NON-TRAPPING ESTIMATES NEAR NORMALLY HYPERBOLIC TRAPPING

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ABSTRACT. In this paper we prove semiclassical resolvent estimates for operators with normally hyperbolic trapping which are lossless relative to non-trapping estimates but take place in weaker function spaces. In particular, we obtain non-trapping estimates in standard L^2 spaces for the resolvent sandwiched between operators which localize away from the trapped set Γ in a rather weak sense, namely whose principal symbols vanish on Γ.

1. Introduction

The purpose of this paper is to obtain lossless, relative to non-trapping, albeit weaker, in terms of function spaces, semiclassical estimates for pseudodiffererential operators $P_h(z)$ with normally hyperbolic trapping for z real. Thus, the main result is an estimate of the form

$$\|u\|_{\mathcal{H}_{\hbar,\Gamma}} \leq Ch^{-1}\|P_h(z)u\|_{\mathcal{H}^*_{\hbar,\Gamma}},$$

with certain function spaces $\mathcal{H}_{\hbar,\Gamma}$ and $\mathcal{H}^*_{\hbar,\Gamma}$, described below; away from the trapped set these are just standard L^2 spaces. As the main application of such estimates is in so-called b-spaces, e.g. Kerr-de Sitter spaces, for which the estimates follow from the semiclassical ones immediately in the presence of dilation invariance, we also prove their counterpart in the general, non-dilation-invariant, b-setting.

So at first we consider a family $P_h(z)$ of semiclassical pseudodifferential operators $P_h(z) \in \Psi_\hbar(X)$ on a closed manifold X, depending smoothly on the parameter $z \in \mathbb{C}$, with normally hyperbolic trapping, with trapped set Γ, and assume that $P_h(z)$ is formally self-adjoint for $z \in \mathbb{R}$. Wunsch and Zworski [13] studied this setting, imposing other global assumptions, the most important one being adding complex absorption W in such a way that all bicharacteristics outside Γ, in both the forward and backward directions, either enter the elliptic set of W in finite time or tend to Γ, and in at least one of the two directions they tend to the elliptic set of W. The bicharacteristics tending to Γ in the forward/backward directions are forward/backward trapped; denote by Γ_-, resp. Γ_+ the forward, resp. backward trapped set,[1] and assume that these are smooth codimension one submanifolds of the semiclassical characteristic set $\Sigma_{\hbar,z}$, intersecting transversally.

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[1]In the notation of Wunsch and Zworski [13], $\Gamma_\pm$ are the backward/forward trapped sets for all (not necessarily null) bicharacteristics near the, say, zero level set of the semiclassical principal symbol $p_{\hbar,z}$, $\Gamma^\lambda_\pm$ are the corresponding sets within the λ-level set of $p_{\hbar,z}$.

In this normally hyperbolic setting Wunsch and Zworski [13] have shown polynomial semiclassical resolvent estimates

$$\|u\| \leq Ch^{-N}\|P_h(z)u\|,\ 0 < h < h_0, \tag{1.1}$$

in small strips $|\operatorname{Im} z| \leq ch$, $c > 0$ sufficiently small, $N > 1$, and indeed for z real, the loss is merely logarithmic, i.e. one has

$$\|u\| \leq Ch^{-1}(\log h^{-1})\|P_h(z)u\|,\ 0 < h < h_0, \tag{1.2}$$

where $\|.\|$ is the L^2-norm. Dyatlov's work [5] has since then improved the estimates in $\operatorname{Im} z < 0$ (for instance, by making c, N explicit).

We are concerned with improved estimates (for z almost real) if one localizes u and $P_h(z)u$ away from the trapping Γ in a rather weak sense, such as by applying pseudodifferential operators with symbols vanishing at Γ. To place this into context, recall that Datchev and Vasy [3, 4] have shown that under our assumptions, with $\operatorname{Im} z = \mathcal{O}(h^\infty)$, if $A, B \in \Psi_\hbar(X)$ with $\mathrm{WF}'_\hbar(A) \cap \Gamma = \mathrm{WF}'_\hbar(B) \cap \Gamma = \emptyset$, B elliptic on $\mathrm{WF}'_\hbar(A)$, then for all M there is N such that

$$\|Au\| \leq Ch^{-1}\|BP_h(z)u\| + C'h^M\|u\| + C''h^{-N}\|(\mathrm{Id} - B)P_h(z)u\|. \tag{1.3}$$

Thus, if $P_h(z)u$ is $\mathcal{O}(h^{N-1})$ at Γ (corresponding to the $\mathrm{Id} - B$ term in the estimate), then on the elliptic set of A, hence off Γ by appropriate choice of A, u satisfies non-trapping semiclassical estimates:

$$\|AP_h(z)^{-1}Av\| \leq Ch^{-1}\|v\|,$$

with A as above (take B as above with $\mathrm{WF}'_\hbar(\mathrm{Id} - B) \cap \mathrm{WF}'_\hbar(A) = \emptyset$). Here the $\mathcal{O}(h^\infty)$ bound on $\operatorname{Im} z$ arises from the a priori estimate, (1.1), and if $1 < N < 3/2$, e.g. as is on, and sufficiently near, the real axis,[2] then one can take $\operatorname{Im} z = \mathcal{O}(h^{-1+2N})$. The purpose of this paper is to improve this result by relaxing the conditions on $\mathrm{WF}'_\hbar(A)$ and $\mathrm{WF}'_\hbar(B)$ in (1.3).

The main point of the theorem below is thus that its estimate degenerates only at, as opposed to near, Γ. The proof given here is closely related to the proof of Wunsch and Zworski [13, Section 4], but being suboptimal in terms of the L^2-estimate, even though it is optimal (non-trapping) when a pseudodifferential operator with vanishing principal symbol at Γ is applied from both sides, it can take place in a significantly simpler, standard, semiclassical pseudodifferential algebra. To set this up, let $Q_\pm \in \Psi_\hbar^{-\infty}(X)$ be self-adjoint and have symbols which are defining functions of $\Gamma_\pm$ near Γ, *within the characteristic set* $\Sigma_{\hbar,z}$, say on a neighborhood O of Γ. Let $Q_0 \in \Psi_\hbar^0(X)$ be a semiclassical operator with $\mathrm{WF}'_\hbar(Q_0) \cap \Gamma = \emptyset$ which is elliptic on O^c (and thus on a neighborhood of O^c), with real principal symbol for convenience. One considers *normally isotropic* spaces at Γ, denoted $\mathcal{H}_{\hbar,\Gamma}$, with squared norms given by

$$\|u\|^2_{\mathcal{H}_{\hbar,\Gamma}} = \|Q_0 u\|^2 + \|Q_+ u\|^2 + \|Q_- u\|^2 + h\|u\|^2;$$

this is just the standard L^2-space microlocally away from Γ as one of Q_+, Q_- or Q_0 is elliptic there, and it does not depend on the choice of Q_0 as on $O \setminus \Gamma$ one of

[2] In the latter case by the Phragmén-Lindelöf theorem.

Q_+ and Q_- is elliptic at every point. The dual space relative to L^2 is then[3]

$$\mathcal{H}^*_{\hbar,\Gamma} = h^{1/2}L^2 + Q_+L^2 + Q_-L^2 + Q_0L^2$$

(which is L^2 as a space, but with this norm); $P_h(z)u$ will then be measured in $\mathcal{H}^*_{\hbar,\Gamma}$.

Theorem 1.1. *Let $P = P_h(z)$, $Q_\pm$ be as above,* $\operatorname{Im} z = \mathcal{O}(h^2)$. *Then*

$$\|Q_+u\| + \|Q_-u\| \le Ch^{-1}\|Pu\|_{\mathcal{H}^*_{\hbar,\Gamma}} + C'h^{1/2}\|u\|, \tag{1.4}$$

and thus by (1.2),

$$\|u\|_{\mathcal{H}_{\hbar,\Gamma}} \le Ch^{-1}\|Pu\|_{\mathcal{H}^*_{\hbar,\Gamma}}. \tag{1.5}$$

In fact, we also obtain a direct proof of (1.5) without using (1.2) at the end of Section 2, as well as the aforementioned b-estimates in Section 3, see Theorem 3.2.

2. Semiclassical resolvent estimates on the real line

2.1. **Notation and definitions.** We will review some definitions of semiclassical analysis, partially in order to fix our notation. For a general reference, see Zworski [14].

Let X be a compact n-dimensional manifold without boundary, and fix a smooth density on X.

- For $u \in L^2(X)$, denote by $\|u\|$ its $L^2(X)$ norm; moreover, denote by $\langle\cdot,\cdot\rangle$ the (sesquilinear) inner product on $L^2(X)$.
- A family of functions $u = (u_h)_{h\in(0,1)}$ on X is *polynomially bounded* if $\|u\| \le Ch^{-N}$ for some N. If $k \in \mathbb{R}$, we say that $u \in \mathcal{O}(h^k)$ if $\|u\| \le C_kh^k$, and $u \in \mathcal{O}(h^\infty)$ if $\|u\| \le C_Nh^N$ for every N.
- For $a = (a_h)_{h\in(0,1)} \in \mathcal{C}^\infty(T^*X)$, we say $a \in h^kS^m(T^*X)$ if a satisfies

  $$|\partial_z^\alpha\partial_\zeta^\beta a_h(z,\zeta)| \le C_{\alpha\beta}h^k\langle\zeta\rangle^{m-|\beta|}$$

  for all multiindices α,β and all $N \in \mathbb{N}$ in any coordinate chart, where the z are coordinates in the base and ζ coordinates in the fiber. We define the *semiclassical quantization* $\operatorname{Op}_h(a)$ of a by

  $$\operatorname{Op}_h(a)u(z) = (2\pi h)^{-n}\int e^{iz\zeta/h}a(z,\zeta)\hat{u}(\zeta/h)\,d\zeta$$

  for $u \in \mathcal{C}^\infty_c(X)$ supported in a chart and for general $u \in \mathcal{C}^\infty_c(X)$ by using a partition of unity. We write $\operatorname{Op}_h(a) \in h^k\Psi^m_\hbar(X)$. The quantization depends on the choice of partition of unity, but the resulting class of operators does not, modulo operators that have Schwartz kernel in $h^\infty\mathcal{C}^\infty(X^2)$. We say that a is a *symbol* of $\operatorname{Op}_h(a)$. The equivalence class of a in $h^kS^m(T^*X)/h^{k-1}S^{m-1}(T^*X)$ is invariantly defined and is called the *principal symbol* of $\operatorname{Op}_h(a)$. All operators below except $Q_0 \in \Psi^0_\hbar(X)$ will in fact have compact microsupport in the sense that they are quantizations of symbols $a \in h^kS^m(T^*X)$ satisfying in addition for all N

  $$|\partial_z^\alpha\partial_\zeta^\beta a_h(z,\zeta)| \le C_Nh^N\langle\zeta\rangle^{-N} \text{ for all multiindices } \alpha,\beta$$

  for ζ outside of a compact subset of T^*X. We denote the class of such symbols by $h^kS(T^*X)$ and the corresponding class of operators by $h^k\Psi_\hbar(X)$.

[3] One really has $Q^*_\pm$ and Q^*_0 in this formula, but the reality of the principal symbols assures that one may replace them by $Q_\pm$ and Q_0 modulo hL^2. See [7, Appendix A] for a general discussion of the underlying functional analysis; also see Footnote 12.

- If $A, B \in \Psi_\hbar(X)$, then $[A, B] \in h\Psi_\hbar(X)$, and its principal symbol is $\frac{h}{i}\mathsf{H}_a b$, where we define the *Hamilton vector field* in a coordinate chart by
$$\mathsf{H}_a = (\partial_\zeta a)\partial_z - (\partial_z a)\partial_\zeta.$$
- By a *bicharacteristic* of A we mean an integral curve of the Hamilton vector field of the principal symbol of A. We denote the integral curve passing through the point $\rho \in T^*X$ by γ_ρ, i.e. $\gamma_\rho(0) = \rho$ and $\gamma_\rho'(s) = \mathsf{H}_a(\gamma_\rho(s))$. We shall also write $\phi^s(\rho) := \gamma_\rho(s)$ for the bicharacteristic flow.
- For a polynomially bounded family $(u_h)_{h\in(0,1)}$ and $k \in \mathbb{R} \cup \{\infty\}$, we say that $u = \mathcal{O}(h^k)$ at a point $\rho \in T^*X$ if there exists $a \in S(T^*X)$ with $a(\rho) \neq 0$ such that $\|\mathrm{Op}_h(a)u\| = \mathcal{O}(h^k)$. We define the *semiclassical wave front set* $\mathrm{WF}_\hbar(u)$ of u as the complement of the set of all $\rho \in T^*X$ at which $u = \mathcal{O}(h^\infty)$.
- The *microsupport* of $A = \mathrm{Op}_h(a) \in h^k\Psi_\hbar(X)$, denoted $\mathrm{WF}_\hbar'(A)$, is the complement of the set of all $\rho \in T^*X$ so that $|\partial^\alpha a| = \mathcal{O}(h^\infty)$ near ρ for every multiindex α, in any (and therefore in every) coordinate chart.
- For $A \in h^k\Psi_\hbar(X)$ with principal symbol $a \in h^kS(T^*X)$, we say that A is *elliptic* at $\rho \in T^*X$ if there is a constant $C > 0$ such that $|a(\rho')| \geq Ch^k$ for ρ' near ρ and h sufficiently small. For a subset $E \Subset T^*X$, we say that A is elliptic on E if A is elliptic at each point of E. If $A \in h^k\Psi_\hbar(X)$ is elliptic on $E \Subset T^*X$ and $Au = f$ with u, f polynomially bounded and f is $\mathcal{O}(1)$ on E, then *microlocal elliptic regularity* states that u is $\mathcal{O}(h^{-k})$ on E.
- The *semiclassical characteristic set* of the semiclassical operator $A \in \Psi_\hbar(X)$ with principal symbol a is defined by $\Sigma_\hbar = \{\rho \in T^*X : a(\rho) = 0\}$.
- If $A \in \Psi_\hbar(X)$ has a principal symbol with non-positive imaginary part, u, f are polynomially bounded, $Au = f$, and $u = \mathcal{O}(h^k)$ at ρ, $f = \mathcal{O}(h^{k+1})$ on $\gamma_\rho([0, T])$ for some $T > 0$, then the *propagation of singularities* states that $u = \mathcal{O}(h^k)$ at $\gamma_\rho(T)$.
- Let $P \in \Psi_\hbar(X)$ be a semiclassical operator. Let $U \subset X$ denote an open subset so that the cotangent bundle over U contains what will be the trapped set, and place complex absorbing potentials in a neighborhood of U^c.[4] We recall the notion of *normal hyperbolicity* from [13]: Define the *backward*, resp. *forward, trapped set* $\tilde{\Gamma}_+$, resp. $\tilde{\Gamma}_-$, by
$$\tilde{\Gamma}_\pm = \{\rho \in T^*X : \gamma_\rho(s) \notin T^*_{U^c}X \text{ for all } \mp s \geq 0\}.$$
Let $\Gamma_\pm^\lambda = \tilde{\Gamma}_\pm \cap p^{-1}(\lambda)$ be the backward/forward trapped set within the energy surface $p^{-1}(\lambda)$, and define the *trapped set* $\Gamma_\lambda := \Gamma_+^\lambda \cap \Gamma_-^\lambda$. In the context of Theorem 1.1, we will only work within the characteristic set of p, hence with $\Gamma_\pm := \Gamma_\pm^0$ and $\Gamma := \Gamma_0$. We say that P is *normally hyperbolically trapping* if:
  (1) There exists $\delta > 0$ such that $dp \neq 0$ on $p^{-1}(\lambda)$ for $|\lambda| < \delta$;
  (2) $\tilde{\Gamma}_\pm \cap p^{-1}(-\delta, \delta)$ are smooth codimension one submanifolds intersecting transversally at $\tilde{\Gamma} \cap p^{-1}(-\delta, \delta)$;
  (3) the flow is hyperbolic in the normal directions to Γ_λ within the energy surface: There exist subbundles $E_\lambda^\pm$ of $T_{\Gamma_\lambda}(\Gamma_\pm^\lambda)$ such that $T_{\Gamma_\lambda}\Gamma_\pm^\lambda = T\Gamma_\lambda \oplus E_\lambda^\pm$, where $d\phi^s : E_\lambda^\pm \to E_\lambda^\pm$, and there exists $\theta > 0$ such that for

[4] See [13, 11] for details; the point here is that the relevant part of our analysis takes place microlocally near the trapped set, and the complex absorbing potentials allow us to 'cut off' the bicharacteristic flow in a neighborhood of the trapped set.

all $|\lambda| < \delta$

$$\|d\phi^s(v)\| \leq Ce^{-\theta|t|}\|v\| \text{ for all } v \in E_\lambda^\mp, \pm t \geq 0.$$

2.2. **Details on the setup and proof of the main result.** Let $p = p_{\hbar,z}$ be the semiclassical principal symbol of $P_\hbar(z)$. Recall from the work of Wunsch and Zworski [13, Lemma 4.1], with a corrected argument in [12], that for defining functions $\phi_\pm$ of $\Gamma_\pm$ (near Γ, namely in a neighborhood O of Γ) within the characteristic set of p one can take $\phi_\pm$ with

$$\mathsf{H}_p\phi_\pm = \mp c_\pm^2\phi_\pm$$

with $c_\pm > 0$ near Γ, and with[5]

$$\{\phi_+, \phi_-\} > 0$$

near Γ. *This is the only relevant feature of normal hyperbolicity for this paper; thus these identitities and estimates could be taken as its definition for our purposes.* By shrinking O if necessary we may assume that this Poisson bracket as well as $c_\pm$ have positive lower bounds on O. Then notice that

$$\mathsf{H}_p\phi_+^2 = -2c_+^2\phi_+^2, \qquad \mathsf{H}_p\phi_-^2 = 2c_-^2\phi_-^2.$$

As indicated in the introduction, we consider *normally isotropic* spaces at Γ, denoted $\mathcal{H}_{\hbar,\Gamma}$, with squared norms given by

$$\|u\|^2_{\mathcal{H}_{\hbar,\Gamma}} = \|Q_0u\|^2 + \|Q_+u\|^2 + \|Q_-u\|^2 + h\|u\|^2;$$

we can take $Q_\pm$ with principal symbol $\phi_\pm$, while Q_0 is elliptic on O^c with real principal symbol. This is just the standard L^2-space microlocally away from Γ as one of Q_+, Q_- or Q_0 is elliptic there, and it does not depend on the choice of Q_0 as on $O \setminus \Gamma$ one of Q_+ and Q_- is elliptic at every point. Notice that in fact

$$(Q_+ - iQ_-)^*(Q_+ - iQ_-) = Q_+^*Q_+ + Q_-^*Q_- - i[Q_+, Q_-]$$

and if $B \in \Psi_\hbar(X)$ with $\mathrm{WF}'_\hbar(B) \subset O$ then

$$h\|Bv\|^2 \leq C\,\mathrm{Re}\langle i[Q_+, Q_-]Bv, Bv\rangle + Ch^{N'}\|v\|^2,$$

$C > 0$, in view of $\{\phi_+, \phi_-\} > 0$ on O, so

$$Q_+^*Q_+ + Q_-^*Q_- = \frac{1}{2}(Q_+^*Q_+ + Q_-^*Q_- + (Q_+ - iQ_-)^*(Q_+ - iQ_-) + i[Q_+, Q_-])$$

shows that, for $h > 0$ small, the norm on $\mathcal{H}_{\hbar,\Gamma}$ is equivalent to just the norm

$$\|u\|^2_{\mathcal{H}_{\hbar,\Gamma},2} = \|Q_0u\|^2 + \|Q_+u\|^2 + \|Q_-u\|^2.$$

As mentioned in the introduction, the dual space relative to L^2 is then

$$\mathcal{H}^*_{\hbar,\Gamma} = h^{1/2}L^2 + Q_+L^2 + Q_-L^2 + Q_0L^2.$$

Then $\Psi_\hbar(X)$ acts on $\mathcal{H}_{\hbar,\Gamma}$, and thus on $\mathcal{H}^*_{\hbar,\Gamma}$, for $B \in \Psi_\hbar(X)$ preserves $h^{-1/2}L^2$ and gives

$$\|Q_+Bu\| \leq \|BQ_+u\| + \|[Q_+, B]u\| \leq C\|Q_+u\| + h\|u\|_{L^2},$$

with a similar result for Q_- and Q_0. We remark that the notation $\mathcal{H}_{\hbar,\Gamma}$ is justified as the space depends only on Γ, not on the particular defining functions $\phi_\pm$ as any

[5]These defining functions exist globally when $\Gamma_\pm$ is orientable; but even if $\Gamma_\pm$ is not such, the square is globally defined. There is only a minor change required below if $\phi_\pm$ are not well defined; see Footnote 6.

other defining functions would change $Q_\pm$ by an elliptic factor modulo an element of $h\Psi_\hbar(X)$, whose contribution to the squared norm can be absorbed into $Ch^2\|u\|^2_{L^2}$, and thus dropped altogether (for h small) in view of the equivalence of the two norms discussed above.

We are now ready to prove Theorem 1.1. Note that this theorem in particular implies the main result of [3] in this setting, in that the estimates are of the same kind, except that in [3] Pu is assumed to be microlocalized away from, and u is estimated microlocally away from, Γ.

We also remark that the microlocal version of the two estimates of the theorem is that given any neighborhood O' of Γ with closure in O, there exist $B_0 \in \Psi_\hbar(X)$ elliptic at Γ, $B_1, B_2 \in \Psi_\hbar(X)$ with $\mathrm{WF}'_\hbar(B_2)\cap\Gamma_+ = \emptyset$, $\mathrm{WF}'_\hbar(B_j) \subset O'$ for $j = 0, 1, 2$ such that

$$\|B_0Q_+u\| + \|B_0Q_-u\| \leq h^{-1}\|B_1Pu\|_{\mathcal{H}^*_{\hbar,\Gamma}} + \|B_2u\|_{L^2} + C'h^{1/2}\|u\|_{L^2}, \tag{2.1}$$

respectively

$$\|B_0u\|_{\mathcal{H}_{\hbar,\Gamma}} \leq h^{-1}\|B_1Pu\|_{\mathcal{H}^*_{\hbar,\Gamma}} + \|B_2u\|_{L^2} + C'h\|u\|_{L^2}; \tag{2.2}$$

see (2.8). The theorem is proved by controlling the B_2u term using the backward non-trapped nature of $\Gamma_- \setminus \Gamma$.

Proof of Theorem 1.1. We first prove (1.4), which proves (1.5) by (1.2). Then we modify the proof slightly to show (1.5) directly, and in particular prove a weaker version of the Wunsch-Zworski estimate (1.2), namely

$$\|u\|_{L^2} \leq Ch^{-2}\|Pu\|_{L^2}.$$

Let $\chi_0(t) = e^{-1/t}$ for $t > 0$, $\chi_0(t) = 0$ for $t \leq 0$, $\chi \in \mathcal{C}^\infty_c([0,\infty))$ be identically 1 near 0 with $\chi' \leq 0$, and indeed with $\chi'\chi = -\chi_1^2$, $\chi_1 \geq 0$, $\chi_1 \in \mathcal{C}^\infty_c([0,\infty))$, and let $\psi \in \mathcal{C}^\infty_c(\mathbb{R})$ be identically 1 near 0. Let

$$a = \chi_0(\phi_+^2 - \phi_-^2 + \kappa)\chi(\phi_+^2)\psi(p),$$

$\kappa > 0$ small. Notice that on $\operatorname{supp} a$, if χ is supported in $[0, R]$,

$$\phi_+^2 \leq R,\ \phi_-^2 \leq \phi_+^2 + \kappa = R + \kappa,$$

so a is localized near Γ if R and κ are taken sufficiently small. Then

$$\begin{aligned}\frac{1}{4}\mathsf{H}_p(a^2) = &-(c_+^2\phi_+^2 + c_-^2\phi_-^2)(\chi_0\chi_0')(\phi_+^2 - \phi_-^2 + \kappa)\chi(\phi_+^2)^2\psi(p)^2 \\ &- c_+^2\phi_+^2(\chi'\chi)(\phi_+^2)\chi_0(\phi_+^2 - \phi_-^2 + \kappa)^2\psi(p)^2.\end{aligned}$$

Now $\chi_0' \geq 0$, so the two terms have opposite signs. Let[6]

$$a_\pm = \phi_\pm\sqrt{(\chi_0\chi_0')(\phi_+^2 - \phi_-^2 + \kappa)}\chi(\phi_+^2)\psi(p),$$

and

$$e_- = c_+\phi_+\chi_1(\phi_+^2)\chi_0(\phi_+^2 - \phi_-^2 + \kappa)\psi(p);$$

then

$$\frac{1}{4}\mathsf{H}_p(a^2) = -c_+^2a_+^2 - c_-^2a_-^2 + e_-^2. \tag{2.3}$$

[6]If $\phi_\pm$ is not defined globally, $a_\pm$ are not defined as stated. (The term e_-^2 need not have a sign, so this issue does not arise for it; see the Weyl quantization argument below.) However, $a_\pm$ need not be real below, so as long as one can choose $\psi_\pm$ complex valued with $|\psi_\pm|^2 = \phi_\pm^2$, replacing the first factor of $\phi_\pm$ with $\psi_\pm$ in the definition of $a_\pm$ allows one to complete the argument in general.

Here

$$\operatorname{supp} e_- \subset \operatorname{supp} a,$$
$$\operatorname{supp} e_- \cap \Gamma_+ = \emptyset,$$

with the last statement following from ϕ_+^2 taking values away from 0 on $\operatorname{supp}\chi_1$; see Figure 1.

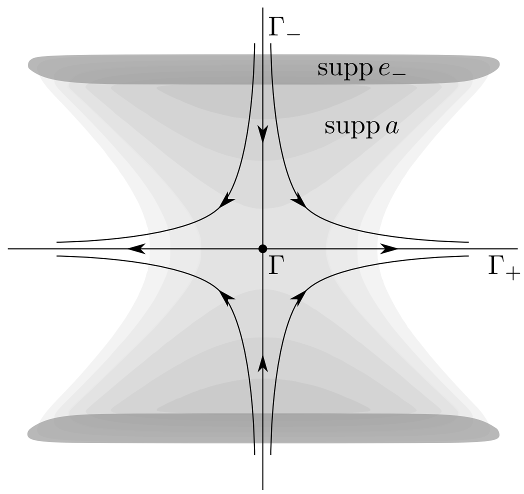

FIGURE 1. Supports of the commutant a and the error term e_- in the positive commutator argument of the non-trapping estimate near the trapped set Γ, Theorem 1.1. The support of a is indicated in light gray; on $\operatorname{supp} a \setminus \operatorname{supp} e_-$, darker colors correspond to larger values of a. Also shown are the forward, resp. backward, trapped set Γ_-, resp. Γ_+, and the bicharacteristic flow nearby. The figure already suggests that $\mathsf{H}_p(a^2)$ is non-positive away from $\operatorname{supp} e_-$, and actually negative away from $\operatorname{supp} e_- \cup \Gamma$; see equation (2.3).

One then takes $A \in \Psi_\hbar(X)$ with principal symbol a, and with $\mathrm{WF}'_\hbar(A) \subset \operatorname{supp} a$, $A_\pm \in \Psi_\hbar(X)$ with principal symbols of $a_\pm$, and with $\mathrm{WF}'_\hbar(A_\pm) \subset \operatorname{supp} a_\pm$, $C_\pm$ have symbol $c_\pm$ and with $\mathrm{WF}'_\hbar(C_\pm) \subset \operatorname{supp} c_\pm$; one similarly lets $E_- \in \Psi_\hbar(X)$ have principal symbol e_-, and wave front set in the support of e_-. This gives that

$$\frac{i}{4h}[P, A^*A] = -(C_+A_+)^*(C_+A_+) - (C_-A_-)^*(C_-A_-) + E_-^*E_- + hF, \qquad (2.4)$$

for some $F \in \Psi_\hbar(X)$ with

$$\mathrm{WF}'_\hbar(F) \subset \operatorname{supp} a.$$

Thus

$$\frac{i}{4h}\langle [P, A^*A]u, u\rangle = -\|C_+A_+u\|^2 - \|C_-A_-u\|^2 + \|E_-u\|^2 + h\langle Fu, u\rangle.$$

Expanding the left hand side gives

$$\begin{aligned}&\langle PA^*Au, u\rangle - \langle A^*APu, u\rangle \\ &\qquad = \langle Au, APu\rangle - \langle APu, Au\rangle + \langle (P-P^*)A^*Au, u\rangle.\end{aligned}$$

As we are assuming that $P - P^*$ is $\mathcal{O}(h^2)$ near Γ, we may also assume that this holds on $\operatorname{supp} a$, thus the last term is $\mathcal{O}(h^2)\|u\|^2$. Thus,

$$\|C_+A_+u\|^2 + \|C_-A_-u\|^2 \le \|E_-u\|^2 + h^{-1}|\langle APu, Au\rangle| + C_1h\|u\|^2. \tag{2.5}$$

Now, by the duality of $\mathcal{H}_{\hbar,\Gamma}$ and $\mathcal{H}^*_{\hbar,\Gamma}$ relative to the L^2 inner product,

$$|\langle APu, Au\rangle| \le \|APu\|_{\mathcal{H}^*_{\hbar,\Gamma}}\|Au\|_{\mathcal{H}_{\hbar,\Gamma}} \le \frac{h\epsilon}{2}\|Au\|^2_{\mathcal{H}_{\hbar,\Gamma}} + \frac{1}{2h\epsilon}\|APu\|^2_{\mathcal{H}^*_{\hbar,\Gamma}}.$$

Further, for $\epsilon > 0$ small, $\epsilon\|Q_+Au\|^2$ can be estimated in terms of $\|C_+A_+u\|^2 + \mathcal{O}(h)\|u\|^2$, as can be seen by comparing the principal symbols, in particular using the ellipticity of C_+ on $\operatorname{supp} a$. One can thus absorb $\frac{\epsilon}{2}\|Au\|^2_{\mathcal{H}_{\hbar,\Gamma}}$ into the left hand side of (2.5). This shows

$$\|C_+A_+u\|^2 + \|C_-A_-u\|^2 \le C\|E_-u\|^2 + Ch^{-2}\|APu\|^2_{\mathcal{H}^*_{\hbar,\Gamma}} + Ch\|u\|^2,$$

which together with the non-trapping control of E_-u (the region $\operatorname{supp} e_-$ is disjoint from Γ_+, so it is backward non-trapped and thus E_-u is controlled by Pu microlocalized off Γ_+, thus by Q_+Pu, modulo higher order in h terms in Pu) proves the first part of Theorem 1.1. Thus, if we have a bound $\|u\| \le C'h^{-1-s}\|Pu\|_{L^2}$, $0 < s < 1/2$, and thus $h\|u\|^2 \le C'h^{-1-2s}\|Pu\|^2_{L^2} \le C''h^{-1-2s}\|Pu\|^2_{\mathcal{H}^*_{\hbar,\Gamma}}$, this implies a non-trapping estimate:

$$\|u\|_{\mathcal{H}_{\hbar,\Gamma}} \le Ch^{-1}\|Pu\|_{\mathcal{H}^*_{\hbar,\Gamma}}.$$

This completes the proof of Theorem 1.1.

In fact, as mentioned earlier, a slight change of point of view proves Theorem 1.1 directly. To see this, we use the Weyl quantization[7] when choosing $a, a_\pm, c_\pm, e_-$; since we are on a manifold, this requires identifying functions with half-densities via trivialization of the half-density bundle by the Riemannian metric; this identification preserves self-adjointness. We also write $P_{\hbar,z}$ as the Weyl quantization of $p_0 + hp_1$ with p_0, p_1 real modulo $\mathcal{O}(h^2)$. Then the principal symbol calculation above holds with p_0 in place of p, and with p_1 included it yields additional terms

$$\begin{aligned}\frac{1}{4}\mathsf{H}_p(a^2) = &-(c_+^2\phi_+^2 + c_-^2\phi_-^2 - h\phi_+\mathsf{H}_{p_1}\phi_+ + h\phi_-\mathsf{H}_{p_1}\phi_-)\\ &\qquad\times(\chi_0\chi_0')(\phi_+^2 - \phi_-^2 + \kappa)\chi(\phi_+^2)^2\psi(p)^2\\ &-(c_+^2\phi_+^2 - h\phi_+\mathsf{H}_{p_1}\phi_+)(\chi'\chi)(\phi_+^2)\chi_0(\phi_+^2 - \phi_-^2 + \kappa)^2\psi(p)^2.\end{aligned}$$

Now, (2.4) becomes

$$\begin{aligned}\frac{i}{4h}[P, A^*A] = &-(C_+A_+)^*(C_+A_+) - (C_-A_-)^*(C_-A_-)\\ &+ h(A_+^*G_+ + G_+^*A_+ + A_-^*G_- + G_-^*A_-) + E + h^2F,\end{aligned} \tag{2.6}$$

with $G_\pm$ being the Weyl quantization of

$$g_\pm = \pm\frac{1}{2}(\mathsf{H}_{p_1}\phi_\pm)\sqrt{(\chi_0\chi_0')(\phi_+^2 - \phi_-^2 + \kappa)}\chi(\phi_+^2)\psi(p),$$

and with $F \in \Psi_\hbar(X)$ with

$$\mathrm{WF}'_\hbar(F) \subset \operatorname{supp} a.$$

[7]In fact, the Weyl quantization is irrelevant. It is straightforward to see that if $A \in \Psi_\hbar(X)$ and if the principal symbol of A is real then the real part of the subprincipal symbol is defined independently of choices. This is all that is needed for the argument below.

Correspondingly, (2.5) becomes

$$\|C_+A_+u\|^2 + \|C_-A_-u\|^2 \leq |\langle Eu, u\rangle| + h^{-1}|\langle APu, Au\rangle| + 2h\|A_+u\|\|G_+u\| + 2h\|A_-u\|\|G_-u\| + C_1h^2\|u\|^2. \tag{2.7}$$

The terms with $G_\pm$ on the right hand side can be estimated by

$$\epsilon\|A_+u\|^2 + \epsilon^{-1}h^2\|G_+u\|^2 + \epsilon\|A_-u\|^2 + \epsilon^{-1}h^2\|G_-u\|^2,$$

and for $\epsilon > 0$ sufficiently small, the $\|A_\pm u\|^2$ terms can now be absorbed into the left hand side of (2.7). Proceeding as above yields

$$\|C_+A_+u\|^2 + \|C_-A_-u\|^2 \leq C|\langle Eu, u\rangle| + Ch^{-2}\|APu\|^2_{\mathcal{H}^*_{\hbar,\Gamma}} + Ch^2\|u\|^2. \tag{2.8}$$

Together with the non-trapping for the E term this gives the global estimate

$$\|u\|^2_{\mathcal{H}_{\hbar,\Gamma}} \leq Ch^{-2}\|Pu\|^2_{\mathcal{H}^*_{\hbar,\Gamma}} + Ch^2\|u\|^2,$$

and now the last term on the right hand side can be absorbed in the left hand side for sufficiently small h, giving the estimate (1.5). □

3. Non-trapping estimates in non-dilation invariant settings

We now transfer Theorem 1.1 into the b-setting; the discussion in the previous section is essentially the dilation invariant special case of this,[8] though in the b-setting there is additional localization near the boundary.

3.1. **Notation and definitions.** For a general reference for b-analysis, see Melrose [6].

Let M be an n-dimensional compact manifold with boundary X.

- Let $\mathcal{V}_b(M)$ be the Lie algebra of *b-vector fields* on M, i.e. of vector fields on M which are tangent to X. Elements of $\mathcal{V}_b(M)$ are sections of a natural vector bundle on M, namely the *b-tangent bundle* ${}^bTM$; in local coordinates (τ, x) near X, the fibers of ${}^bTM$ are spanned by $\tau\partial_\tau$ and ∂_x. The fibers of the dual bundle ${}^bT^*M$, called *b-cotangent bundle*, are spanned by $\frac{d\tau}{\tau}$ and dx.

  It is often convenient to consider the fiber compactification $\overline{{}^bT^*}M$ of ${}^bT^*M$, where the fibers are replaced by their radial compactification. The new boundary of $\overline{{}^bT^*}M$ at fiber infinity is the *b-cosphere bundle* ${}^bS^*M$; it still possesses the compactification of the 'old' boundary $\overline{{}^bT^*}_XM$, see Figure 2. ${}^bS^*M$ is naturally the quotient of ${}^bT^*M \setminus o$ by the $\mathbb{R}^+$-action of dilation in the fibers of the cotangent bundle. Many sets that we will consider below are conic subsets of ${}^bT^*M \setminus o$, and we will often view them as subsets of ${}^bS^*M$.
- For $a \in \mathcal{C}^\infty({}^bT^*M)$, we say $a \in S^m({}^bT^*M)$ if a satisfies

$$|\partial_z^\alpha\partial_\zeta^\beta a(z,\zeta)| \leq C_{\alpha\beta}\langle\zeta\rangle^{m-|\beta|} \text{ for all multiindices } \alpha, \beta$$

  in any coordinate chart, where z are coordinates in the base and ζ coordinates in the fiber; more precisely, in local coordinates (τ, x) near X, we take $\zeta = (\sigma, \xi)$, where we write b-covectors as

$$\sigma\,\frac{d\tau}{\tau} + \sum_j \xi_j\,dx_j.$$

[8] See [11, Section 3.1] for a discussion of the relationship between b- and semiclassical analysis.

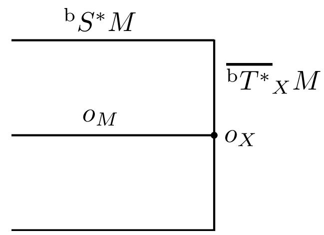

FIGURE 2. The radially compactified cotangent bundle $\overline{{}^{\mathrm{b}}T^*}M$ near $\overline{{}^{\mathrm{b}}T^*}_X M$; the cosphere bundle ${}^{\mathrm{b}}S^*M$, which is the boundary at fiber infinity of $\overline{{}^{\mathrm{b}}T^*}M$, is also shown, as well as the zero section $o_M \subset \overline{{}^{\mathrm{b}}T^*}M$ and the zero section over the boundary $o_X \subset \overline{{}^{\mathrm{b}}T^*}_X M$.

We define the quantization $\mathrm{Op}(a)$ of a, acting on smooth functions u supported in a coordinate chart, by

$$\mathrm{Op}(a)u(\tau,x) = (2\pi)^{-n} \int e^{i(\tau-\tau')\tilde\sigma + i(x-x')\xi} \phi\left(\frac{\tau-\tau'}{\tau}\right) \times a(\tau, x, \tau\tilde\sigma, \xi) u(\tau', x')\, d\tau'\, dx'\, d\tilde\sigma\, d\xi,$$

where the τ'-integral is over $[0,\infty)$, and $\phi \in \mathcal{C}^\infty_{\mathrm{c}}((-1/2, 1/2))$ is identically 1 near 0.[9] For general u, define $\mathrm{Op}(a)u$ using a partition of unity. We write $\mathrm{Op}(a) \in \Psi^m_{\mathrm{b}}(M)$. We say that a is a *symbol* of $\mathrm{Op}(a)$. The equivalence class of a in $S^m({}^{\mathrm{b}}T^*M)/S^{m-1}({}^{\mathrm{b}}T^*M)$ is invariantly defined on ${}^{\mathrm{b}}T^*M$ and is called the *principal symbol* of $\mathrm{Op}(a)$. We will tacitly assume that all our operators have homogeneous principal symbols.

- If $A \in \Psi^{m_1}_{\mathrm{b}}(M)$ and $B \in \Psi^{m_2}_{\mathrm{b}}(M)$, then $[A,B] \in \Psi^{m_1+m_2-1}_{\mathrm{b}}(M)$, and its principal symbol is $\frac{1}{i}\mathsf{H}_a b \equiv \frac{1}{i}\{a,b\}$, where the Hamilton vector field H_a of the principal symbol a of A is the extension of the Hamilton vector field from $T^*M^\circ \setminus o$ to ${}^{\mathrm{b}}T^*M \setminus o$, which is a homogeneous degree $m-1$ vector field on ${}^{\mathrm{b}}T^*M \setminus o$ tangent to the boundary ${}^{\mathrm{b}}T^*_X M$. In local coordinates (τ, x, σ, ξ) on ${}^{\mathrm{b}}T^*M$ as above, this has the form
$$\mathsf{H}_a = (\partial_\sigma a)(\tau\partial_\tau) - (\tau\partial_\tau a)\partial_\sigma + \sum_j \big((\partial_{\xi_j} a)\partial_{x_j} - (\partial_{x_j} a)\partial_{\xi_j}\big). \tag{3.1}$$
- We define bicharacteristics completely analogously to the semiclassical setting.
- The *microsupport* $\mathrm{WF}'_{\mathrm{b}}(A) \subset {}^{\mathrm{b}}T^*M \setminus o$ of $A = \mathrm{Op}(a) \in \Psi^m_{\mathrm{b}}(M)$ is the complement of the set of all $\rho \in {}^{\mathrm{b}}T^*M \setminus o$ such that a is rapidly decaying in a conic neighborhood around ρ. Note that $\mathrm{WF}'_{\mathrm{b}}(A)$ is conic, hence we will also view it as a subset of ${}^{\mathrm{b}}S^*M$.
- Fix a *b-density* on M, which is locally of the form $a\left|\frac{d\tau}{\tau}\, dz\right|$, $a > 0$.
- Define the *b-Sobolev space* $H^k_{\mathrm{b}}(M)$ for $k \in \mathbb{Z}_{\geq 0}$ by
$$H^k_{\mathrm{b}}(M) = \{u \in L^2(M) \colon X_1 \cdots X_k u \in L^2(M), X_1, \ldots, X_j \in \mathcal{V}_{\mathrm{b}}(M)\},$$
and for general $k \in \mathbb{R}$ by duality and interpolation. Moreover, define the weighted b-Sobolev spaces $H^{s,\alpha}_{\mathrm{b}}(M) := \tau^\alpha H^s_{\mathrm{b}}(M)$ for $s, \alpha \in \mathbb{R}$, where τ is

[9] The cutoff ϕ ensures that these operators lie in the 'small b-calculus' of Melrose, in particular that such quantizations act on weighted b-Sobolev spaces, defined below.

a boundary defining function, i.e. $\tau = 0$ at X and $d\tau \neq 0$ there. Every b-pseudodifferential operator $A \in \Psi_{\mathrm{b}}^m(M)$ defines a bounded map $A\colon H_{\mathrm{b}}^{s,\alpha}(M) \to H_{\mathrm{b}}^{s-m,\alpha}(M)$, $s, \alpha \in \mathbb{R}$.

- For $A \in \Psi_{\mathrm{b}}^m(M)$ with principal symbol $a \in S^m({}^{\mathrm{b}}T^*M)$, we say that A is *elliptic* at $\rho \in {}^{\mathrm{b}}T^*M \setminus o$ if there is a constant $C > 0$ such that $|a(z,\zeta)| \geq C|\zeta|^m$ for (z,ζ) in a conic neighborhood of ρ. The *characteristic set* of A is the complement (in ${}^{\mathrm{b}}T^*M \setminus o$) of the set of all ρ at which A is elliptic.
- For $u \in H_{\mathrm{b}}^{-\infty,\alpha}(M)$, define its $H_{\mathrm{b}}^{s,\alpha}$ *wave front set* $\mathrm{WF}_{\mathrm{b}}^{s,\alpha}(u) \subset {}^{\mathrm{b}}T^*M \setminus o$ as the complement of the set of all $\rho \in {}^{\mathrm{b}}T^*M \setminus o$ for which there exists $a \in S^0({}^{\mathrm{b}}T^*M)$ elliptic at ρ such that $\mathrm{Op}(a)u \in H_{\mathrm{b}}^{s,\alpha}(M)$. In particular, $\mathrm{WF}_{\mathrm{b}}^{s,\alpha}(u) = \emptyset$ if and only if $u \in H_{\mathrm{b}}^{s,\alpha}(M)$.
- *Microlocal elliptic regularity* states that if $Au = f$ with $A \in \Psi_{\mathrm{b}}^m(M)$, $u, f \in H_{\mathrm{b}}^{-\infty,\alpha}(M)$, $\rho \notin \mathrm{WF}_{\mathrm{b}}^{s-m,\alpha}(f)$ and A is elliptic at ρ, then $\rho \notin \mathrm{WF}_{\mathrm{b}}^{s,\alpha}(u)$.
- If $A \in \Psi_{\mathrm{b}}^m(M)$ has a principal symbol with non-positive imaginary part, $u, f \in H_{\mathrm{b}}^{-\infty,\alpha}(M)$, $Au = f$, moreover $\rho \notin \mathrm{WF}_{\mathrm{b}}^{s,\alpha}(u)$ and $\gamma_\rho([0,T]) \cap \mathrm{WF}_{\mathrm{b}}^{s-m+1,\alpha}(f) = \emptyset$ for some $T > 0$, then the *propagation of singularities* states that $\gamma_\rho(T) \notin \mathrm{WF}_{\mathrm{b}}^{s,\alpha}(u)$.

3.2. **Setup, statement and proof of the result.** Suppose $\mathcal{P} \in \Psi_{\mathrm{b}}^m(M)$, $\mathcal{P} - \mathcal{P}^* \in \Psi_{\mathrm{b}}^{m-2}(M)$. Let p be the principal symbol of $\mathcal{P}$, which is thus a homogeneous degree m function on ${}^{\mathrm{b}}T^*M \setminus o$, which we assume to be *real-valued*. Let $\tilde\rho$ denote a homogeneous degree -1 defining function of ${}^{\mathrm{b}}S^*M$. Then the rescaled Hamilton vector field

$$V = \tilde\rho^{m-1}\mathsf{H}_p$$

is a $\mathcal{C}^\infty$ vector field on $\overline{{}^{\mathrm{b}}T^*}M$ away from the 0-section, and it is tangent to all boundary faces. The characteristic set Σ is the zero-set of the smooth function $\tilde\rho^m p$ in ${}^{\mathrm{b}}S^*M$. We will, somewhat imprecisely, refer to the flow of V in $\Sigma \subset {}^{\mathrm{b}}S^*M$ as the Hamilton, or (null)bicharacteristic flow; its integral curves, the (null)bicharacteristics, are reparameterizations of those of the Hamilton vector field H_p, projected by the quotient map ${}^{\mathrm{b}}T^*M \setminus o \to {}^{\mathrm{b}}S^*M$.

We first work microlocally near the trapped set, namely assume that

(1) $\Gamma \subset \Sigma \cap {}^{\mathrm{b}}S^*_X M$ is a smooth submanifold disjoint from the image of $T^*X \setminus o$ (so τD_τ is elliptic near Γ),
(2) Γ_+ is a smooth submanifold of $\Sigma \cap {}^{\mathrm{b}}S^*_X M$ in a neighborhood U_1 of Γ,
(3) Γ_- is a smooth submanifold of Σ transversal to $\Sigma \cap {}^{\mathrm{b}}S^*_X M$ in U_1,
(4) Γ_+ has codimension 2 in Σ, Γ_- has codimension 1,
(5) Γ_+ and Γ_- intersect transversally in Σ with $\Gamma_+ \cap \Gamma_- = \Gamma$,
(6) the rescaled Hamilton vector field $V = \tilde\rho^{m-1}\mathsf{H}_{p_0}$ is tangent to both Γ_+ and Γ_-, and thus to Γ.

We assume that Γ_+ is backward trapped for the Hamilton flow (i.e. bicharacteristics in Γ_+ near Γ tend to Γ as the parameter goes to $-\infty$), i.e. is the unstable manifold of Γ, while Γ_- is forward trapped, i.e. is the stable manifold of Γ, see Figure 3; indeed, we assume a quantitative version of this. (There is a completely analogous statement if Γ_+ is forward trapped and Γ_- is backward trapped: replacing $\mathcal{P}$ by $-\mathcal{P}$ preserves all assumptions, but reverses the Hamilton flow.) To state this, let ϕ_- be a defining function of Γ_-, and let $\phi_+ \in \mathcal{C}^\infty({}^{\mathrm{b}}S^*M)$ be a defining function of Γ_+ in ${}^{\mathrm{b}}S^*_X M$; thus Γ_+ is defined within ${}^{\mathrm{b}}S^*M$ by $\tau = 0, \phi_+ = 0$. Notice

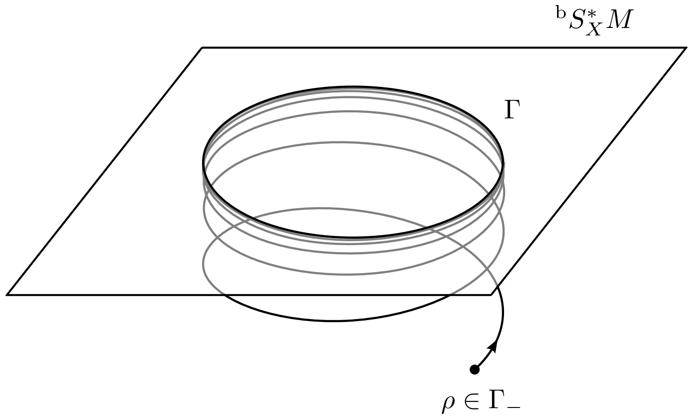

FIGURE 3. An examplary situation with trapping: Shown are the (projection from ${}^{\mathrm{b}}S^*M$ to the base M of the) trapped set Γ, the b-cosphere bundle over X as well as a forward bicharacteristic starting at a point $\rho \in \Gamma_-$.

that V being to tangent to ${}^{\mathrm{b}}S^*_X M$ (due to (3.1)) implies that $V\tau$ is a multiple of τ; we assume that, near Γ,

$$V\tau = -c_\partial^2 \tau,\ c_\partial > 0; \tag{3.2}$$

this is consistent with the stability of Γ_-. By the tangency requirement, with

$$\hat{p}_0 = \tilde{\rho}^m p_0,$$

$V\phi_- = \alpha_- \phi_- + \nu_- \hat{p}_0$, α_- smooth; notice that changing ϕ_- by a smooth non-zero multiple f gives $V(f\phi_-) = \alpha_- f\phi_- + \nu_- f\hat{p}_0 + (Vf)\phi_-$, so α_- depends on the choice of ϕ_-. On the other hand, the tangency requirement gives $V\phi_+ = \alpha_+\phi_+ + \beta_+\tau + \nu_+\hat{p}_0$. For the sake of conciseness, rather than stating the assumptions on the Hamilton flow as in [13], we assume directly that $\phi_\pm$ satisfy

$$V\phi_- = c_-^2 \phi_- + \nu_- \hat{p}_0,\ V\phi_+ = -c_+^2 \phi_+ + \beta_+ \tau + \nu_+ \hat{p}_0, \tag{3.3}$$

with $c_\pm > 0$ smooth near Γ, $\beta_+, \nu_\pm$ smooth near Γ and

$$\{\phi_+, \phi_-\} > 0 \tag{3.4}$$

near Γ. However, if we merely assume the normal hyperbolicity within ${}^{\mathrm{b}}S^*_X M$ as in [13, Section 1.2], [13, Lemma 4.1], as corrected in [12], actually gives such defining functions $\phi_\pm^0$ *within* ${}^{\mathrm{b}}S^*_X M$ (i.e. letting $\tau = 0$); taking an arbitrary extension in case of ϕ_+, and an extension which is a defining function in case of Γ_-, all the requirements above are satisfied. Let $U_0 \subset \overline{U_0} \subset U_1$ be a neighborhood of Γ such that the Poisson bracket in (3.4) as well as $c_\pm$ have positive lower bounds.

There is an asymmetry between the roles of $\phi_\pm$ and τ, and thus we consider the parabolic defining function

$$\rho_+ = \phi_+^2 + \mathsf{M}\tau$$

for Γ_+, $\mathsf{M} > 0$, to be chosen. Then, near Γ,

$$\begin{aligned}\hat{\rho}_+ = V\rho_+ &= -2c_+^2\phi_+^2 + 2\beta_+\phi_+\tau + 2\nu_+\phi_+\hat{p}_0 - \mathsf{M}c_\partial^2\tau \\ &= -2c_+^2\phi_+^2 - (\mathsf{M}c_\partial^2 - 2\beta_+\phi_+)\tau + 2\nu_+\phi_+\hat{p}_0 \\ &\leq -\tilde{c}_+^2\rho_+ + 2\nu_+\phi_+\hat{p}_0, \qquad \tilde{c}_+ > 0,\end{aligned} \tag{3.5}$$

if $\mathsf{M} > 0$ is chosen sufficiently large, consistently with the forward trapped nature of Γ_-. (Here the term with $\hat{p}_0$ is considered harmless as one essentially restricts to the characteristic set, $\hat{p}_0 = 0$.) Also, note that one can use[10] the reciprocal $\tilde{\rho} = |\sigma|^{-1}$ of the principal symbol σ of τD_τ as the local defining function of ${}^{\mathrm{b}}S^*M$ as fiber-infinity in ${}^{\mathrm{b}}T^*M$ near Γ; then

$$V\tilde{\rho} = \tilde{\alpha}\tilde{\rho}\tau \tag{3.6}$$

for some $\tilde{\alpha}$ smooth in view of (3.1).

Similar to the normally isotropic spaces in the semiclassical setting, we introduce spaces which are *normally isotropic at* Γ.[11] Concretely, let $Q_\pm \in \Psi_{\mathrm{b}}^0(M)$ have principal symbol $\phi_\pm$ as before, $\hat{P}_0 \in \Psi_{\mathrm{b}}^0(M)$ have principal symbol $\hat{p}_0$ and let $Q_0 \in \Psi_{\mathrm{b}}^0(M)$ be elliptic, with real principal symbol for convenience, on U_0^c (and thus nearby). Define the (global) b-normally isotropic spaces at Γ of order s, $\mathcal{H}_{\mathrm{b},\Gamma}^s$, by the norm

$$\|u\|_{\mathcal{H}_{\mathrm{b},\Gamma}^s}^2 = \|Q_0u\|_{H_{\mathrm{b}}^s}^2 + \|Q_+u\|_{H_{\mathrm{b}}^s}^2 + \|Q_-u\|_{H_{\mathrm{b}}^s}^2 + \|\tau^{1/2}u\|_{H_{\mathrm{b}}^s}^2 + \|\hat{P}_0u\|_{H_{\mathrm{b}}^s}^2 + \|u\|_{H_{\mathrm{b}}^{s-1/2}}^2, \tag{3.7}$$

and let $\mathcal{H}_{\mathrm{b},\Gamma}^{*,-s}$ be the dual space relative to L^2, which is thus[12]

$$Q_0H_{\mathrm{b}}^{-s} + Q_+H_{\mathrm{b}}^{-s} + Q_-H_{\mathrm{b}}^{-s} + \tau^{1/2}H_{\mathrm{b}}^{-s} + \hat{P}_0H_{\mathrm{b}}^{-s} + H_{\mathrm{b}}^{-s+1/2}.$$

Note that microlocally away from Γ, $\mathcal{H}_{\mathrm{b},\Gamma}^s$ is just the standard H_{b}^s space while $\mathcal{H}_{\mathrm{b},\Gamma}^{*,-s}$ is H_{b}^{-s} since at least one of Q_0, $Q_\pm$, τ is elliptic. Moreover, $\Psi_{\mathrm{b}}^k(M) \ni A : \mathcal{H}_{\mathrm{b},\Gamma}^s \to \mathcal{H}_{\mathrm{b},\Gamma}^{s-k}$ is continuous since $[Q_+, A] \in \Psi_{\mathrm{b}}^{k-1}(M)$ etc.; the analogous statement also holds for the dual spaces. Further, the last term in (3.7) can be replaced by $\|u\|_{H_{\mathrm{b}}^{s-1}}^2$ as $i[Q_+, Q_-] = B^*B + R$, $B \in \Psi_{\mathrm{b}}^{-1/2}(M)$, $R \in \Psi_{\mathrm{b}}^{-2}(M)$, using the same argument as in the semiclassical setting (however, it cannot be dropped altogether unlike in the semiclassical setting!).

Remark 3.1. The notation $\mathcal{H}_{\mathrm{b},\Gamma}^s(M)$ is justified for the space is independent of the particular defining functions $\phi_\pm$ chosen; near Γ any other choice would replace $\phi_\pm$

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[10] Indeed, in the semiclassical setting, after Mellin transforming this problem, $|\sigma|^{-1}$ plays the role of the semiclassical parameter h, which in that case *commutes* with the operator.

[11] Note that ${}^{\mathrm{b}}T^*M$ is *not* a symplectic manifold (in the natural way) since the symplectic form on ${}^{\mathrm{b}}T^*_{M^\circ}M$ does not extend smoothly to ${}^{\mathrm{b}}T^*M$. Thus, the word 'normally isotropic' is not completely justified; we use it since it reflects that in the analogous semiclassical setting, see [13], the set Γ is symplectic, and the origin in the symplectic orthocomplement $(T_\alpha\Gamma)^\perp$ of $T_\alpha\Gamma$, which is also symplectic, is isotropic within $(T_\alpha\Gamma)^\perp$.

[12] We refer to [7, Appendix A] for a general discussion of the underlying functional analysis. In particular, Lemma A.3 there essentially gives the density of $\dot{\mathcal{C}}^\infty(M)$ in $\mathcal{H}_{\mathrm{b},\Gamma}^s(M)$: one can simply drop the subscript 'e' in the statement of that lemma to conclude that $H_{\mathrm{b}}^\infty(M)$ (so in particular $H_{\mathrm{b}}^{s-1/2}(M)$) is dense in $\mathcal{H}_{\mathrm{b},\Gamma}^s(M)$, and then the density of $\dot{\mathcal{C}}^\infty(M)$ in $H_{\mathrm{b}}^{s'}(M)$ for any s' completes the argument. The completeness of $\mathcal{H}_{\mathrm{b},\Gamma}^s(M)$ follows from the continuity of $\Psi_{\mathrm{b}}^0(M)$ on $H_{\mathrm{b}}^{s-1/2}(M)$.

by smooth non-degenerate linear combinations plus a multiple of τ and of $\hat{p}$, denote these by $\tilde{\phi}_\pm$, and thus the corresponding $\tilde{Q}_\pm$ can be expressed as

$$B_+Q_+ + B_-Q_- + B_\partial\tau + \hat{B}\hat{P} + B_0Q_0 + R, \qquad B_\pm, B_0, B_\partial, \hat{B} \in \Psi_b^0(M),\ R \in \Psi_b^{-1}(M),$$

so the new norm can be controlled by the old norm, and conversely in view of the non-degeneracy.

Our result is then:

Theorem 3.2. *With $\mathcal{P}, \mathcal{H}^s_{b,\Gamma}, \mathcal{H}^{*,s}_{b,\Gamma}$ as above, for any neighborhood U of Γ and for any N there exist $B_0 \in \Psi_b^0(M)$ elliptic at Γ and $B_1, B_2 \in \Psi_b^0(M)$ with $\mathrm{WF}_b'(B_j) \subset U$, $j = 0, 1, 2$, $\mathrm{WF}_b'(B_2) \cap \Gamma_+ = \emptyset$ and $C > 0$ such that*

$$\|B_0u\|_{\mathcal{H}^s_{b,\Gamma}} \le \|B_1\mathcal{P}u\|_{\mathcal{H}^{*,s-m+1}_{b,\Gamma}} + \|B_2u\|_{H^s_b} + C\|u\|_{H_b^{-N}}, \tag{3.8}$$

i.e. if all the functions on the right hand side are in the indicated spaces: $B_1\mathcal{P}u \in \mathcal{H}^{*,s-m+1}_{b,\Gamma}$, etc., then $B_0u \in \mathcal{H}^s_{b,\Gamma}$, and the inequality holds.

The same conclusion also holds if we assume $\mathrm{WF}_b'(B_2) \cap \Gamma_- = \emptyset$ instead of $\mathrm{WF}_b'(B_2) \cap \Gamma_+ = \emptyset$.

Finally, if $r < 0$, then, with $\mathrm{WF}_b'(B_2) \cap \Gamma_+ = \emptyset$, (3.8) becomes

$$\|B_0u\|_{H_b^{s,r}} \le \|B_1\mathcal{P}u\|_{H_b^{s-m+1,r}} + \|B_2u\|_{H_b^{s,r}} + C\|u\|_{H_b^{-N,r}}, \tag{3.9}$$

while if $r > 0$, then, with $\mathrm{WF}_b'(B_2) \cap \Gamma_- = \emptyset$,

$$\|B_0u\|_{H_b^{s,r}} \le \|B_1\mathcal{P}u\|_{H_b^{s-m+1,r}} + \|B_2u\|_{H_b^{s,r}} + C\|u\|_{H_b^{-N,r}}, \tag{3.10}$$

Remark 3.3. Note that the weighted versions (3.9)-(3.10) use *standard* weighted b-Sobolev spaces; this corresponds to non-trapping semiclassical estimates if the subprincipal symbol has the correct, definite, sign at Γ.

Proof. We may assume that $U \subset U_0$ is disjoint from a neighborhood of $\mathrm{WF}_b'(Q_0)$, and thus ignore Q_0 in the definition of $\mathcal{H}^s_{b,\Gamma}$ below.

We first prove that there exist B_0, B_1, B_2 as above and $B_3 \in \Psi_b^0(M)$ with $\mathrm{WF}_b'(B_3) \subset U$ such that

$$\|B_0u\|_{\mathcal{H}^s_{b,\Gamma}} \le \|B_1\mathcal{P}u\|_{\mathcal{H}^{*,s-m+1}_{b,\Gamma}} + \|B_2u\|_{H^s_b} + \|B_3u\|_{H_b^{s-1}} + C\|u\|_{H_b^{-N}}. \tag{3.11}$$

An iterative argument will then prove the theorem.

The proof is a straightforward modification of the construction in the semiclassical setting above, replacing ϕ_+^2 by $\phi_+^2 + \mathsf{M}\tau$, $\mathsf{M} > 0$ large, in accordance with (3.5).

We start by pointing out that for any $\tilde{B}_0 \in \Psi_b^0(M)$ and any $\tilde{B}_3 \in \Psi_b^0(M)$ elliptic on $\mathrm{WF}_b'(\tilde{B}_0)$, $\|\hat{P}_0\tilde{B}_0u\|_{H^s_b} \le C\|\tilde{B}_0\mathcal{P}u\|_{H_b^{s-m}} + C'\|\tilde{B}_3u\|_{H_b^{s-1}}$, by simply using that $\hat{P}_0$ is an elliptic multiple of P modulo $\Psi_b^{-1}(M)$. Since $\|\tilde{B}_0\mathcal{P}u\|_{H_b^{s-m}} \le C\|\tilde{B}_0\mathcal{P}u\|_{\mathcal{H}^{*,s-m}_{b,\Gamma}}$, the $\hat{P}_0$ contribution to $\|\tilde{B}_0u\|_{\mathcal{H}^s_{b,\Gamma}}$ in (3.11) is thus automatically controlled.

So let $\chi_0(t) = e^{-\digamma/t}$ for $t > 0$, $\chi_0(t) = 0$ for $t \le 0$, with $\digamma > 0$ (large) to be specified, $\chi \in \mathcal{C}_c^\infty([0,\infty))$ be identically 1 near 0 with $\chi' \le 0$, and indeed with $\chi'\chi = -\chi_1^2$, $\chi_1 \ge 0$, $\chi_1 \in \mathcal{C}_c^\infty([0,\infty))$, and let $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$ be identically 1 near

0. As we use the Weyl quantization,[13] we write $\mathcal{P}$ as the Weyl quantization of $p = p_0 + \tilde\rho p_1$, with $\tilde\rho p_1$ of order $m-1$. Let

$$a = \tilde\rho^{-s+(m-1)/2}\chi_0(\rho_+ - \phi_-^2 + \kappa)\chi(\rho_+)\psi(\tilde\rho^m p), \tag{3.12}$$

$\kappa > 0$ small. Notice that on $\operatorname{supp} a$, if χ is supported in $[0, R]$,

$$\rho_+ \le R,\ \phi_-^2 \le \rho_+ + \kappa = R + \kappa,$$

so a is localized near Γ if R and κ are taken sufficiently small. In particular, the argument of χ_0 is bounded above by $R+\kappa$, so given any $\mathsf{M}_0 > 0$ one can take $\digamma > 0$ large so that

$$\chi_0'\chi_0 - \mathsf{M}_0\chi_0^2 = b^2\chi_0'\chi_0,$$

with $b \ge 1/2$, $\mathcal{C}^\infty$, on the range of the argument of χ_0.

In fact, we also need to regularize, namely introduce

$$a_\epsilon = (1 + \epsilon\tilde\rho^{-1})^{-2}a,\ \epsilon \in [0, 1], \tag{3.13}$$

which is a symbol of order $s-(m-1)/2-2$ for $\epsilon > 0$, and is uniformly bounded in symbols of order $s-(m-1)/2$ as ϵ varies in $[0, 1]$. In order to avoid more cumbersome notation below, we ignore the regularizer and work directly with a; since the regularizer gives the same kind of contributions to the commutator as the weight $\tilde\rho^{-s+(m-1)/2}$, these contributions can be dominated in exactly the same way.

Then, with $p = p_0 + \tilde\rho p_1$ as above, $W = \tilde\rho^{m-2}\mathsf{H}_{\tilde\rho p_1}$, which is a smooth vector field near ${}^{\mathrm{b}}S^*M$ as $\tilde\rho p_1$ is order $m-1$, noting $W\tilde\rho = \tilde\alpha_1\tau\tilde\rho$ similarly to (3.6), and $W\tau = \alpha_{\partial,1}\tau$ by the tangency of W to $\tau = 0$,

$$\begin{aligned}\frac{1}{4}\mathsf{H}_p(a^2) = &-(-\hat\rho_+/2 + c_-^2\phi_-^2 + \nu_-\phi_-\hat p_0 - \tilde\rho\phi_+(W\phi_+) - \tilde\rho\mathsf{M}\alpha_{\partial,1}\tau + \tilde\rho\phi_-(W\phi_-))\\ &\qquad\times\tilde\rho^{-2s}(\chi_0\chi_0')(\rho_+ - \phi_-^2 + \kappa)\chi(\rho_+)^2\psi(\tilde\rho^m p)^2\\ &+\frac{1}{4}(-2s+m-1)\tilde\rho^{-2s}(\tilde\alpha + \tilde\rho\tilde\alpha_1)\tau\chi_0(\rho_+ - \phi_-^2 + \kappa)^2\chi(\rho_+)^2\psi(\tilde\rho^m p)^2\\ &+\frac{1}{2}\tilde\rho^{-2s}(\hat\rho_+ + \tilde\rho W\rho_+)(\chi'\chi)(\rho_+)\chi_0(\rho_+ - \phi_-^2 + \kappa)^2\psi(\tilde\rho^m p)^2\\ &+\frac{m}{2}(\tilde\alpha + \tilde\rho\tilde\alpha_1)\tilde\rho^{-2s}(\tilde\rho^m p)\tau\chi_0(\rho_+ - \phi_-^2 + \kappa)^2\chi(\rho_+)^2(\psi\psi')(\tilde\rho^m p).\end{aligned} \tag{3.14}$$

A key point is that the second term on the right hand side, given by the weight $\tilde\rho^{-2s+m-1}$ being differentiated, can be absorbed into the first by making $\digamma > 0$ large so that $\hat\rho_+\chi_0'(\rho_+ - \phi_-^2 + \kappa)$ dominates

$$|-2s+m-1||\tilde\alpha|\tau\chi_0(\rho_+ - \phi_-^2 + \kappa)$$

on $\operatorname{supp} a$, which can be arranged as $|-2s+m-1||\tilde\alpha|\tau$ is bounded by a sufficiently large multiple of $\hat\rho_+$ there. Thus,

$$\frac{1}{4}\mathsf{H}_p(a^2) = -c_+^2a_+^2 - c_-^2a_-^2 - a_\partial^2 + 2g_+a_+ + 2g_-a_- + e + \tilde e + 2a_+j_+p + 2a_-j_-p \tag{3.15}$$

with

$$a_\pm = \tilde\rho^{-s}\phi_\pm\sqrt{(\chi_0\chi_0')(\rho_+ - \phi_-^2 + \kappa)}\chi(\rho_+)\psi(\tilde\rho^m p),$$

[13] Again, the Weyl quantization is irrelevant: If $A \in \Psi_{\mathrm{b}}^m(X)$ and the principal symbol of A is real then the real part of the subprincipal symbol is defined independently of choices, which suffices below.

$$\begin{aligned}
a_\partial &= \tilde\rho^{-s}\tau^{1/2}\Big((\mathsf{M}(c_\partial^2/2) - \beta_+\phi_+ - \tilde\rho\mathsf{M}a_{\partial,1})(\chi_0\chi_0')(\rho_+ - \phi_-^2 + \kappa) \\
&\qquad\qquad - \frac{1}{4}(-2s+m-1)(\tilde\alpha + \tilde\rho\tilde\alpha_1)\chi_0(\rho_+ - \phi_-^2 + \kappa)^2\Big)^{1/2}\chi(\rho_+)\psi(\tilde\rho^m p), \\
g_\pm &= \pm\frac{1}{2}\tilde\rho^{-s+1}((W\phi_\pm) - \nu_\pm\tilde\rho^{m-1}p_1)\sqrt{(\chi_0\chi_0')(\rho_+ - \phi_-^2 + \kappa)}\chi(\rho_+)\psi(\tilde\rho^m p), \\
e &= -\frac{1}{2}\tilde\rho^{-2s}(\hat\rho_+ + \hat\rho W\rho_+)\chi_1(\rho_+)^2\chi_0(\rho_+ - \phi_-^2 + \kappa)^2\psi(\tilde\rho^m p)^2, \\
\tilde e &= \frac{m}{2}\tilde\rho^{-2s}(\tilde\rho^m p)(\tilde\alpha + \tilde\rho\tilde\alpha_1)\tau\chi_0(\rho_+ - \phi_-^2 + \kappa)^2\chi(\rho_+)^2(\psi\psi')(\tilde\rho^m p), \\
j_\pm &= \pm\frac{1}{2}\nu_\pm\tilde\rho^{-s+m}\sqrt{(\chi_0\chi_0')(\rho_+ - \phi_-^2 + \kappa)}\chi(\rho_+)\psi(\tilde\rho^m p),
\end{aligned}$$

the square root in a_∂ is that of a non-negative quantity and is $\mathcal{C}^\infty$ for M large (so that $\beta_+\phi_+$ can be absorbed into $\mathsf{M}(c_\partial^2/2)$) and $\digamma$ large (so that a small multiple of χ_0' can be used to dominate χ_0), as discussed earlier, and

$$\begin{aligned}
&\operatorname{supp} e \subset \operatorname{supp} a,\ \operatorname{supp} e \cap \Gamma_+ = \emptyset, \\
&\operatorname{supp} \tilde e \subset \operatorname{supp} a,\ \operatorname{supp} \tilde e \cap \Sigma = \emptyset.
\end{aligned}$$

This gives, with the various operators being Weyl quantizations of the corresponding lower case symbols,

$$\begin{aligned}
\frac{i}{4}[\mathcal{P}, A^*A] = &-(C_+A_+)^*(C_+A_+) - (C_-A_-)^*(C_-A_-) - A_\partial^*A_\partial \\
&+ G_+^*A_+ + A_+^*G_+ + G_-^*A_- + A_-^*G_- \\
&+ E + \tilde E + A_+^*J_+\mathcal{P} + \mathcal{P}^*J_+^*A_+ + A_-^*J_-\mathcal{P} + \mathcal{P}^*J_-^*A_- + F
\end{aligned} \tag{3.16}$$

where now $A \in \Psi_{\mathrm{b}}^{s-(m-1)/2}(M)$, $A_\pm, A_\partial \in \Psi_{\mathrm{b}}^s(M)$, $G_\pm \in \Psi_{\mathrm{b}}^{s-1}(M)$, $E \in \Psi_{\mathrm{b}}^{2s}(M)$, $\tilde E \in \Psi_{\mathrm{b}}^{2s}(M)$, $J_\pm \in \Psi_{\mathrm{b}}^{s-m}(M)$, $F \in \Psi_{\mathrm{b}}^{2s-2}(M)$ with $\mathrm{WF}_{\mathrm{b}}'(F) \subset \operatorname{supp} a$.

After this point the calculations repeat the semiclassical argument: First using $\mathcal{P} - \mathcal{P}^* \in \Psi_{\mathrm{b}}^{m-2}(M)$,

$$\begin{aligned}
&\|C_+A_+u\|^2 + \|C_-A_-u\|^2 + \|A_\partial u\|^2 \\
&\quad \le |\langle Eu, u\rangle| + |\langle \tilde Eu, u\rangle| + |\langle A\mathcal{P}u, Au\rangle| + 2\|A_+u\|\|G_+u\| + 2\|A_-u\|\|G_-u\| \\
&\qquad\quad + 2|\langle J_+\mathcal{P}u, A_+u\rangle| + 2|\langle J_-\mathcal{P}u, A_-u\rangle| + C_1\|\tilde F_1 u\|^2_{H_{\mathrm{b}}^{s-1}} + C_1\|u\|^2_{H_{\mathrm{b}}^{-N}},
\end{aligned} \tag{3.17}$$

where we took $\tilde F_1 \in \Psi_{\mathrm{b}}^0(M)$ elliptic on $\mathrm{WF}_{\mathrm{b}}'(F)$ and with $\mathrm{WF}_{\mathrm{b}}'(\tilde F_1)$ near Γ. Noting that $\mathrm{WF}_{\mathrm{b}}'(\tilde E) \cap \Sigma = \emptyset$, the elliptic estimates give

$$|\langle \tilde Eu, u\rangle| \le C\|B_1\mathcal{P}u\|^2_{H_{\mathrm{b}}^{s-m}} + C\|u\|^2_{H_{\mathrm{b}}^{-N}}$$

if $B_1 \in \Psi_{\mathrm{b}}^0(M)$ is elliptic on $\operatorname{supp}\tilde e$. Let $\Lambda \in \Psi_{\mathrm{b}}^{(m-1)/2}(M)$ be elliptic with real principal symbol λ, and let $\Lambda^- \in \Psi_{\mathrm{b}}^{-(m-1)/2}(M)$ be a parametrix for it so that $\Lambda\Lambda^- - \mathrm{Id} = R_0 \in \Psi_{\mathrm{b}}^{-\infty}(M)$. Then

$$\begin{aligned}
|\langle A\mathcal{P}u, Au\rangle| &\le |\langle \Lambda^- A\mathcal{P}u, \Lambda^* Au\rangle\| + |\langle R_0 A\mathcal{P}u, Au\rangle| \\
&\le \frac{1}{2\epsilon}\|\Lambda^- A\mathcal{P}u\|^2_{\mathcal{H}_{\mathrm{b},\Gamma}^{*,0}} + \frac{\epsilon}{2}\|\Lambda^* Au\|^2_{\mathcal{H}_{\mathrm{b},\Gamma}^{0}} + C'\|u\|^2_{H_{\mathrm{b}}^{-N}}
\end{aligned}$$

As $\Lambda^* A \in \Psi_b^s(M)$, for sufficiently small $\epsilon > 0$, $\frac{\epsilon}{2}\|\Lambda^* A u\|^2_{\mathcal{H}^0_{b,\Gamma}}$ can be absorbed into[14] $\|C_+A_+u\|^2 + \|C_-A_-u\|^2 + \|A_\partial u\|^2$ plus $\|\tilde{B}_0\hat{P}_0 u\|^2_{H^s_b}$, and as discussed above, the latter already has the control required for (3.11). On the other hand, taking $B_1 \in \Psi_b^0(M)$ elliptic on $\mathrm{WF}'_b(A)$, as $\Lambda^- A \in \Psi_b^{s-m+1}(M)$,

$$\|\Lambda^- A\mathcal{P}u\|^2_{\mathcal{H}^{*,0}_{b,\Gamma}} \leq C''\|B_1\mathcal{P}u\|^2_{\mathcal{H}^{*,s-m+1}_{b,\Gamma}} + C''\|u\|^2_{H_b^{-N}}.$$

Similarly, to deal with the $J_\pm$ terms on the right hand side of (3.17), one writes

$$\begin{aligned}|\langle J_\pm \mathcal{P}u, A_\pm u\rangle| &\leq \frac{1}{2\epsilon}\Big(\|B_1\mathcal{P}u\|^2_{H_b^{s-m}} + C''\|u\|^2_{H_b^{-N}}\Big) + \frac{\epsilon}{2}\|A_\pm u\|^2_{L^2} \\ &\leq \frac{1}{2\epsilon}\Big(\|B_1\mathcal{P}u\|^2_{\mathcal{H}^{*,s-m}_{b,\Gamma}} + C''\|u\|^2_{H_b^{-N}}\Big) + \frac{\epsilon}{2}\|A_\pm u\|^2_{L^2},\end{aligned}$$

while the $G_\pm$ terms can be estimated by

$$\epsilon\|A_+u\|^2 + \epsilon^{-1}\|G_+u\|^2 + \epsilon\|A_-u\|^2 + \epsilon^{-1}\|G_-u\|^2,$$

and for $\epsilon > 0$ sufficiently small, the $\|A_\pm u\|^2$ terms in both cases can be absorbed into the left hand side of (3.17) while the $G_\pm$ into the error term. This gives, with $\tilde{F}_2$ having properties as $\tilde{F}_1$,

$$\begin{aligned}&\|C_+A_+u\|^2 + \|C_-A_-u\|^2 + \|A_\partial u\|^2 \\ &\quad\leq |\langle Eu, u\rangle| + C\|B_1\mathcal{P}u\|^2_{\mathcal{H}^{*,s-m+1}_{b,\Gamma}} + C_2\|\tilde{F}_2 u\|^2_{H_b^{s-1}} + C_2\|u\|^2_{H_b^{-N}}.\end{aligned}$$

By the remark before the statement of the theorem, if $B_0 \in \Psi_b^0(M)$ is such that $\chi_0(\rho_+ - \phi_-^2 + \kappa)\chi(\rho_+)\psi(p) > 0$ on $\mathrm{WF}'_b(B_0)$, $\|B_0 u\|^2_{H_b^{s-1/2}}$ can be added to the left hand side at the cost of changing the constant in front of $\|\tilde{F}_2 u\|^2_{H_b^{s-1}} + \|u\|^2_{H_b^{-N}}$ on the right hand side. Taking such $B_0 \in \Psi_b^0(M)$, and B_1 elliptic on $\mathrm{WF}'_b(A)$ as before, $B_2 \in \Psi_b^0(M)$ elliptic on $\mathrm{WF}'_b(E)$ but with $\mathrm{WF}'_b(B_2)$ disjoint from Γ_+, we conclude that

$$\|B_0 u\|^2_{\mathcal{H}^s_{b,\Gamma}} \leq C\|B_1\mathcal{P}u\|^2_{\mathcal{H}^{*,s-m+1}_{b,\Gamma}} + C\|B_2 u\|^2_{H_b^s} + C\|\tilde{F}_2 u\|^2_{H_b^{s-1}} + C\|u\|^2_{H_b^{-N}},$$

proving (3.11), up to redefining B_j by multiplication by a positive constant. Recall that unless one makes sufficient a priori assumptions on the regularity of u, one actually needs to regularize, but as mentioned after (3.13), the regularizer is handled in exactly the same manner as the weight.

Now in general, with χ as before, but supported in $[0,1]$ instead of $[0,R]$, writing $\chi_R = \chi(./R)$, letting $a = a_{R,\kappa}$ to emphasize its dependence on these quantities, when R and κ are decreased, $\mathrm{supp}\, a_{R,\kappa}$ also decreases in Σ in the strong sense that $0 < R < R'$, $0 < \kappa < \kappa'$ implies that $a_{R',\kappa'}$ is elliptic on $\mathrm{supp}\, a_{R,\kappa}$ within Σ, and indeed globally if the cutoff ψ is suitably adjusted as well. Thus, if $u \in H_b^{-N}$, say, one uses first (3.11) with $s = -N+1$, and with B_j given by the proof above, so the $B_3 u$ term is a priori bounded, to conclude that $B_0 u \in \mathcal{H}^s_{b,\Gamma}$ and the estimate holds, so in particular, u is in $H_b^{-N+1/2}$ microlocally near Γ (concretely, on the elliptic set of B_0). Now one decreases κ and R by an arbitrarily small amount and

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[14] The point being that $A_+^*C_+^*C_+A_+ - \epsilon A^*\Lambda Q_+^*Q_+\Lambda^*A$ has principal symbol $c_+^2a_+^2 - \epsilon a^2\phi_+^2\lambda^2$ which can be written as the square of a real symbol for $\epsilon > 0$ small in view of the main difference in vanishing factors in the two terms being that χ_0' in a_+^2 is replaced by χ_0 in a, and thus the corresponding operator can be expressed as $\tilde{C}^*\tilde{C}$ for suitable $\tilde{C}$, modulo an element of $\Psi_b^{2s-2}(M)$, with the latter contributing to the H_b^{s-1} error term on the right hand side of (3.11).

applies (3.11) with $s = -N + 3/2$; the $B_3 u$ term is now a priori bounded by the microlocal membership of u in $H_{\mathrm{b}}^{-N+1/2}$, and one concludes that $B_0 u \in \mathcal{H}_{\mathrm{b},\Gamma}^{-N+3/2}$, so in particular u is microlocally in H_{b}^{-N+1}. Proceeding inductively, one deduces the first statement of the theorem, (3.8).

If one reverses the role of Γ_+ and Γ_- in the statement of the theorem, one simply reverses the roles of $\rho_+ = \phi_+^2 + \mathsf{M}\tau$ and ϕ_-^2 in the definition of a in (3.12). This reverses the signs of all terms on the right hand side of (3.14) whose sign mattered below, and thus the signs of the first three terms on the right hand side of (3.16), which then does not affect the rest of the argument.

In order to prove (3.9), one simply adds a factor τ^{-2r} to the definition of a in (3.12). This adds a factor τ^{-2r} to every term on the right hand side of (3.16), as well as an additional term

$$\frac{r}{2}\tau^{-2r}\tilde{\rho}^{-2s}c_\partial^2\chi_0(\rho_+ - \phi_-^2 + \kappa)^2\chi(\rho_+)^2\psi(p)^2,$$

which for $r < 0$ has the same sign as the terms whose sign was used above, and indeed can be written as the negative of a square. Thus (3.15) becomes

$$\begin{aligned}\frac{1}{4}\mathsf{H}_p(a^2) = &- c_+^2 a_+^2 - c_-^2 a_-^2 - a_\partial^2 - a_r^2 \\ &+ 2g_+ a_+ + 2g_- a_- + e + \tilde{e} + 2j_+ a_+ p + 2j_- a_- p\end{aligned} \tag{3.18}$$

with

$$a_r = \sqrt{\frac{-r}{2}}\tau^{-r}\tilde{\rho}^{-s}c_\partial\chi_0(\rho_+ - \phi_-^2 + \kappa)\chi(\rho_+)\psi(p),$$

and all other terms as above apart from the additional factor of τ^{-r} in the definition of $a_\pm$, etc. Since a_r is actually elliptic at Γ when $r \neq 0$, this proves the desired estimate (and one does not need to use the improved properties given by the Weyl calculus!).

When the role of Γ_+ and Γ_- is reversed, there is an overall sign change, and thus $r > 0$ gives the advantageous sign; the rest of the argument is unchanged. □

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