{\mathrm{sc}}(M)$. Again, we can define scattering differential operators by taking finite sums of finite products of these. A typical scattering differential operator is

$$(x^2D_x)^2 + (xD_y)^2 - \lambda,$$

where $\lambda \in \mathbb{C}$. Up to an inessential first order term, this is the spectral family of the Laplacian of a conic metric on the 'large end' of a cone, such as Euclidean space near infinity. To see this, recall that an exact conic metric has the form

$$dr^2 + r^2\, h(y, dy),$$

and thus, up to first order terms, its Laplacian has the form $D_r^2 + r^{-2}\Delta_h$. If we let $x = r^{-1}$, $dr = -\frac{dx}{x^2}$ and $D_r = -x^2D_x$, so this becomes of the form described above if we bordify by adding $x = 0$ to $(0, r_0^{-1})_x \times Y$. Note that the identification

$$(r_0, \infty) \times Y \to (0, r_0^{-1}) \times Y$$

is now via

$$(r, y) \to (r^{-1}, y),$$

so if the bordified space is denoted by M, elements of $\mathcal{C}^\infty(M)$ have an expansion in powers of x, i.e. of r^{-1}, unlike e^{-r} above. Thus, one has to pay attention to the bordification/compactification one uses. Now, if $\lambda = 0$, we can factor this operator as

$$x^2L,\ L = D_x x^2 D_x + \Delta_y \in \mathrm{Diff}_{\mathrm{b}}^2(M),$$

and thus b-analysis is applicable, but if $\lambda \neq 0$ this is no longer the case. Thus, for $\lambda \neq 0$, one needs to use the scattering framework to analyze the problem. (If $\lambda = 0$, the scattering framework can be used as a starting point, but the operator is degenerate then in an appropriate sense, which forces one to work with the b-framework at least implicitly.)

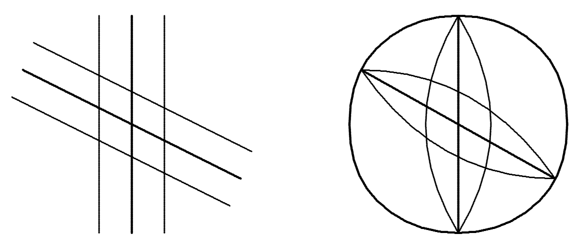

FIGURE 2. Left: two families of parallel lines in $\mathbb{R}^2$. Right: the compactification $\overline{\mathbb{R}^2}$, and the image of the two families of parallel lines. Note that parallel families end at the same two points at $\partial\overline{\mathbb{R}^2}$, while conic sectors, such as on the upper left between the two thick lines, map to wedge shaped domains.

In the case of Euclidean space, a straightforward calculation shows that compactification of $\mathbb{R}^n$ by reciprocal polar coordinates, i.e. identifying $\mathbb{R}^n \setminus \{0\}$ with

$(0,\infty)_r \times \mathbb{S}^{n-1}$, and compactifying to $M = \overline{\mathbb{R}^n} = \overline{\mathbb{B}^n}$ (a closed ball) using the map $x = r^{-1}$, see Figure 2, means not only that the spectral family of the Laplacian becomes an element of $\mathrm{Diff}^2_{\mathrm{sc}}(M)$, but more generally translation invariant differential operators become elements of $\mathrm{Diff}^m_{\mathrm{sc}}(M)$, and indeed become a basis, over $\mathcal{C}^\infty(M)$, of $\mathrm{Diff}^m_{\mathrm{sc}}(M)$. In particular, $\Box_g$, where g is the Minkowski metric, satisfies $\Box_g \in \mathrm{Diff}^2_{\mathrm{sc}}(M)$. Further, much like the Euclidean Laplacian, due to its homogeneity under dilations (which is what the b-structure relates to), $\Box_g = x^2 L$ with $L \in \mathrm{Diff}^2_{\mathrm{b}}(M)$. This means that the Minkowski wave operator, and indeed more general operators of similar form, called d'Alembertians of Lorentzian scattering metrics, are amenable to b-analysis. Note, however, that the Klein-Gordon operator $\Box_g - \lambda \in \mathrm{Diff}^2_{\mathrm{sc}}(M)$ does not factor in this way for $\lambda \neq 0$, and it is thus analyzable in the scattering framework.

The general approach we present to the analysis of differential operators P is via Fredholm estimates, such as

$$\|u\|_{\mathcal{X}} \leq C(\|Pu\|_{\mathcal{Y}} + \|u\|_{\tilde{\mathcal{X}}})$$

and

$$\|v\|_{\mathcal{Y}^*} \leq C(\|P^* v\|_{\mathcal{X}^*} + \|v\|_{\mathcal{Z}}),$$

where the inclusion of the spaces $\mathcal{X}$ into $\tilde{\mathcal{X}}$ and $\mathcal{Y}^*$ into $\mathcal{Z}$ is compact. Since our spaces correspond to geometrically complete non-compact manifolds, or compact manifolds but with differentiability encoded via complete vector fields such as $\mathcal{V}_{\mathrm{b}}(M)$, such a compact inclusion has two ingredients: gain in differentiability and gain in decay. (The simplest example is weighted Sobolev spaces $H^{s,r}(\mathbb{R}^n)$, with $H^s(\mathbb{R}^n)$ the standard Sobolev space, which is a weighted L^2-space on the Fourier transform side: $H^s(\mathbb{R}^n) = \mathcal{F}^{-1}\langle . \rangle^{-s} L^2(\mathbb{R}^n)$, and $H^{s,r}(\mathbb{R}^n) = \langle . \rangle^{-r} H^s(\mathbb{R}^n)$, with $\langle z \rangle = (1+|z|^2)^{1/2}$; see (3.46). Then the inclusion $H^{s,r}(\mathbb{R}^n) \to H^{s',r'}(\mathbb{R}^n)$ being compact requires both $s > s'$ and $r > r'$. These spaces in fact turn out to be the scattering Sobolev spaces $H^{s,r}_{\mathrm{sc}}(\overline{\mathbb{R}^n})$ and $H^{s',r'}_{\mathrm{sc}}(\overline{\mathbb{R}^n})$.) The gain in differentiability, much as in standard local (pseudo)differential analysis, is based on properties of principal symbols, which capture the operators modulo lower (differential) order terms. However, the gain in decay is more subtle.

To illustrate this, notice a structural difference between the Lie algebras $\mathcal{V}_{\mathrm{b}}(M)$ and $\mathcal{V}_{\mathrm{sc}}(M)$:

$$[\mathcal{V}_{\mathrm{b}}(M), \mathcal{V}_{\mathrm{b}}(M)] \subset \mathcal{V}_{\mathrm{b}}(M),\ [\mathcal{V}_{\mathrm{sc}}(M), \mathcal{V}_{\mathrm{sc}}(M)] \subset \rho\mathcal{V}_{\mathrm{sc}}(M),$$

i.e. while $\mathcal{V}_{\mathrm{b}}(M)$ is merely a Lie algebra, $\mathcal{V}_{\mathrm{sc}}(M)$ is commutative to leading (zeroth) order at ∂M. In order to analyze differential operators in $\mathrm{Diff}_{\mathrm{b}}(M)$ or $\mathrm{Diff}_{\mathrm{sc}}(M)$, we develop pseudodifferential algebras $\Psi_{\mathrm{b}}(M)$ and $\Psi_{\mathrm{sc}}(M)$. The leading order commutativity of $\mathcal{V}_{\mathrm{sc}}(M)$ means that $\Psi_{\mathrm{sc}}(M)$ has an other symbol at ∂M which admits similar constructions and estimates as the standard principal symbol; the two symbols are indeed related via the Fourier transform on $\mathbb{R}^n$ as we discuss in the next sections covering standard and scattering pseudodifferential operators. On the other hand, to $\Psi_{\mathrm{b}}(M)$ corresponds an analytic family of operators on ∂M, called the indicial or normal family. For instance, on cylindrical ends, one conjugates the operator by the Mellin transform in x, i.e. the Fourier transform in $\log x = -r$:

$$(\mathcal{M}u)(\sigma, y) = \int x^{-i\sigma} u(x,y)\,\frac{dx}{x},$$

with

$$(\mathcal{M}^{-1}v)(x,y) = \frac{1}{2\pi}\int_{\operatorname{Im}\sigma=-\alpha} x^{i\sigma} v(\sigma,y)\,d\sigma,$$

where the choice of α corresponds to the weighted space we are working with. Here we are interested in functions supported near $x = 0$ (i.e. r near $+\infty$), so the behavior of u as $x \to +\infty$ is irrelevant, and the Mellin transform gives a result which is holomorphic in an upper half plane: the more decay one has at $x = 0$, the larger the half plane is. The Mellin transform changes xD_x to multiplication by the b-dual variable σ, so the indicial family of

$$P = (xD_x)^2 + \Delta_Y - \lambda$$

is

$$\hat{N}(P)(\sigma) = \sigma^2 + \Delta_Y - \lambda.$$

This family has a meromorphic inverse, with poles at $\sigma = \sigma_j$, $\sigma_j = \pm\sqrt{\lambda - \lambda_j}$, as λ_j runs over the eigenvalues of Δ_Y (here we are assuming Y compact). Note that we are inverting operators (i.e. working with a non-commutative algebra), but on a manifold without boundary, i.e. on a simpler space. The Fredholm properties of P then will depend on the weighted b-Sobolev space we use, with the choice of weight corresponding to the choice of the imaginary part of σ: one needs to choose a weight x^α such that no poles $\hat{N}(P)(\sigma)^{-1}$ lie on the line $\operatorname{Im}\sigma = -\alpha$. (There are only finitely many poles in a strip $|\operatorname{Im}\sigma| < C$, so there are many good choices, but there are also a few bad choices.) Here the b-Sobolev spaces of non-negative integer differentiability order s and of real weight α are defined by

$$u \in H_{\mathrm{b}}^{s,\alpha}(M) \text{ iff } x^{-\alpha}u \in H_{\mathrm{b}}^{s,0}(M) = H_{\mathrm{b}}^{s}(M);$$
$$u \in H_{\mathrm{b}}^{s}(M) \text{ iff } \forall L \in \mathrm{Diff}_{\mathrm{b}}^{s}(M),\ Lu \in L_{\mathrm{b}}^{2}(M),$$

where $L^2_{\mathrm{b}}(M)$ is the L^2-space with respect to any non-degenerate b-density (see below), such as $|\frac{dx}{x}\,dy_1 \dots dy_{n-1}|$. Further, even if we have a Fredholm choice of space, the invertibility properties, even the Fredholm index, depend on this choice. This contrasts much with say formally self-adjoint elliptic problems on compact manifolds, for which any pair of Sobolev spaces, with orders differing by the order of the operator, give rise to a Fredholm pair of the same invertibility properties. One indication of this is that the kernel (nullspace) of elliptic operators on such manifolds is in $\mathcal{C}^\infty(M)$, i.e. in every Sobolev space, while in the b-setting, it is *not* in every weighted space, though the differentiability order does not matter. In fact, if $Pu \in \dot{\mathcal{C}}^\infty(M)$, say, i.e. is $\mathcal{C}^\infty$ and vanishes with all derivatives at ∂M, then typically u has an asymptotic expansion at ∂M of the form

$$\sum_{j:\ \operatorname{Im}\sigma_j<-\alpha} x^{i\sigma_j}u_j,\ u_j \in \mathcal{C}^\infty(M),$$

(with additional logarithmic terms, such as $\sum_{\ell=0}^{k_j}(\log x)^\ell x^{i\sigma_j}u_{j,\ell}$ in the general case), where α is such that $u \in H_{\mathrm{b}}^{s,\alpha}(M)$ for some s, and there are no σ_j with $\operatorname{Im}\sigma_j = -\alpha$. The σ_j are also called resonances; they thus determine the asymptotic behavior/expansion of solutions of P.

While we have discussed linear problems, with a bit of work the analysis can be extended to non-linear problems as well. In the non-elliptic context (our main interest here) small data problems are quite-well behaved: for semilinear problems one can even use Picard iteration typically (much as for ODEs) to obtain global solutions and asymptotic expansions; for quasilinear problems a bit more care is needed.

We now return to $\mathcal{V}_{\mathrm{b}}(M)$; many of the following considerations apply, with simple modifications, to $\mathcal{V}_{\mathrm{sc}}(M)$. Much like the set of all vector fields $\mathcal{V}(M)$ is the set of all smooth sections of TM, $\mathcal{V}_{\mathrm{b}}(M)$ is the set of all smooth sections of a vector bundle ${}^{\mathrm{b}}TM$, called the b-tangent bundle. Indeed, a local basis of ${}^{\mathrm{b}}TM$ is given by $x\partial_x, \partial_{y_1}, \ldots, \partial_{y_{n-1}}$, i.e. in terms of coordinates (x, y) as above, $(x, y_1, \ldots, y_{n-1}, a_0, a_1, \ldots, a_{n-1})$ are local coordinates on ${}^{\mathrm{b}}TM$, cf. (2.1). In M°, ${}^{\mathrm{b}}TM$ can be naturally identified with TM (corresponding to tangency to ∂M being vacuous there); globally $\mathcal{V}_{\mathrm{b}}(M) \subset \mathcal{V}(M)$ gives a fiber-preserving map

$$\iota : {}^{\mathrm{b}}TM \to TM;$$

with $(x, y_1, \ldots, y_{n-1}, b_0, b_1, \ldots, b_{n-1})$ coordinates on TM as in the above parameterization of vector fields, then the map is, in local coordinates

$$\iota(x, y_1, \ldots, y_{n-1}, a_0, a_1, \ldots, a_{n-1}) = (x, y_1, \ldots, y_{n-1}, xa_0, a_1, \ldots, a_{n-1}),$$

corresponding to

$$b_0 = xa_0,\ b_j = a_j,\ j = 1, \ldots, n-1.$$

Note that ι is not injective at $p \in \partial M$; its kernel ${}^{\mathrm{b}}N_p\partial M$ is the span of $x\partial_x$, the space of normal b-vector fields to the boundary, while its range is $T_p\partial M$, the tangent space to ∂M (as a subspace of T_pM). Note that, unlike for T_pM, the natural subspace of ${}^{\mathrm{b}}T_pM$ is *not* vector fields tangent to ∂M, but vector fields b-normal to ∂M.

The dual bundle, ${}^{\mathrm{b}}T^*M$, called the b-cotangent bundle, of ${}^{\mathrm{b}}TM$, then has a local basis $\frac{dx}{x}, dy_1, \ldots, dy_{n-1}$, i.e. smooth sections locally have the form

$$\sigma\,\frac{dx}{x} + \sum \eta_j\,dy_j,$$

with $\sigma, \eta \in \mathcal{C}^\infty(M)$. The adjoint of ι is the map $\pi : T^*M \to {}^{\mathrm{b}}T^*M$, which is sometimes (especially for boundary value problems) called the compressed cotangent bundle map, and ${}^{\mathrm{b}}T^*M$ the compressed cotangent bundle. In coordinates, with $(\xi, \zeta_1, \ldots, \zeta_{n-1})$ being dual coordinates to $(x, y_1, \ldots, y_{n-1})$,

$$\pi(x, y_1, \ldots, y_{n-1}, \xi, \zeta_1, \ldots, \zeta_{n-1}) = (x, y_1, \ldots, y_{n-1}, x\xi, \zeta_1, \ldots, \zeta_{n-1}),$$

corresponding to

$$\xi\,dx + \sum \zeta_j\,dy_j = (x\xi)\,\frac{dx}{x} + \sum_j \zeta_j\,dy_j.$$

The kernel of π on T^*_pM is now $N^*_p\partial M$, the conormal bundle fiber of ∂M, and the range is the span of the dy_j, which is the cotangent bundle of ∂M within ${}^{\mathrm{b}}T^*_pM$. Note that the natural subspace of ${}^{\mathrm{b}}T^*_pM$ is the cotangent, not the conormal, bundle of the boundary.

Intuitively, π projects out the normal momentum (momentum being dual to position), but preserves the tangential momentum, and thus can be used to encode the law of reflection for boundary problems. In fact, if we had more time in this minicourse, in its final part we would study exactly this problem (even on manifolds with corners): how do waves on, say, $M = Y \times \mathbb{R}_t$, (Y, h) Riemannian with boundary, $g = dt^2 - h$, behave at ∂M? Notice that $\Box_g \in \mathrm{Diff}^2(M)$, but is not in $\mathrm{Diff}^2_{\mathrm{b}}(M)$. Nonetheless, the tools we use to analyze $\Box_g$ are in the b-category. The reason for this is that 'standard' ps.d.o's do not quite make sense on manifolds with boundary, and even in the cases they do, they typically do not preserve boundary conditions. Thus, we use a two-algebra approach: the operator to analyze is in $\mathrm{Diff}(M)$, but the toolkit we use is in $\Psi_{\mathrm{b}}(M)$. We refer to [52] for details and further references. In fact, we point out that b-pseudodifferential operators originated in Melrose's work on propagation problems on manifolds with boundary [33].

We now turn to tensorial considerations. A smooth non-degenerate symmetric bilinear form on ${}^{\mathrm{b}}TM$, i.e. a non-degenerate section of $\mathrm{Sym}^2\,{}^{\mathrm{b}}T^*M = {}^{\mathrm{b}}T^*M \odot {}^{\mathrm{b}}T^*M$ is, for each $p \in M$, a (symmetric, bilinear) map $g_p : {}^{\mathrm{b}}T_pM \times {}^{\mathrm{b}}T_pM \to \mathbb{R}$ such that if $V \in {}^{\mathrm{b}}T_pM$ and $g_p(V, W) = 0$ for all $W \in {}^{\mathrm{b}}T_pM$, then $V = 0$. Equivalently, any $g \in \mathrm{Sym}^2\,{}^{\mathrm{b}}T^*M$ gives a map ${}^{\mathrm{b}}T_pM \to {}^{\mathrm{b}}T_p^*M$, and then non-degeneracy is equivalent to the injectivity of this map. In view of the equality of dimensions of the domain and target spaces, this in turn is equivalent to its invertibility. We call such a non-degenerate symmetric bilinear form a metric. The signature of such a metric is $(k, n-k)$ if the maximal subspace on which it is positive definite is k; note that the signature is constant. Taking the negative of g reverses the signature, so e.g. $(n, 0)$ and $(0, n)$ are essentially equivalent signatures, corresponding to Riemannian metrics. Lorentzian metrics have signature $(1, n-1)$ or $(n-1, 1)$; we mostly use the former convention here (so time has a positive sign, space negative signs). One can also takes metrics of other signatures (if the dimension n is at least 4); these can arise naturally.

In view of the invertibility, one obtains a dual bilinear form on ${}^{\mathrm{b}}T^*M$, as well as on tensorially related bundles, such as the form bundles or the symmetric bundles. One also has a metric density, which is a non-degenerate section of the density bundle ${}^{\mathrm{b}}\Omega M$. The latter has basis $|\frac{dx}{x}\,dy_1 \,\ldots\, d_{y_{n-1}}|$ in local coordinates; the metric density is

$$|dg| = |\det g|\,|\frac{dx}{x}\,dy_1 \,\ldots\, d_{y_{n-1}}|,$$

where one takes the determinant of the matrix of coefficients of g with respect to the b-frame $x\partial_x, \partial_{y_1}, \ldots, \partial_{y_{n-1}}$. In particular, this gives rise to a positive definite inner product on $\dot{\mathcal{C}}^\infty(M)$, the space of $\mathcal{C}^\infty$ functions vanishing to infinite order at ∂M:

$$\langle u, v\rangle = \int u\overline{v}\,|dg|,$$

as well as a non-positive definite inner product on, say, differential forms

$$\langle u, v\rangle = \int G_p(u, \overline{v})\,|dg|,$$

where G is the dual metric on differential forms. While this inner product is not positive definite, this is only due to the fiber inner product G_p (on finite dimensional spaces), rather than to $|dg|$, so in particular formal adjoints of differential operators are well-defined differential operators. Moreover, the exterior derivative d maps $\dot{\mathcal{C}}^\infty(M)$ to $\dot{\mathcal{C}}^\infty(M, {}^{\mathrm{b}}T^*M)$, and more generally $\dot{\mathcal{C}}^\infty(M, {}^{\mathrm{b}}\Lambda^k M)$ to $\dot{\mathcal{C}}^\infty(M, {}^{\mathrm{b}}\Lambda^{k+1} M)$, and thus given a metric g we can define the Laplacian or d'Alembertian

$$\Box = d^*d + dd^*,$$

with adjoint $*$ taken relative to g.