PROPAGATION OF SINGULARITIES FOR THE WAVE EQUATION ON MANIFOLDS WITH CORNERS

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Abstract. In this paper we describe the propagation of $\mathcal{C}^\infty$ and Sobolev singularities for the wave equation on $\mathcal{C}^\infty$ manifolds with corners M equipped with a Riemannian metric g. That is, for $X = M \times \mathbb{R}_t$, $P = D_t^2 - \Delta_M$, and $u \in H^1_{\mathrm{loc}}(X)$ solving $Pu = 0$ with homogeneous Dirichlet or Neumann boundary conditions, we show that $\mathrm{WF}_{\mathrm{b}}(u)$ is a union of maximally extended generalized broken bicharacteristics. This result is a $\mathcal{C}^\infty$ counterpart of Lebeau's results for the propagation of analytic singularities on real analytic manifolds with appropriately stratified boundary, [11]. Our methods rely on b-microlocal positive commutator estimates, thus providing a new proof for the propagation of singularities at hyperbolic points even if M has a smooth boundary (and no corners).

1. Introduction

In this paper we describe the propagation of $\mathcal{C}^\infty$ and Sobolev singularities for the wave equation on a manifold with corners M equipped with a smooth Riemannian metric g. We first recall the basic definitions from [12] – and refer to [20, Section 2] as a more accessible reference. Thus, a *tied (or t-) manifold with corners* X of dimension n is a paracompact Hausdorff topological space with a $\mathcal{C}^\infty$ structure with corners. The latter simply means that the local coordinate charts map into $[0,\infty)^k \times \mathbb{R}^{n-k}$ rather than into $\mathbb{R}^n$. Here k varies with the coordinate chart. We write $\partial_\ell X$ for the set of points $p \in X$ such that in any local coordinates $\phi = (\phi_1, \ldots, \phi_k, \phi_{k+1}, \ldots, \phi_n)$ near p, with k as above, precisely ℓ of the first k coordinate functions vanish at $\phi(p)$. We usually write such local coordinates as $(x_1, \ldots, x_k, y_1, \ldots, y_{n-k})$. A *boundary face* of codimension ℓ is the closure of a connected component of $\partial_\ell X$. A boundary face of codimension 1 is called a *boundary hypersurface*. A *manifold with corners* is a tied manifold with corners such that all boundary hypersurfaces are embedded submanifolds. This implies the existence of global defining functions ρ_H for each boundary hypersurface H (so $\rho_H \in \mathcal{C}^\infty(X)$, $\rho_H \geq 0$, ρ_H vanishes exactly on H and $d\rho_H \neq 0$ on H) – in each local coordinate chart intersecting H we may take one of the x_j's ($j = 1, \ldots, k$) to be ρ_H. While our results are local, and hence hold for t-manifolds with corners, it is convenient to use the embeddedness occasionally to avoid overburdening the notation. Moreover, in a given coordinate system, we often write H_j for the boundary hypersurface whose restriction to the given coordinate patch is given by $x_j = 0$, so the notation H_j depends on a particular coordinate system having been chosen (but we usually

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ignore this point). If X is a manifold with corners, X° denotes its interior, which is thus a $\mathcal{C}^\infty$ manifold (without boundary).

Returning to the wave equation, let M be a manifold with corners equipped with a smooth Riemannian metric g. Let $\Delta = \Delta_g$ be the positive Laplacian of g, let $X = M \times \mathbb{R}_t$, $P = D_t^2 - \Delta$, and consider the Dirichlet boundary condition for P:

$$Pu = 0,\ u|_{\partial X} = 0,$$

with the boundary condition meaning more precisely that $u \in H^1_{0,\mathrm{loc}}(X)$. Here $H^1_0(X)$ is the completion of $\dot{\mathcal{C}}^\infty_c(X)$ (the vector space of $\mathcal{C}^\infty$ functions of compact support on X, vanishing with all derivatives at ∂X) with respect to $\|u\|^2_{H^1(X)} = \|du\|_{L^2(X)} + \|u\|_{L^2(X)}$, $L^2(X) = L^2(X, dg\,dt)$, and $H^1_{0,\mathrm{loc}}(X)$ is its localized version, i.e. $u \in H^1_0(X)$ if for all $\phi \in \mathcal{C}^\infty_c(X)$, $\phi u \in H^1_0(X)$. At the end of the introduction we also consider Neumann boundary conditions.

The statement of the propagation of singularities of solutions has two additional ingredients: locating singularities of a distribution, as captured by the wave front set, and describing the curves along which they propagate, namely the bicharacteristics. Both of these are closely related to an appropropriate notion of phase space, in which both the wave front set and the bicharacteristics are located. On manifolds without boundary, this phase space is the standard cotangent bundle. In the presence of boundaries the phase space is the b-cotangent bundle, ${}^{\mathrm{b}}T^*X$, ('b' stands for boundary) which we now briefly describe following [19], which mostly deals with the $\mathcal{C}^\infty$ boundary case, and especially [20].

Thus, $\mathcal{V}_{\mathrm{b}}(X)$ is, by definition, the Lie algebra of $\mathcal{C}^\infty$ vector fields on X tangent to every boundary face of X. In local coordinates as above, such vector fields have the form

$$\sum a_j(x,y) x_j \partial_{x_j} + \sum_j b_j(x,y)\partial_{y_j}$$

wit a_j, b_j smooth. Correspondingly, $\mathcal{V}_{\mathrm{b}}(X)$ is the set of all $\mathcal{C}^\infty$ sections of a vector bundle ${}^{\mathrm{b}}TX$ over X: locally $x_j\partial_{x_j}$ and ∂_{y_j} generate $\mathcal{V}_{\mathrm{b}}(X)$ (over $\mathcal{C}^\infty(X)$), and thus (x, y, a, b) are local coordinates on ${}^{\mathrm{b}}TX$.

The dual bundle of ${}^{\mathrm{b}}TX$ is ${}^{\mathrm{b}}T^*X$; this is the phase space in our setting. Sections of these have the form

$$\sum \sigma_j(x,y)\,\frac{dx_j}{x_j} + \sum_j \zeta_j(x,y)\,dy_j, \tag{1.1}$$

and correspondingly (x, y, σ, ζ) are local coordinates on it. Let o denote the zero section of ${}^{\mathrm{b}}T^*X$ (as well as other related vector bundles below). Then ${}^{\mathrm{b}}T^*X \setminus o$ is equipped with an $\mathbb{R}^+$-action (fiberwise multiplication) which has no fixed points. It is often natural to take the quotient with the $\mathbb{R}^+$-action, and work on the b-cosphere bundle, ${}^{\mathrm{b}}S^*X$.

The differential operator algebra generated by $\mathcal{V}_{\mathrm{b}}(X)$ is denoted by $\mathrm{Diff}_{\mathrm{b}}(X)$, and its microlocalization is $\Psi_{\mathrm{b}}(X)$, the algebra of b-, or totally characteristic, pseudodifferential operators. For $A \in \Psi^m_{\mathrm{b}}(X)$, $\sigma_{b,m}(A)$ is a homogeneous degree m function on ${}^{\mathrm{b}}T^*X \setminus o$. Since X is not compact, even if M is, we always understand that $\Psi^m_{\mathrm{b}}(X)$ stands for properly supported ps.d.o's, so its elements define continuous maps $\dot{\mathcal{C}}^\infty(X) \to \dot{\mathcal{C}}^\infty(X)$ as well as $\mathcal{C}^{-\infty}(X) \to \mathcal{C}^{-\infty}(X)$. Here $\dot{\mathcal{C}}^\infty(X)$ denotes the subspace of $\mathcal{C}^\infty(X)$ consisting of functions vanishing at ∂X with all derivatives,

$\dot{\mathcal{C}}^\infty_c(X)$ the subspace of $\dot{\mathcal{C}}^\infty(X)$ consisting of functions of compact support. Moreover, $\mathcal{C}^{-\infty}(X)$ is the dual space of $\dot{\mathcal{C}}^\infty_c(X)$; we may call its elements 'tempered' or 'extendible' distributions. Thus, $\mathcal{C}^\infty_c(X^\circ) \subset \dot{\mathcal{C}}^\infty(X)$ and $\mathcal{C}^{-\infty}(X) \subset \mathcal{C}^{-\infty}(X^\circ)$.

We are now ready to define the wave front set $\mathrm{WF}_{\mathrm{b}}(u)$ for $u \in H^1_{\mathrm{loc}}(X)$. This measures if u has additional regularity, locally in ${}^{\mathrm{b}}T^*X$, relative to H^1. For $u \in H^1_{\mathrm{loc}}(X)$, $q \in {}^{\mathrm{b}}T^*X \setminus o$, $m \geq 0$, we say that $q \notin \mathrm{WF}^{1,m}_{\mathrm{b}}(u)$ if there is $A \in \Psi^m_{\mathrm{b}}(X)$ such that $\sigma_{b,m}(A)(q) \neq 0$ and $Au \in H^1(X)$. Since compactly supported elements of $\Psi^0_{\mathrm{b}}(X)$ preserve $H^1_{\mathrm{loc}}(X)$, it follows that for $u \in H^1_{\mathrm{loc}}(X)$, $\mathrm{WF}^{1,0}_{\mathrm{b}}(u) = \emptyset$. For any m, $\mathrm{WF}^{1,m}_{\mathrm{b}}(u)$ is a conic subset of ${}^{\mathrm{b}}T^*X \setminus o$; hence it is natural to identify it with a subset of ${}^{\mathrm{b}}S^*X$. Its intersection with ${}^{\mathrm{b}}T^*_{X^\circ}X \setminus o$, which can be naturally identified with $T^*X^\circ \setminus o$, is $\mathrm{WF}^{m+1}(u)$. Thus, in the interior of X, $\mathrm{WF}^{1,m}_{\mathrm{b}}(u)$ measures if u is microlocally in H^{m+1}. The main result of this paper, stated at the end of this section, is that for $u \in H^1_0(X)$ with $Pu = 0$, $\mathrm{WF}^{1,m}_{\mathrm{b}}(u)$ is a union of maximally extended generalized broken bicharacteristics, which are defined below. In fact, the requirement $u \in H^1_0(X)$ can be relaxed and m can be allowed to be negative, see Definitions 3.15-3.17. We also remark that for such u, the $H^1(X)$-based b-wave front set, $\mathrm{WF}^{1,m}_{\mathrm{b}}(u)$, could be replaced by an $L^2(X)$-based b-wave front set, see Lemma 6.1. In addition, our methods apply, a fortiori, for elliptic problems such as Δ_g on (M, g), e.g. showing that $u \in H^1_{0,\mathrm{loc}}(M)$ and $(\Delta_g - \lambda)u = 0$ imply $u \in H^{1,\infty}_{\mathrm{b},\mathrm{loc}}(M)$, so u is conormal – see the end of Section 4.

This propagation result is the $\mathcal{C}^\infty$ (and Sobolev space) analogue of Lebeau's result [11] for analytic singularities of u when M and g are real analytic. Thus, the geometry is similar in the two settings, but the analytic techniques are rather different: Lebeau uses complex scaling and the analytic wave front set of the extension of u as 0 to a neighborhood of X (in an extension $\tilde{X}$ of the manifold X), while we use positive commutator estimates and b-microlocalization relative to the form domain of the Laplacian. It should be kept in mind though that positive commutator estimates can often be thought of as an infinitesimal version of complex scaling (if complex scaling is available at all), although this is more of a moral than a technical statement, for the techiniques involved in working infinitesimally are quite different from what one can do if one has room to deform contours of integration! In fact, our microlocalization techniques, especially the positive commutator constructions, are very closely related to the methods used in N-body scattering, [24], to prove the propagation of singularities (meaning microlocal lack of decay at infinity) there. Although Lebeau allows more general singularities than corners for X, provided that X sits in a real analytic manifold $\tilde{X}$ with g extending to $\tilde{X}$, we expect to generalize our results to settings where no analogous $\mathcal{C}^\infty$ extension is available, see the remarks at the end of the introduction.

We now describe the setup in more detail so that our main theorem can be stated in a precise fashion. Let F_i, $i \in I$, be the closed boundary faces of M (including M), $\mathcal{F}_i = F_i \times \mathbb{R}$, $\mathcal{F}_{i,\mathrm{reg}}$ the interior ('regular part') of $\mathcal{F}_i$. Note that for each $p \in X$, there is a unique i such that $p \in \mathcal{F}_{i,\mathrm{reg}}$. Although we work on both M and X, and it is usually clear which one we mean even in the local coordinate discussions, to make matters clear we write local coordinates on M, as in the introduction, as (x, y) (with $x = (x_1, \ldots, x_k)$, $y = (y_1, \ldots, y_{\dim M - k})$), with $x_j \geq 0$ $(j = 1, \ldots, k)$ on M, and then local coordinates on X, induced by the product $M \times \mathbb{R}_t$, as $(x, \bar{y})$, $\bar{y} = (y, t)$ (so X is given by $x_j \geq 0$, $j = 1, \ldots, k$).

Let $p \in \partial X$, and let $\mathcal{F}_i$ be the closed face of X with the smallest dimension that contains p, so $p \in \mathcal{F}_{i,\mathrm{reg}}$. Then we may choose local coordinates $(x, y, t) = (x, \bar{y})$ near p in which $\mathcal{F}_i$ is defined by $x_1 = \ldots = x_k = 0$, and the other boundary faces through p are given by the vanishing of a subset of the collection $x_1, \ldots, x_k$ of functions – in particular, the k boundary hypersurfaces H_j through p are locally given by $x_j = 0$ for $j = 1, \ldots, k$. (This may require shrinking a given coordinate chart $(x', \bar{y}')$ that contains p so that the x'_j that do not vanish identically on $\mathcal{F}_i$ do not vanish at all on the smaller chart, and can be relabelled as one of the coordinates y_ℓ.)

Now, there is a natural non-injective 'inclusion' $\pi : T^*X \to {}^{\mathrm{b}}T^*X$ induced by identifying ${}^{\mathrm{b}}TX$ with TX (and hence also their dual bundles) with each other in the interior of X, where the condition on tangency to boundary faces is vacuous. In view of (1.1), in the canonical local coordinates $(x, \bar{y}, \xi, \bar{\zeta})$ on T^*X (so one forms are $\sum \xi_j \, dx_j + \sum \bar{\zeta}_j \, d\bar{y}_j$), and canonical local coordinates $(x, \bar{y}, \sigma, \bar{\zeta})$ on ${}^{\mathrm{b}}T^*X$, π takes the form

$$\pi(x, \bar{y}, \xi, \bar{\zeta}) = (x, \bar{y}, x\xi, \bar{\zeta}), \text{ with } x\xi = (x_1\xi_1, \ldots, x_k\xi_k).$$

Thus, π is a $\mathcal{C}^\infty$ map, but at the boundary of X, it is not a local diffeomorphism. Moreover, the range of π over the interior of a face $\mathcal{F}_i$ lies in $T^*\mathcal{F}_i$ (which is well-defined as a subspace of ${}^{\mathrm{b}}T^*X$) while its kernel is $N^*\mathcal{F}_i$, the conormal bundle of $\mathcal{F}_i$ in X. In local coordinates as above, in which $\mathcal{F}_i$ is given by $x = 0$, the range $T^*\mathcal{F}_i$ over $\mathcal{F}_i$ is given by $x = 0$, $\sigma = 0$ (i.e. by $x_1 = \ldots = x_k = 0$, $\sigma_1 = \ldots = \sigma_k = 0$), while the kernel $N^*\mathcal{F}_i$ is given by $x = 0$, $\bar{\zeta} = 0$. Then we define the compressed b-cotangent bundle ${}^{\mathrm{b}}\dot{T}^*X$ to be the range of π:

$${}^{\mathrm{b}}\dot{T}^*X = \pi(T^*X) = \cup_{i \in I} T^*\mathcal{F}_{i,\mathrm{reg}} \subset {}^{\mathrm{b}}T^*X.$$

We write o for the 'zero section' of ${}^{\mathrm{b}}\dot{T}^*X$ as well, so

$${}^{\mathrm{b}}\dot{T}^*X \setminus o = \cup_{i \in I} T^*\mathcal{F}_{i,\mathrm{reg}} \setminus o,$$

and then π restricts to a map

$$T^*X \setminus \cup_i N^*\mathcal{F}_i \to {}^{\mathrm{b}}\dot{T}^*X \setminus o.$$

Now, the characteristic set $\mathrm{Char}(P) \subset T^*X \setminus o$ of P is defined by $p^{-1}(\{0\})$, where $p \in \mathcal{C}^\infty(T^*X \setminus o)$ is the principal symbol of P, which is homogeneous degree 2 on $T^*X \setminus o$. Notice that $\mathrm{Char}(P) \cap N^*\mathcal{F}_i = \emptyset$ for all i, i.e. the boundary faces are all non-characteristic for P. Thus, $\pi(\mathrm{Char}(P)) \subset {}^{\mathrm{b}}\dot{T}^*X \setminus o$. We define the elliptic, glancing and hyperbolic sets by

$$\begin{aligned}
\mathcal{E} &= \{q \in {}^{\mathrm{b}}\dot{T}^*X \setminus o : \ \pi^{-1}(q) \cap \mathrm{Char}(P) = \emptyset\},\\
\mathcal{G} &= \{q \in {}^{\mathrm{b}}\dot{T}^*X \setminus o : \ \mathrm{Card}(\pi^{-1}(q) \cap \mathrm{Char}(P)) = 1\},\\
\mathcal{H} &= \{q \in {}^{\mathrm{b}}\dot{T}^*X \setminus o : \ \mathrm{Card}(\pi^{-1}(q) \cap \mathrm{Char}(P)) \geq 2\},
\end{aligned}$$

with Card denoting the cardinality of a set; each of these is a conic subset of ${}^{\mathrm{b}}\dot{T}^*X \setminus o$. Note that in T^*X°, π is the identity map, so every point $q \in T^*X^\circ$ is either in $\mathcal{E}$ or $\mathcal{G}$ depending on whether $q \notin \mathrm{Char}(P)$ or $q \in \mathrm{Char}(P)$.

Local coordinates on the base induce local coordinates on the cotangent bundle, namely $(x, y, t, \xi, \zeta, \tau)$ on T^*X near $\pi^{-1}(q)$, $q \in T^*\mathcal{F}_{i,\mathrm{reg}}$, and corresponding coordinates (y, t, ζ, τ) on a neighborhood $\mathcal{U}$ of q in $T^*\mathcal{F}_{i,\mathrm{reg}}$. The metric function on

T^*M has the form

$$g(x,y,\xi,\zeta)=\sum_{i,j}A_{ij}(x,y)\xi_i\xi_j+\sum_{i,j}2C_{ij}(x,y)\xi_i\zeta_j+\sum_{i,j}B_{ij}(x,y)\zeta_i\zeta_j$$

with A,B,C smooth. Moreover, these coordinates can be chosen (i.e. the y_j can be adjusted) so that $C(0,y)=0$. Thus,

$$p|_{x=0}=\tau^2-\xi\cdot A(y)\xi-\zeta\cdot B(y)\zeta,$$

with A, B positive definite matrices depending smoothly on y, so

$$\begin{aligned}\mathcal{E}\cap\mathcal{U}&=\{(y,t,\zeta,\tau):\ \tau^2<\zeta\cdot B(y)\zeta,\ (\zeta,\tau)\neq 0\},\\ \mathcal{G}\cap\mathcal{U}&=\{(y,t,\zeta,\tau):\ \tau^2=\zeta\cdot B(y)\zeta,\ (\zeta,\tau)\neq 0\},\\ \mathcal{H}\cap\mathcal{U}&=\{(y,t,\zeta,\tau):\ \tau^2>\zeta\cdot B(y)\zeta,\ (\zeta,\tau)\neq 0\}.\end{aligned}$$

The compressed characteristic set is

$$\dot\Sigma=\pi(\mathrm{Char}(P))=\mathcal{G}\cup\mathcal{H},$$

and

$$\hat\pi:\mathrm{Char}(P)\to\dot\Sigma$$

is the restriction of π to $\mathrm{Char}(P)$. Then $\dot\Sigma$ has the subspace topology of ${}^{\mathrm{b}}T^*X$, and it can also be topologized by $\hat\pi$, i.e. requiring that $C\subset\dot\Sigma$ is closed (or open) if and only if $\hat\pi^{-1}(C)$ is closed (or open). These two topologies are equivalent, though the former is simpler in the present setting – e.g. it is immediate that $\dot\Sigma$ is metrizable. Lebeau [11] (following Melrose's original approach in the $\mathcal{C}^\infty$ boundary setting, see [17]) uses the latter; in extensions of the present work, to allow e.g. iterated conic singularities, that approach will be needed. Again, an analogous situation arises in N-body scattering, though that is in many respects more complicated if some subsystems have bound states [24, 25].

We are now ready to define generalized broken bicharacteristics, essentially following Lebeau [11]. We say that a function f on $T^*X\backslash o$ is π-invariant if $f(q)=f(q')$ whenever $\pi(q)=\pi(q')$. In this case f induces a function f_π on ${}^{\mathrm{b}}\dot{T}^*X$ which satisfies $f=f_\pi\circ\pi$. Moreover, if f is continuous, then so is f_π. Notice that if $f=\pi^*f_0$, $f_0\in\mathcal{C}^\infty({}^{\mathrm{b}}T^*X)$, then $f\in\mathcal{C}^\infty(T^*X)$ is certainly π-invariant.

Definition 1.1. A generalized broken bicharacteristic of P is a continuous map $\gamma:I\to\dot\Sigma$, where $I\subset\mathbb{R}$ is an interval, satisfying the following requirements:

(i) If $q_0=\gamma(t_0)\in\mathcal{G}$ then for all π-invariant functions $f\in\mathcal{C}^\infty(T^*X)$,

$$\frac{d}{dt}(f_\pi\circ\gamma)(t_0)=H_pf(\tilde q_0),\ \tilde q_0=\hat\pi^{-1}(q_0). \tag{1.2}$$

(ii) If $q_0=\gamma(t_0)\in\mathcal{H}\cap T^*\mathcal{F}_{i,\mathrm{reg}}$ then there exists $\epsilon>0$ such that

$$t\in I,\ 0<|t-t_0|<\epsilon\Rightarrow\gamma(t)\notin T^*\mathcal{F}_{i,\mathrm{reg}}. \tag{1.3}$$

(iii) If $q_0=\gamma(t_0)\in\mathcal{G}\cap T^*\mathcal{F}_{i,\mathrm{reg}}$, and $\mathcal{F}_i$ is a boundary hypersurface (i.e. has codimension 1), then in a neighborhood of t_0, γ is a generalized broken bicharacteristic in the sense of Melrose-Sjöstrand [13], see also [4, Definition 24.3.7].

Remark 1.2. Note that for $q_0 \in \mathcal{G}$, $\hat{\pi}^{-1}(\{q_0\})$ consists of a single point, so (1.2) makes sense. Moreover, (iii) implies (i) if q_0 is in a boundary hypersurface, but it is stronger at diffractive points, see [4, Section 24.3]. The propagation of analytic singularities, as in Lebeau's case, does not distinguish between gliding and diffractive points, hence (iii) can be dropped to define what we may call analytic generalized broken bicharacteristics. It is an interesting question whether in the $\mathcal{C}^\infty$ setting there are also analogous diffractive phenomena at higher codimension boundary faces, i.e. whether the following theorem can be strengthened at certain points.

We remark also that there is an equivalent definition (presented in lecture notes about the present work, see [26]), which is more directly motivated by microlocal analysis and which also works in other settings such as N-body scattering in the presence of bound states.

Our main result is:

Theorem. *(See Corollary 8.4.) Suppose that* $Pu = 0$, $u \in H^1_{0,loc}(X)$. *Then* $\mathrm{WF}_b^{1,\infty}(u) \subset \dot{\Sigma}$, *and it is a union of maximally extended generalized broken bicharacteristics of* P *in* $\dot{\Sigma}$.

The analogue of this theorem was proved in the real analytic setting by Lebeau [11], and in the $\mathcal{C}^\infty$ setting with $\mathcal{C}^\infty$ boundaries (and no corners) by Melrose, Sjöstrand and Taylor [13, 14, 22]. In addition, Ivrii [8] has obtained propagation results for systems. Moreover, a special case with codimension 2 corners in $\mathbb{R}^2$ had been considered by P. Gérard and Lebeau [3] in the real analytic setting, and by Ivrii [5] in the smooth setting. It should be mentioned that due to its relevance, this problem has a long history, and has been studied extensively by Keller in the 1940s and 1950s in various special settings, see e.g. [1, 10]. The present work (and ongoing projects continuing it, especially joint work with Melrose and Wunsch [15], see also [2, 16]) can be considered a justification of Keller's work in the general geometric setting (curved edges, variable coefficient metrics, etc).

A more precise version of this theorem, with microlocal assumptions on Pu, is stated in Theorem 8.1. In particular, one can allow $Pu \in \mathcal{C}^\infty(X)$, which immediately implies that the theorem holds for solutions of the wave equation with inhomogeneous $\mathcal{C}^\infty$ Dirichlet boundary conditions that match across the boundary hyperfaces, see Remark 8.2. In addition, this theorem generalizes to the wave operator with Neumann boundary conditions, which need to be interpreted in terms of the quadratic form of P (i.e. the Dirichlet form). That is, if $u \in H^1_{\mathrm{loc}}(X)$ satisfies

$$\langle d_M u, d_M v\rangle_X - \langle \partial_t u, \partial_t v\rangle_X = 0$$

for all $v \in H^1_c(X)$, then $\mathrm{WF}_{\mathrm{b}}^{1,\infty}(u) \subset \dot{\Sigma}$, and it is a union of maximally extended generalized broken bicharacteristics of P in $\dot{\Sigma}$. In fact, the proof of the theorem for Dirichlet boundary conditions also utilizes the quadratic form of P. It is slightly simpler in presentation only to the extent that one has more flexibility to integrate by parts, etc., but in the end the proof for Neumann boundary conditions simply requires a slightly less conceptual (in terms of the traditions of microlocal analysis) reorganization, e.g. not using commutators $[P, A]$ directly, but commuting A through the exterior derivative d_M and ∂_t directly.

It is expected that these results will generalize to iterated edge-type structures (under suitable hypotheses), whose simplest example is given by (isolated) conic points, recently analyzed by Melrose and Wunsch [16], extending the product cone analysis of Cheeger and Taylor [2]. This is subject of an ongoing project with Richard Melrose and Jared Wunsch [15].

It is an interesting question whether this propagation theorem can be improved in the sense that, under certain 'non-focussing' assumptions for a solution u of the wave equation, if a bicharacteristic segment carrying a singularity of u hits a corner, then the reflected singularity is weaker along 'non-geometrically related' generalized broken bicharacteristics continuing the aforementioned segment than along 'geometrically related' ones – roughly, 'geometrically related' continuations should be limits of bicharacteristics just missing the corner. In the setting of (isolated) conic points, such a result was obtained by Cheeger, Taylor, Melrose and Wunsch [2, 16]. While the analogous result (including its precise statement) for manifolds with corners is still some time away, significant progress has been made, since this original version of this manuscript was written, on analyzing edge-type metrics (on manifolds with boundaries) in the project [15]. The outline of these results, including a discussion how it relates to the problem under consideration here, is written up in the lecture notes of the author on the present paper [26].

To make it clear what the main theorem states, we remark that the propagation statement means that if u solves $Pu = 0$ (with, say, Dirichlet boundary condition), and $q \in {}^{\mathrm{b}}T^*_{\partial X}X \setminus o$ is such that u has no singularities on bicharacteristics entering q (say, from the past), then we conclude that u has no singularities at q, in the sense that $q \notin \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u)$, i.e. we only gain b-derivatives (or totally characteristic derivatives) microlocally. In particular, even if $\mathrm{WF}_{\mathrm{b}}^{1,\infty}(u)$ is empty, we can only conclude that u is conormal to the boundary, in the precise sense that $V_1 \dots V_k u \in H^1_{\mathrm{loc}}(X)$ for any $V_1, \dots, V_k \in \mathcal{V}_{\mathrm{b}}(X)$, and not that $u \in H^k_{\mathrm{loc}}(X)$ for all k. Indeed, the latter cannot be expected to hold, as can be seen by considering e.g. the wave equation (or even elliptic equations) in 2-dimensional conic sectors.

This already illustrates that from a technical point of view a major challange is to combine two differential (and pseudodifferential) algebras: $\mathrm{Diff}(X)$ and $\mathrm{Diff}_{\mathrm{b}}(X)$ (or $\Psi_{\mathrm{b}}(X)$). The wave operator P lies in $\mathrm{Diff}(X)$, but microlocalization needs to take place in $\Psi_{\mathrm{b}}(X)$: if $\Psi(\tilde{X})$ is the algebra of usual pseudodifferential operators on an extension $\tilde{X}$ of X, its elements do not even act on $\mathcal{C}^\infty(X)$: see [4, Section 18.2] when X has a smooth boundary (and no corners). In addition, one needs an algebra whose elements A respect the boundary conditions, so e.g. $Au|_{\partial X}$ depends only on $u|_{\partial X}$ – this is exactly the origin of the algebra of totally characteristic pseudodifferential operators, denoted by $\Psi_{\mathrm{b}}(X)$, in the $\mathcal{C}^\infty$ boundary setting [18]. The interaction of these two algebras also explains why we prove even microlocal elliptic regularity via the quadratic form of P (the Dirichlet form), rather than by standard arguments, valid if one studies microlocal elliptic regularity for an element of an algebra (such as $\Psi_{\mathrm{b}}(X)$) with respect to the same algebra.

The ideas of the positive commutator estimates, in particular the construction of the commutants, are very similar to those arising in the proof of the propagation of singularities in N-body scattering in previous works of the author – the wave equation corresponds to the relatively simple scenario there when no proper subsystems have bound states [24]. Indeed, the author has indicated many times in lectures that there is a close connection between these two problems, and it is a pleasure to finally spell out in detail how the N-body methods can be adapted to the present setting.

The organization of the paper is as follows. In Section 2 we recall basic facts about $\Psi_{\mathrm{b}}(X)$ and analyze its commutation properties with $\mathrm{Diff}(X)$. In Section 3 we describe the mapping properties of $\Psi_{\mathrm{b}}(X)$ on $H^1(X)$-based spaces. We also define and discuss the b-wave front set based on $H^1(X)$ there. The following section is devoted to the elliptic estimates for the wave equation. These are obtained from the microlocal positivity of the Dirichlet form, which implies in particular that in this region commutators are negligible for our purposes. In Section 5 we describe basic properties of bicharacteristics, mostly relying on Lebeau's work [11]. In Sections 6 and 7, we prove propagation estimates at hyperbolic, resp. glancing, points, by positive commutator arguments. Similar arguments were used by Melrose and Sjöstrand [13] for the analysis of propagation at glancing points for manifolds with smooth boundaries. In Section 8 these results are combined to prove our main theorems. The arguments presented there are very close to those of Melrose, Sjöstrand and Lebeau.

Here we point out that Ivrii [8, 6, 7, 9] also used microlocal energy estimates to obtain propagation results of a different flavor for symmetric systems in the smooth boundary setting, including at hyperbolic points. Roughly, Ivrii's results give conditions for hypersurfaces Σ through a point q_0 under which the following conclusion holds: the point q_0 is absent from the wave front set of a solution provided that, in a neighborhood of q_0, one side of Σ is absent from the wave front set – with further restrictions on the hypersurface in the presence of smooth boundaries. In some circumstances, using other known results, Ivrii could strengthen the conclusion further.

Since the changes for Neumann boundary conditions are minor, and the arguments for Dirichlet boundary conditions can be stated in a form closer to those found in classical microlocal analysis (essentially, in the Neumann case one has to pay a price for integrating by parts, so one needs to present the proofs in an appropriately rearranged, and less transparent, form) the proofs in the body of the paper are primarily written for Dirichlet boundary conditions, and the required changes are pointed out at the end of the various sections.

In addition, the hypotheses of the propagation of singularities theorem can be relaxed to $u \in H^{1,m}_{\mathrm{b},0,\mathrm{loc}}(X)$, $m \leq 0$, defined in Definition 3.15. Since this simply requires replacing the $H^1(X)$ norms by the $H^{1,m}_{\mathrm{b}}$ norms (which are only locally well defined), we suppress this point except in the statement of the final result, to avoid overburdening the notation. No changes are required in the argument to deal with this more general case. See Remark 8.3 for more details.

To give the reader a guide as to what the real novelty is, Sections 2-3 should be considered as variations on a well-developed theme. While some of the features of microlocal analysis, especially wave front sets, is not discussed on manifolds with corners elsewhere, the modifications needed are essentially trivial (cf. [4, Chapter 18]). A slight novelty is using $H^1(X)$ as the point of reference for the b-wave front sets (rather than simply weighted L^2 spaces), which is very useful later in the paper, but again only demands minimal changes to standard arguments. The discussions of bicharacteristics in Section 5 essentially quotes Lebeau's paper [11, Section III]. Moreover, given the results of Sections 4, 6 and 7, the proof of propagation of singularities in Section 8 is standard, essentially due to Melrose and Sjöstrand [14, Section 3]. Indeed, as presented by Lebeau [11, Proposition VII.1], basically no changes are necessary at all in this proof.

The novelty is thus the use of the Dirichlet form (hence the H^1-based wave front set) for the proof of both the elliptic and hyperbolic/glancing estimates, and the systematic use of positive commutator estimates in the hyperbolic/glancing regions, with the commutants arising from an intrinsic pseudodifferential operator algebra, $\Psi_{\mathrm{b}}(X)$. This approach is quite robust, hence significant extensions of the results can be expected, as was already indicated.

I would like to thank Richard Melrose for his interest in this project, for reading, and thereby improving, parts of the paper, and for numerous helpful and stimulating discussions, especially for the wave equation on forms. While this topic did not become a part of the paper, it did play a role in the presentation of the arguments here. I am also grateful to Jared Wunsch for helpful discussions and his willingness to read large parts of the manuscript at the early stages, when the background material was still mostly absent; his help significantly improved the presentation here. I would also like to thank Rafe Mazzeo for his continuing interest in this project and for his patience when I tried to explain him the main ideas in the early days of this project, and Victor Ivrii for his interest in, and his support for, this work. At last, but not least, I am very grateful to the anonymous referee for a thorough reading of the manuscript and for many helpful suggestions.

2. Interaction of $\mathrm{Diff}(X)$ with the b-calculus

One of the main technical issues in proving our main theorem is that unless $\partial X = \emptyset$, the wave operator P is *not* a b-differential operator: $P \notin \mathrm{Diff}^2_{\mathrm{b}}(X)$. In this section we describe the basic properties of how $\mathrm{Diff}^k(X)$, which includes P for $k = 2$, interacts with $\Psi_{\mathrm{b}}(X)$. We first recall though that for $p \in \mathcal{F}_{i,\mathrm{reg}}$, local coordinates in ${}^{\mathrm{b}}T^*X$ over a neighborhood of p are given by $(x, y, t, \sigma, \zeta, \tau)$ with $\sigma_j = x_j \xi_j$. Thus, the map π in local coordinates is $(x, y, t, \xi, \zeta, \tau) \mapsto (x, y, t, x\xi, \zeta, \tau)$, where by $x\xi$ we mean the vector $(x_1\xi_1, \ldots, x_k\xi_k)$.

In fact, in this section y and t play a completely analogous role, hence there is no need to distinguish them at all. The difference will only arise when we start studying the wave operator P in Section 4. Thus, we let $\bar{y} = (y, t)$ and $\bar{\zeta} = (\zeta, \tau)$ here to simplify the notation.

We briefly recall basic properties of the set of 'classical' (one-step polyhomogeneous, in the sense that the full symbols are such on the fibers of ${}^{\mathrm{b}}T^*X$) pseudodifferential operators $\Psi_{\mathrm{b}}(X) = \cup_m \Psi^m_{\mathrm{b}}(X)$ and the set of standard (conormal) b-pseudodifferential operators, $\Psi_{\mathrm{bc}}(X) = \cup_m \Psi^m_{\mathrm{bc}}(X)$. The difference between these two classes is in terms of the behavior of their (full) symbols at fiber-infinity of ${}^{\mathrm{b}}T^*X$: elements of $\Psi_{\mathrm{bc}}(X)$ have full symbols that satisfy the usual symbol estimates, while elements of $\Psi_{\mathrm{b}}(X)$ have in addition an asymptotic expansion in terms of homogeneous functions, so $\Psi^m_{\mathrm{b}}(X) \subset \Psi^m_{\mathrm{bc}}(X)$. Conceptually, these are best defined via the Schwartz kernel of $A \in \Psi^m_{\mathrm{bc}}(X)$ in terms of a certain blow-up X^2_{b} of $X \times X$, see [20] – the Schwartz kernel is conormal to the lift $\mathrm{diag}_{\mathrm{b}}$ of the diagonal of X^2 to X^2_{b} with infinite order vanishing on all boundary faces of X^2_{b} which are disjoint from $\mathrm{diag}_{\mathrm{b}}$. Modulo $\Psi^{-\infty}_{\mathrm{b}}(X)$, however, the explicit quantization map we give below describes $\Psi^m_{\mathrm{bc}}(X)$ and $\Psi^m_{\mathrm{b}}(X)$. Here $\Psi^{-\infty}_{\mathrm{bc}}(X) = \Psi^{-\infty}_{\mathrm{b}}(X) = \cap_m \Psi^m_{\mathrm{bc}}(X) = \cap_m \Psi^m_{\mathrm{b}}(X)$ is the ideal of smoothing operators. The topology of $\Psi_{\mathrm{bc}}(X)$ is given in terms the conormal seminorms of the Schwartz kernel K of its elements; these seminorms can be stated in terms of the Besov space norms of $L_1 L_2 \ldots L_k K$ as k runs over non-negative integers, and the L_j

over first order differential operators tangential to $\mathrm{diag}_{\mathrm{b}}$, see [4, Definition 18.2.6]. Recall in particular that these seminorms are (locally) equivalent to the $\mathcal{C}^\infty$ seminorms away from the lifted diagonal $\mathrm{diag}_{\mathrm{b}}$.

There is a principal symbol map $\sigma_{b,m} : \Psi^m_{\mathrm{bc}}(X) \to S^m({}^{\mathrm{b}}T^*X)/S^{m-1}({}^{\mathrm{b}}T^*X)$; here, for a vector bundle E over X, $S^k(E)$ denotes the set of symbols of order k on E (i.e. these are symbols in the fibers of E, smoothly varying over X). Its restriction to $\Psi^m_{\mathrm{b}}(X)$ can be re-interpreted as a map $\sigma_{b,m} : \Psi^m_{\mathrm{b}}(X) \to \mathcal{C}^\infty({}^{\mathrm{b}}T^*X\setminus o)$ with values in homogeneous functions of degree m; the range can of course also be identified with $\mathcal{C}^\infty({}^{\mathrm{b}}S^*X)$ if $m = 0$ (and with sections of a line bundle over ${}^{\mathrm{b}}S^*X$ in general). There is a short exact sequence

$$0 \longrightarrow \Psi^{m-1}_{\mathrm{b}}(X) \longrightarrow \Psi^m_{\mathrm{b}}(X) \longrightarrow S^m({}^{\mathrm{b}}T^*X)/S^{m-1}({}^{\mathrm{b}}T^*X) \longrightarrow 0$$

as usual; the last non-trivial map is $\sigma_{b,m}$. There are also quantization maps (which depend on various choices) $q = q_m : S^m({}^{\mathrm{b}}T^*X) \to \Psi^m_{\mathrm{bc}}(X)$, which restrict to $q : S^m_{\mathrm{cl}}({}^{\mathrm{b}}T^*X) \to \Psi^m_{\mathrm{b}}(X)$, cl denoting classical symbols, and $\sigma_{b,m} \circ q_m$ is the quotient map $S^m \to S^m/S^{m-1}$. For instance, over a local coordinate chart U as above, with a supported in ${}^{\mathrm{b}}T^*_KX$, $K \subset U$ compact, we may take, with $n = \dim X$,

$$\begin{aligned} &q(a)u(x,\bar{y}) \\ &\qquad = (2\pi)^{-n} \int e^{i(x-x')\cdot\xi + (\bar{y}-\bar{y}')\cdot\bar{\zeta}} \phi(\frac{x-x'}{x}) a(x,y,x\xi,\bar{\zeta}) u(x',\bar{y}')\,dx'\,d\bar{y}'\,d\xi\,d\zeta, \end{aligned} \tag{2.1}$$

understood as an oscillatory integral, where $\phi \in \mathcal{C}^\infty_c((-1/2,1/2)^k)$ is identically 1 near 0 and $\frac{x-x'}{x} = (\frac{x_1-x_1'}{x_1},\ldots,\frac{x_k-x_k'}{x_k})$, and the integral in x' is over $[0,\infty)^k$. Here the role of ϕ is to ensure the infinite order vanishing at the boundary hypersurfaces of X^2_{b} disjoint from $\mathrm{diag}_{\mathrm{b}}$; it is irrelevant as far as the behavior of Schwartz kernels near the diagonal is concerned (it is identically 1 there). This can be extended to a global map via a partition of unity, as usual. Locally, for $q(a)$, $\mathrm{supp}\, a \subset {}^{\mathrm{b}}T^*_KX$ as above, the conormal seminorms of the Schwartz kernel of $q(a)$ (i.e. the Besov space norms described above) can be bounded in terms of the symbol seminorms of a, see the beginning of [4, Section 18.2], and conversely. Moreover, any $A \in \Psi_{\mathrm{bc}}(X)$ with *properly supported Schwartz kernel* defines continuous linear maps $A : \dot{\mathcal{C}}^\infty(X) \to \dot{\mathcal{C}}^\infty(X)$, $A : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$.

Remark 2.1. We often do not state it below, but in general most pseudodifferential operators have compact support in this paper. Sometimes we use properly supported ps.d.o's, only for not having to state precise support conditions; these are always composed with compactly supported ps.d.o's or applied to compactly supported distributions, so effectively they can be treated as compactly supported. See also Remark 4.1.

With $\tilde{g}$ being any $\mathcal{C}^\infty$ Riemannian metric on X, and $K \subset X$ compact, any $A \in \Psi^0_{\mathrm{bc}}(X)$ with Schwartz kernel supported in $K \times K$ defines a bounded operator on $L^2(X) = L^2(X, d\tilde{g})$, with norm bounded by a seminorm of A in $\Psi^0_{\mathrm{bc}}(X)$. Indeed, this is true for $A \in \Psi^{-\infty}_{\mathrm{b}}(X)$ with compact support, as follows from the Schwartz lemma and the explicit description of the Schwartz kernel of A on X^2_{b}. The standard square root argument then shows the boundedness for $A \in \Psi^0_{\mathrm{bc}}(X)$, with norm bounded by a seminorm of A in $\Psi^0_{\mathrm{bc}}(X)$ – see [20, Equation (2.16)]. In fact, we get more from the argument: letting $a = \sigma_{b,0}(A)$, there exists $A' \in \Psi^{-1}_{\mathrm{b}}(X)$ such that for all $v \in L^2(X)$,

$$\|Av\| \leq 2 \sup |a| \, \|v\| + \|A'v\|.$$

(The factor 2 of course can be improved, as can the order of A'.) This estimate will play an important role in our propagation estimates – it will take the place of constructing a square root of the commutator, which would be difficult here as we will commute P with an element of $\Psi_{\mathrm{b}}(X)$, so the commutator will not lie in $\Psi_{\mathrm{b}}(X)$. We remark here that it is more usual to take a 'b-density' in place of $d\tilde{g}$, i.e. a globally non-vanishing section of $\Omega^1_{\mathrm{b}} X = \Omega_{\mathrm{b}} X$, which thus takes the form $(x_1 \dots x_k)^{-1} \, d\tilde{g}$ locally near a codimension k corner, to define an L^2-space, namely $L^2_{\mathrm{b}}(X) = L^2(X, \frac{d\tilde{g}}{x_1 \dots x_k})$; then $L^2(X) = x_1^{-1/2} \dots x_k^{-1/2} L^2_{\mathrm{b}}(X)$ appears as a weighted space. Elements of $\Psi^0_{\mathrm{bc}}(X)$ are bounded on both L^2 spaces, in the manner stated above. The two boundedness results are very closely related, for if $A \in \Psi^0_{\mathrm{bc}}(X)$, then so is $x_j^{\lambda} A x_j^{-\lambda}$, $\lambda \in \mathbb{C}$.

There is an operator wave front set associated to $\Psi_{\mathrm{bc}}(X)$ as well: for $A \in \Psi^m_{\mathrm{bc}}(X)$, $\mathrm{WF}'_{\mathrm{b}}(A)$ is a conic subset of ${}^{\mathrm{b}}T^*X \setminus o$, and has the interpretation that A is 'in $\Psi^{-\infty}_{\mathrm{bc}}(X)$' outside $\mathrm{WF}'_{\mathrm{b}}(A)$. (We caution the reader that unlike the previous material, as well as the rest of the background in the next three paragraphs, $\mathrm{WF}'_{\mathrm{b}}$ is not discussed in [20]. This discussion, however, is standard; see e.g. [4, Section 18.1], esp. after Definition 18.1.25, in the boundariless case, and [4, Section 18.3] for the case of a $\mathcal{C}^\infty$ boundary, where one simply says that the operator is order $-\infty$ on certain open cones, see e.g. the proof of Theorem 18.3.27 there.) In particular, if $\mathrm{WF}'_{\mathrm{b}}(A) = \emptyset$, then $A \in \Psi^{-\infty}_{\mathrm{b}}(X)$. For instance, if $A = q(a)$, $a \in S^m({}^{\mathrm{b}}T^*X)$, q as in (2.1), $\mathrm{WF}'_{\mathrm{b}}(A)$ is defined by the requirement that if $p \notin \mathrm{WF}'_{\mathrm{b}}(A)$ then p has a conic neighborhood U in ${}^{\mathrm{b}}T^*X \setminus o$ such that $A = q(a)$, a is rapidly decreasing in U, i.e. $|a(x, \bar{y}, \sigma, \bar{\zeta})| \leq C_N(1+|\sigma|+|\bar{\zeta}|)^{-N}$ for all N. Thus, $\mathrm{WF}'_{\mathrm{b}}(A)$ is a closed conic subset of ${}^{\mathrm{b}}T^*X \setminus o$. Moreover, if $K \subset {}^{\mathrm{b}}S^*X$ is compact, and U is a neighborhood of K, there exists $A \in \Psi^0_{\mathrm{b}}(X)$ such that A is the identity on K and vanishes outside U, i.e. $\mathrm{WF}'_{\mathrm{b}}(A) \subset U$, $\mathrm{WF}'_{\mathrm{b}}(\mathrm{Id} - A) \cap K = \emptyset$ – we can construct a to be homogeneous degree zero outside a neighborhood of o, such that this homogeneous function regarded as a function on ${}^{\mathrm{b}}S^*X$ (and still denoted by a) satisfies $a \equiv 1$ near K, $\operatorname{supp} a \subset U$, and then let $A = q(a)$. (This roughly says that $\Psi_{\mathrm{b}}(X)$ can be used to localize in ${}^{\mathrm{b}}S^*X$, i.e. to b-microlocalize.)

$\Psi_{\mathrm{bc}}(X)$ forms a filtered $*$-algebra, so $A_j \in \Psi^{m_j}_{\mathrm{bc}}(X)$, $j = 1, 2$, implies $A_1 A_2 \in \Psi^{m_1+m_2}_{\mathrm{bc}}(X)$, and $A_j^* \in \Psi^{m_j}_{\mathrm{bc}}(X)$ with

$$\sigma_{b,m_1+m_2}(A_1 A_2) = \sigma_{b,m_1}(A_1)\sigma_{b,m_2}(A_2), \ \sigma_{b,m_j}(A_j^*) = \overline{\sigma_{b,m_j}(A)}.$$

Here the formal adjoint is defined with respect to $L^2(X)$, the L^2-space of any $\mathcal{C}^\infty$ Riemannian metric on X; the same statements hold with respect to $L^2_{\mathrm{b}}(X)$ as well, since conjugation by $x_1 \dots x_k$ preserves $\Psi^m_{\mathrm{bc}}(X)$ (as well as $\Psi^m_{\mathrm{b}}(X)$), as already remarked for $m = 0$. Moreover, $[A_1, A_2] \in \Psi^{m_1+m_2-1}_{\mathrm{bc}}(X)$ with

$$\sigma_{b,m_1+m_2-1}([A_1, A_2]) = \frac{1}{i}\{a_1, a_2\}, \ a_j = \sigma_{b,m_j}(A_j);$$

$\{\cdot, \cdot\}$ is the Poisson bracket lifted from T^*X via the identification of T^*X° with ${}^{\mathrm{b}}T^*_{X^\circ}X$. If $A_j \in \Psi^{m_j}_{\mathrm{b}}(X)$, then $A_1 A_2 \in \Psi^{m_1+m_2}_{\mathrm{b}}(X)$, $A_j^* \in \Psi^{m_j}_{\mathrm{b}}(X)$, and $[A_1, A_2] \in \Psi^{m_1+m_2-1}_{\mathrm{b}}(X)$. In addition, operator composition satisfies

$$\mathrm{WF}'_{\mathrm{b}}(A_1 A_2) \subset \mathrm{WF}'_{\mathrm{b}}(A_1) \cap \mathrm{WF}'_{\mathrm{b}}(A_2).$$

If $A \in \Psi^m_{\mathrm{bc}}(A)$ is elliptic, i.e. $\sigma_{b,m}(A)$ is invertible as a symbol (with inverse in $S^{-m}({}^{\mathrm{b}}T^*X \setminus o)/S^{-m-1}({}^{\mathrm{b}}T^*X \setminus o)$), then there is a parametrix $G \in \Psi^{-m}_{\mathrm{bc}}(X)$ for A, i.e. $GA - \mathrm{Id}, AG - \mathrm{Id} \in \Psi^{-\infty}_{\mathrm{bc}}(X)$. This construction microlocalizes, so if $\sigma_{b,m}(A)$ is elliptic at $q \in {}^{\mathrm{b}}T^*X \setminus o$, i.e. $\sigma_{b,m}(A)$ is invertible as a symbol in an open cone around q, then there is a *microlocal parametrix* $G \in \Psi^{-m}_{\mathrm{bc}}(X)$ for A at q, so $q \notin \mathrm{WF}'_{\mathrm{b}}(GA - \mathrm{Id})$, $q \notin \mathrm{WF}'_{\mathrm{b}}(AG - \mathrm{Id})$, so GA, AG are microlocally the identity operator near q. More generally, if $K \subset {}^{\mathrm{b}}S^*X$ is compact, and $\sigma_{b,m}(A)$ is elliptic on K then there is $G \in \Psi^{-m}_{\mathrm{bc}}(X)$ such that $K \cap \mathrm{WF}'_{\mathrm{b}}(GA - \mathrm{Id}) = \emptyset$, $K \cap \mathrm{WF}'_{\mathrm{b}}(AG - \mathrm{Id}) = \emptyset$. For $A \in \Psi^m_{\mathrm{b}}(X)$, $\sigma_{b,m}(A)$ can be regarded as a homogeneous degree m function on ${}^{\mathrm{b}}T^*X \setminus o$, and ellipticity at q means that $\sigma_{b,m}(A)(q) \neq 0$. For such A, one can take $G \in \Psi^{-m}_{\mathrm{b}}(X)$ in all the cases described above.

The other important ingredient, which however rarely appears in the following discussion, although when it appears it is crucial, is the notion of the indicial operator. This captures the mapping properties of $A \in \Psi_{\mathrm{b}}(X)$ in terms of gaining any decay at ∂X. It plays a role here as $P \notin \mathrm{Diff}_{\mathrm{b}}(X)$, so even if we do not expect to gain any decay for solutions u of $Pu = 0$ say, we need to understand the commutation properties of $\mathrm{Diff}_{\mathrm{b}}(X)$ with $\Psi_{\mathrm{b}}(X)$, which will in turn follow from properties of the indicial operator. There is an indicial operator map (which can also be considered as a non-commutative analogue of the principal symbol), denoted by $\hat{N}_i$, for each boundary face $\mathcal{F}_i$, $i \in I$, and $\hat{N}_i$ maps $\Psi^m_{\mathrm{bc}}(X)$ to a family of b-pseudodifferential operators on $\mathcal{F}_i$. For us, only the indicial operators associated to boundary hypersurfaces H_j (given by $x_j = 0$) will be important; in this case the family is parameterized by σ_j, the b-dual variable of x_j. It is characterized by the property that if $f \in \mathcal{C}^\infty(H_j)$ and $u \in \mathcal{C}^\infty(X)$ is any extension of f, i.e. $u|_{H_j} = f$, then

$$\hat{N}_j(A)(\sigma_j)f = (x_j^{-i\sigma_j} A x_j^{i\sigma_j} u)|_{H_j},$$

where $x_j^{-i\sigma_j} A x_j^{i\sigma_j} \in \Psi^m_{\mathrm{bc}}(X)$, hence $x_j^{-i\sigma_j} A x_j^{i\sigma_j} u \in \mathcal{C}^\infty(X)$, and the right hand side does not depend on the choice of u. (In this formulation, we need to fix x_j, at least mod $x_j^2\mathcal{C}^\infty(X)$, to fix $\hat{N}_j(A)$. Note that the radial vector field, $x_j D_{x_j}$, is independent of this choice of x_j, at least modulo $x_j\mathcal{V}_{\mathrm{b}}(X)$.) If $A \in \Psi^m_{\mathrm{bc}}(X)$ and $\hat{N}_i(A) = 0$, then in fact $A \in \mathcal{C}^\infty_{\mathcal{F}_i}(X)\,\Psi^m_{\mathrm{bc}}(X)$, where $\mathcal{C}^\infty_{\mathcal{F}_i}(X)$ is the ideal of $\mathcal{C}^\infty(X)$ consisting of functions that vanish at $\mathcal{F}_i$. In particular, for a boundary hypersurface H_j defined by x_j, if $A \in \Psi^m_{\mathrm{bc}}(X)$ and $\hat{N}_j(A) = 0$, then $A = x_j A'$ with $A' \in \Psi^m_{\mathrm{bc}}(X)$. The indicial operators satisfy $\hat{N}_i(AB) = \hat{N}_i(A)\hat{N}_i(B)$. The indicial family of $x_j D_{x_j}$ at H_j is multiplication by σ_j, while the indicial family of $x_k D_{x_k}$, $k \neq j$, is $x_k D_{x_k}$ and that of $D_{\bar{y}_k}$ is $D_{\bar{y}_k}$. In particular, $\hat{N}_j([x_j D_{x_j}, A]) = [\hat{N}_j(x_j D_{x_j}), \hat{N}_j(A)] = 0$, so

$$[x_j D_{x_j}, A] \in x_j\,\Psi^m_{\mathrm{bc}}(X), \tag{2.2}$$

which plays a role below. All of the above statements also hold with $\Psi_{\mathrm{bc}}(X)$ replaced by $\Psi_{\mathrm{b}}(X)$.

The key point in analyzing smooth vector fields on X, and thereby differential operators such as P is that while $D_{x_j} \notin \mathcal{V}_{\mathrm{b}}(X)$, for any $A \in \Psi^m_{\mathrm{b}}(X)$ there is an operator $\tilde{A} \in \Psi^m_{\mathrm{b}}(X)$ such that

$$D_{x_j} A - \tilde{A} D_{x_j} \in \Psi^m_{\mathrm{b}}(X), \tag{2.3}$$

and analogously for $\Psi^m_{\mathrm{b}}(X)$ replaced by $\Psi^m_{\mathrm{bc}}(X)$. Indeed,

$$D_{x_j}A = x_j^{-1}(x_jD_{x_j})A = x_j^{-1}[x_jD_{x_j}, A] + x_j^{-1}Ax_jD_{x_j}.$$

By (2.2), applied for Ψ_{b} rather than Ψ_{bc},

$$x_j^{-1}[x_jD_{x_j}, A] \in \Psi^m_{\mathrm{b}}(X).$$

Thus, we may take $\tilde{A} = x_j^{-1}Ax_j$, proving (2.3). We also have, more trivially, that

$$D_{\bar{y}_j}A - \tilde{A}D_{\bar{y}_j} \in \Psi^m_{\mathrm{b}}(X),\ \tilde{A} \in \Psi^m_{\mathrm{b}}(X),\ \sigma_{b,m}(A) = \sigma_{b,m}(\tilde{A}). \tag{2.4}$$

Since $\sigma_{b,m}(A) = \sigma_{b,m}(x_j^{-1}Ax_j)$, we deduce the following lemma.

Lemma 2.2. *Suppose* $V \in \mathcal{V}(X)$*,* $A \in \Psi^m_b(X)$*. Then* $[V, A] = \sum A_jV_j + B$ *with* $A_j \in \Psi^{m-1}_b(X)$*,* $V_j \in \mathcal{V}(X)$*,* $B \in \Psi^m_b(X)$*.*

Similarly, $[V, A] = \sum V_jA'_j + B'$ *with* $A'_j \in \Psi^{m-1}_b(X)$*,* $V_j \in \mathcal{V}(X)$*,* $B' \in \Psi^m_b(X)$*.*

Analogous results hold with $\Psi_b(X)$ *replaced by* $\Psi_{bc}(X)$*.*

Proof. It suffices to prove this for the coordinate vector fields, and indeed just for the D_{x_j}. Then with the notation of (2.3),

$$D_{x_j}A - AD_{x_j} = (\tilde{A} - A)D_{x_j} + B,$$

and $\sigma_{b,m}(\tilde{A}) = \sigma_{b,m}(A)$, so $\tilde{A} - A \in \Psi^{m-1}_{\mathrm{b}}(X)$, proving the claim. $\square$

More generally, we make the definition:

Definition 2.3. $\mathrm{Diff}^k\,\Psi^s_{\mathrm{b}}(X)$ is the vector space of operators of the form

$$\sum_j P_jA_j,\ P_j \in \mathrm{Diff}^k(X),\ A_j \in \Psi^s_{\mathrm{b}}(X), \tag{2.5}$$

where the sum is locally finite in X.

Remark 2.4. Since any point $q \in {}^{\mathrm{b}}T^*X \setminus o$ has a conic neighborhood U in ${}^{\mathrm{b}}T^*X \setminus o$ on which some vector field $V \in \mathcal{V}_{\mathrm{b}}(X)$ is elliptic, i.e. $\sigma_{b,1}(V) \neq 0$ on U, we can always write $A_j \in \Psi^{s+k-k_j}_{\mathrm{b}}(X)$ with $\mathrm{WF}'_{\mathrm{b}}(A) \subset U$, $k_j \leq k$, as $A_j = Q_jA'_j + R_j$ with $Q_j \in \mathrm{Diff}^{k-k_j}_{\mathrm{b}}(X)$, $A'_j \in \Psi^s_{\mathrm{b}}(X)$, $R_j \in \Psi^{-\infty}_{\mathrm{b}}(X)$. Thus, any operator which is given by a locally finite sum of the form

$$\sum_j P_jA_j,\ P_j \in \mathrm{Diff}^{k_j}(X),\ A_j \in \Psi^{s+k-k_j}_{\mathrm{b}}(X),$$

can in fact be written in the form (2.5). In particular, $\mathrm{Diff}^{k'}\,\Psi^{s'}_{\mathrm{bc}}(X) \subset \mathrm{Diff}^k\,\Psi^s_{\mathrm{bc}}(X)$ provided that $k' \leq k$ and $k' + s' \leq k + s$, and $\mathrm{Diff}^{k'}\,\Psi^{s'}_{\mathrm{b}}(X) \subset \mathrm{Diff}^k\,\Psi^s_{\mathrm{b}}(X)$ provided that $k' \leq k$, $k' + s' \leq k + s$ and $s - s'$ is an integer.

Lemma 2.5. $\mathrm{Diff}^*\,\Psi^*_b(X)$ *is filtered algebra with respect to operator composition, with* $B_j \in \mathrm{Diff}^{k_j}\,\Psi^{s_j}_b(X)$*,* $j = 1, 2$*, implying* $B_1B_2 \in \mathrm{Diff}^{k_1+k_2}\,\Psi^{s_1+s_2}_b(X)$*. Moreover, with* B_1, B_2 *as above,*

$$[B_1, B_2] \in \mathrm{Diff}^{k_1+k_2}\,\Psi^{s_1+s_2-1}_b(X).$$

Proof. To prove that $\mathrm{Diff}^*\,\Psi^*_{\mathrm{b}}(X)$ is an algebra, we only need to prove that if $A \in \Psi^s_{\mathrm{b}}(X)$, $P \in \mathrm{Diff}^k(X)$, then $AP \in \mathrm{Diff}^k(X)\,\Psi^s_{\mathrm{b}}(X)$. Writing P as a sum of products of vector fields in $\mathcal{V}(X)$, the claim follows from Lemma 2.2.

Writing $B_j = V_{j,1}\ldots V_{j,k_1}A_j$, $A_j \in \Psi^{s_j}_{\mathrm{b}}(X)$, $V_{j,i} \in \mathcal{V}(X)$, and expanding the commutator $[B_1, B_2]$, one gets a finite sum, each of which is a product of the factors $V_{j,1},\ldots V_{j,k_1},A_j$ with two factors (one with $j=1$ and one with $j=2$) removed and replaced by a commutator. In view of the first part of the lemma, it suffices to note that

$$\begin{aligned}&[V_{1,i},V_{2,i'}]\in\mathcal{V}(X),\ \mathrm{Diff}^{k_1+k_2-1}\Psi_{\mathrm{b}}^{s_1+s_2}(X)\subset\mathrm{Diff}^{k_1+k_2}\Psi_{\mathrm{b}}^{s_1+s_2-1}(X),\\&[A_1,A_2]\in\Psi_{\mathrm{b}}^{s_1+s_2-1}(X)\\&[V_{j,i},A_{3-j}]\in\mathrm{Diff}^1\Psi_{\mathrm{b}}^{s_{3-j}-1}(X),\end{aligned}$$

where the last statement is a consequence of Lemma 2.2, taking into account that $\Psi_{\mathrm{b}}^m(X)\subset\mathrm{Diff}^1\Psi_{\mathrm{b}}^{m-1}(X)$. □

We can also define the principal symbol on $\mathrm{Diff}^k\Psi_{\mathrm{b}}^s(X)$. Thus, using $\pi:T^*X\to{}^{\mathrm{b}}T^*X$, we can pull pack $\sigma_{b,s}(A)$, $A\in\Psi_{\mathrm{b}}^s(X)$, to T^*X, and define:

Definition 2.6. Suppose $B=\sum P_jA_j\in\mathrm{Diff}^k\Psi_{\mathrm{b}}^s(X)$, $P_j\in\mathrm{Diff}^k(X)$, $A_j\in\Psi_{\mathrm{b}}^s(X)$. The principal symbol of B is the $\mathcal{C}^\infty$ homogeneous degree $k+s$ function on $T^*X\setminus o$ defined by

$$\sigma_{k+s}(B)=\sum\sigma_k(P_j)\pi^*\sigma_{b,s}(A_j). \tag{2.6}$$

Lemma 2.7. *$\sigma_{k+s}(B)$ is independent of all choices.*

Proof. Away from ∂X, B is a pseudodifferential operator of order $k+s$, and $\sigma_{k+s}(B)$ is its invariantly defined symbol. Since the right hand side of (2.6) is continuous up to ∂X, and is independent of all choices in T^*X°, it is independent of all choices in T^*X. □

We are now ready to compute the principal symbol of the commutator of $A\in\Psi_{\mathrm{b}}^m(X)$ with D_{x_j}.

Lemma 2.8. *Let ∂_{x_j}, ∂_{σ_j} denote local coordinate vector fields on ${}^bT^*X$ in the coordinates $(x,\bar{y},\sigma,\bar{\zeta})$. For $A\in\Psi_b^m(X)$ with Schwartz kernel supported in the coordinate patch, $a=\sigma_{b,m}(A)\in\mathcal{C}^\infty({}^bT^*X\setminus o)$, we have $[D_{x_j},A]=A_1D_{x_j}+A_0\in\mathrm{Diff}^1\Psi_b^{m-1}(X)$ with $A_0\in\Psi_b^m(X)$, $A_1\in\Psi_b^{m-1}(X)$ and*

$$\sigma_{b,m-1}(A_1)=\frac{1}{i}\partial_{\sigma_j}a,\ \sigma_{b,m}(A_0)=\frac{1}{i}\partial_{x_j}a. \tag{2.7}$$

This result also holds with $\Psi_b(X)$ replaced by $\Psi_{bc}(X)$ everywhere.

Remark 2.9. Notice that $\sigma_m([D_{x_j},A])=\frac{1}{i}\{\xi_j,\pi^*a\}=\frac{1}{i}\partial_{x_j}|_\xi$, $\{.,.\}$ denoting the Poisson bracket on T^*X and $\partial_{x_j}|_\xi$ denoting the appropriate coordinate vector field on T^*X, i.e. where ξ is held fixed (rather than σ), since both sides are continuous functions on $T^*X\setminus o$ which agree on $T^*X^\circ\setminus o$. A simple calculation shows that the lemma is consistent with this result. The statement of the lemma would follow from this observation if we showed that the kernel of σ_m on $\mathrm{Diff}^1\Psi_{\mathrm{b}}^{m-1}(X)$ is $\mathrm{Diff}^1\Psi_{\mathrm{b}}^{m-2}(X)$ – the proof given below avoids this point by reducing the calculation to $\Psi_{\mathrm{b}}(X)$.

Proof. The lemma follows from

$$D_{x_j}A-AD_{x_j}=x_j^{-1}[x_jD_{x_j},A]+x_j^{-1}[A,x_j]D_{x_j}.$$

Indeed, letting

$$A_0=x_j^{-1}[x_jD_{x_j},A]\in\Psi_{\mathrm{b}}^m(X),\ A_1=x_j^{-1}[A,x_j]\in\Psi_{\mathrm{b}}^{m-1}(X), \tag{2.8}$$

the principal symbols can be calculated in the b-calculus. Since they are given by the standard Poisson bracket in T^*X°, hence in ${}^{\mathrm{b}}T^*_{X^\circ}X$, by continuity the same calculation gives a valid result in ${}^{\mathrm{b}}T^*X$. As $\partial_{\xi_j} = x_j\partial_{\sigma_j}$, $\partial_{x_j}|_\xi = \partial_{x_j}|_\sigma + \xi_j\partial_{\sigma_j}$, we see that for $b = \sigma_j$ or $b = x_j$, the Poisson bracket $\{b, a\}$ is given by

$$\begin{aligned} &x_j(\partial_{\sigma_j}b)(\partial_{x_j}|_\sigma a + \xi_j\partial_{\sigma_j}a) - x_j(\partial_{\sigma_j}a)(\partial_{x_j}|_\sigma b + \xi_j\partial_{\sigma_j}b)\\ &\qquad = x_j(\partial_{\sigma_j}b)\partial_{x_j}|_\sigma a - x_j(\partial_{\sigma_j}a)\partial_{x_j}|_\sigma b \end{aligned}$$

so we get

$$\{\sigma_j, a\} = x_j\partial_{x_j}|_\sigma a,\ \{x_j, a\} = -x_j\partial_{\sigma_j}a,$$

so (2.7) follows from (2.8). □

3. Function spaces and microlocalization

We now turn to action of $\Psi_{\mathrm{b}}(X)$ on function spaces related to differential operators in $\mathrm{Diff}(X)$, and in particular $H^1(X)$ which corresponds to first order differential operators, such as the exterior derivative d. We first recall that $\mathcal{C}^\infty_c(X)$ is the space of $\mathcal{C}^\infty$ functions of compact support on X (which may thus be non-zero at ∂X), while $\dot{\mathcal{C}}^\infty_c(X)$ is the subspace of $\mathcal{C}^\infty_c(X)$ consisting of functions which vanish to infinite order at ∂X. Although we will mostly consider local results, and any $\mathcal{C}^\infty$ Riemannian metric can be used to define $L^2_{\mathrm{loc}}(X)$, $L^2_c(X)$ (as different choices give the same space), it is convenient to fix a global Riemmanian metric, $\tilde{g} = g + dt^2$, on X, where g is the metric on M. With this choice, $L^2(X)$ is well-defined as a Hilbert space. For $u \in \mathcal{C}^\infty_c(X)$, we let

$$\|u\|^2_{H^1(X)} = \|du\|^2_{L^2(X)} + \|u\|^2_{L^2(X)}.$$

We then let $H^1(X)$ be the completion of $\mathcal{C}^\infty_c(X)$ with respect to the $H^1(X)$ norm. Then we define $H^1_0(X)$ as the closure of $\dot{\mathcal{C}}^\infty_c(X)$ inside $H^1(X)$.

Remark 3.1. We recall alternative viewpoints of these Sobolev spaces. Good references for the $\mathcal{C}^\infty$ boundary case (and no corners) include [4, Appendix B.2] and [23, Section 4.4]; only minor modifications are needed to deal with the corners for the special cases we discuss below.

We can define $H^1(X^\circ)$ as the subspace of $L^2(X)$ consisting of functions u such that du, defined as the distributional derivative of u in X°, lying in $L^2(X, \Lambda^1 X)$; we then equip it with the above norm − this is locally equivalent to saying that $Vu \in L^2_{\mathrm{loc}}(X)$ for all $\mathcal{C}^\infty$ vector fields V on X, where Vu refers to the distributional derivative of u on X°.

In fact, $H^1(X^\circ) = H^1(X)$, since $H^1(X^\circ)$ is complete with respect to the H^1 norm and $\mathcal{C}^\infty_c(X)$ is easily seen to be dense in it. For instance, locally, if X is given by $x_j \geq 0$, $j = 1, \dots, k$, and u is supported in such a coordinate chart, one can take $u_s(x, \bar{y}) = u(x_1{+}s, \dots, x_k{+}s, \bar{y})$ for $s > 0$, and see that $u_s|_X \to u$ in $H^1_c(X^\circ)$. Then a standard regularization argument on $\mathbb{R}^n$, $n = \dim X$, gives the claimed density of $\mathcal{C}^\infty_c(X)$ in $H^1_c(X^\circ)$. Thus, $H^1(X^\circ) = H^1(X)$ indeed, which shows in particular that $H^1(X) \subset L^2(X)$. (Note that $\|u\|_{L^2(X)} \leq \|u\|_{H^1(X)}$ only guarantees that there is a continuous 'inclusion' $H^1(X) \hookrightarrow L^2(X)$, not that it is injective, although that can be proved easily by a direct argument, cf. the Friedrichs extension method for operators, see e.g. [21, Theorem X.23].)

If $\tilde{X}$ is a manifold without boundary, and X is embedded into it, one can also extend elements of $H^1(X)$ to elements $H^1_{\mathrm{loc}}(\tilde{X})$ exactly as in the $\mathcal{C}^\infty$ boundary case (or simply locally extending in x_1 first, then in x_2, etc., and using the $\mathcal{C}^\infty$

boundary result), see [23, Section 4.4]. Thus, with the notation of [4, Appendix B.2], $H^1_{\mathrm{loc}}(X) = \bar{H}^1_{\mathrm{loc}}(X^\circ)$. As is clear from the completion definition, $H^1_{0,\mathrm{loc}}(X)$ can be identified with the subset of $H^1_{\mathrm{loc}}(\tilde{X})$ consisting of functions supported in X. Thus, $H^1_{0,\mathrm{loc}}(X) = \dot{H}^1_{\mathrm{loc}}(X)$ with the notation of [4, Appendix B.2].

All of the above discussion can be easily modified for H^m in place of H^1, $m \geq 0$ an integer.

We are now ready to state the action on Sobolev spaces. These results would be valid, with similar proofs, if we replace $H^1(X)$ by $H^m(X)$, $m \geq 0$ integer. We also refer to [4, Theorem 18.3.13] for further extensions when X has a $\mathcal{C}^\infty$ boundary (and no corners).

Lemma 3.2. *Any $A \in \Psi^0_{bc}(X)$ with compact support defines a continuous linear maps $A : H^1(X) \to H^1(X)$, $A : H^1_0(X) \to H^1_0(X)$, with norms bounded by a seminorm of A in $\Psi^0_{bc}(X)$.*

Moreover, for any $K \subset X$ compact, any $A \in \Psi^0_{bc}(X)$ with proper support defines a continuous map from the subspace of $H^1(X)$ (resp. $H^1_0(X)$) consisting of distributions supported in K to $H^1_c(X)$ (resp. $H^1_{0,c}(X)$).

Remark 3.3. Note that all smooth vector fields V of compact support define a continuous operator $H^1(X) \to L^2(X)$, so in particular $V \in \mathcal{V}_{\mathrm{b}}(X)$ do so. Now, any $A \in \Psi^1_{\mathrm{bc}}(X)$ can be written as $\sum (D_{x_j} x_j) A_j + \sum D_{\bar{y}_j} A'_j + A''$ with $A_j, A'_j, A'' \in \Psi^0_{\mathrm{bc}}(X)$ by writing $\sigma_{b,1}(A) = \sum \sigma_j a_j + \sum \bar{\zeta}_j a'_j$, and taking A_j, A'_j with principal symbol a_j, a'_j. Therefore the lemma implies that any $A \in \Psi^1_{\mathrm{bc}}(X)$ defines a continuous linear operator $H^1(X) \to L^2(X)$, and in particular restricts to a map $H^1_0(X) \to L^2(X)$.

Proof. For $A \in \Psi^0_{\mathrm{bc}}(X)$, by (2.3) $D_{x_j} A u = \tilde{A} D_{x_j} u + Bu$, with $\tilde{A} \in \Psi^0_{\mathrm{bc}}(X)$, $B \in \Psi^0_{\mathrm{bc}}(X)$ the seminorms of both in $\Psi^0_{\mathrm{bc}}(X)$ bounded by seminorms of A in $\Psi^0_{\mathrm{bc}}(X)$, so by the first half of the proof

$$\|D_{x_j} A u\|_{L^2(X)} \leq \|\tilde{A}\|_{\mathcal{B}(L^2(X),L^2(X))} \|D_{x_j} u\|_{L^2(X)} + \|B\|_{\mathcal{B}(L^2(X),L^2(X))} \|u\|_{L^2(X)}.$$

Since there is an analogous formula for D_{x_j} replaced by $D_{\bar{y}_j}$, we deduce that for some $C > 0$, depending only on a seminorm of A in $\Psi^0_{\mathrm{bc}}(X)$,

$$\|d_X A u\|_{L^2(X)} \leq C(\|d_X u\|_{L^2(X)} + \|u\|_{L^2(X)}).$$

Thus, $A \in \Psi^0_{\mathrm{bc}}(X)$ extends to a continuous linear map from the completion of $\mathcal{C}^\infty_c(X)$ with respect to the $H^1(X)$ norm to itself, i.e. from $H^1(X)$ to itself as claimed. As it maps $\dot{\mathcal{C}}^\infty_c(X) \to \dot{\mathcal{C}}^\infty_c(X)$, it also maps the H^1-closure of $\dot{\mathcal{C}}^\infty(X)$ to itself, i.e. it defines a continuous linear map $H^1_0(X) \to H^1_0(X)$, finishing the proof of the first half of the lemma.

For the second half, we only need to note that $Au = A\phi u$ if $\phi \equiv 1$ near K and has compact support; now $A\phi$ has compact support so the first half of the lemma is applicable. □

Note that $H^1(X) \subset L^2(X) \subset \mathcal{C}^{-\infty}(X)$, with $\mathcal{C}^{-\infty}(X)$ denoting the dual space of $\dot{\mathcal{C}}^\infty_c(X)$, i.e. the space of extendible distributions. Since for any m, $A \in \Psi^m_{\mathrm{bc}}(X)$ maps $\mathcal{C}^{-\infty}(X) \to \mathcal{C}^{-\infty}(X)$, we could view A already defined as a map $H^1(X) \to \mathcal{C}^{-\infty}(X)$; then the above lemma is a continuity result for $m = 0$.

We let $H^{-1}(X)$ be the dual of $H^1_0(X)$ and $\dot{H}^{-1}(X)$ be the dual of $H^1(X)$, with respect to an extension of the sesquilinear form $\langle u, v\rangle = \int_X u\,\overline{v}\,d\tilde{g}$, i.e. the L^2 inner product. As $H^1_0(X)$ is a closed subspace of $H^1(X)$, $H^{-1}(X)$ is the quotient of $\dot{H}^{-1}(X)$ by the annihilator of $H^1_0(X)$. In terms of the identification of the H^1 spaces in the penultimate paragraph of Remark 3.1, $H^{-1}_{\mathrm{loc}}(X) = \bar{H}^{-1}_{\mathrm{loc}}(X^\circ)$ in the notation of [4, Appendix B.2], i.e. its elements are the restrictions to X° of elements of $H^{-1}_{\mathrm{loc}}(\tilde{X})$. Analogously, $\dot{H}^{-1}_{\mathrm{loc}}(X)$ consists of those elements of $H^{-1}_{\mathrm{loc}}(\tilde{X})$ which are supported in X.

Any $V \in \mathrm{Diff}^1(X)$ of compact support defines a continuous map $L^2(X) \to H^{-1}(X)$ via $\langle Vu, v\rangle = \langle u, V^* v\rangle$ for $u \in L^2(X)$, $v \in H^1_0(X)$; this is the same map as induced by extending V to an element $\tilde{V}$ of $\mathrm{Diff}^1(\tilde{X})$, extending u to $\tilde{X}$, say as 0, and letting $Vu = \tilde{V}\tilde{u}|_{X^\circ}$. Thus, any $P \in \mathrm{Diff}^2(X)$ of compact support defines continuous maps $H^1(X) \to H^{-1}(X)$, and in particular $H^1_0(X) \to H^{-1}(X)$, since we can write $P = \sum V_j W_j$ with $V_j, W_j \in \mathrm{Diff}^1(X)$. Similarly, any $P \in \mathrm{Diff}^2(X)$ defines continuous maps $H^1_{\mathrm{loc}}(X) \to H^{-1}_{\mathrm{loc}}(X)$, and in particular $H^1_{0,\mathrm{loc}}(X) \to H^{-1}_{\mathrm{loc}}(X)$. Thus, for $P = \Delta_{\tilde{g}} + 1$, $\langle u, v\rangle_{H^1(X)} = \langle u, Pv\rangle$ if $u \in H^1_0(X)$ and $v \in H^1(X)$. Similarly, for $P = D_t^2 - \Delta_g$, $\langle D_t u, D_t v\rangle - \langle d_M u, d_M v\rangle = \langle u, Pv\rangle$, if $u \in H^1_0(X)$ and $v \in H^1(X)$.

We also remark that as $H^1(X)$ and $H^1_0(X)$ are Hilbert spaces, their duals are naturally identified with themselves via the inner product. Thus, if f is a continuous linear functional on $H^1_0(X)$, then there is a $v \in H^1_0(X)$ such that $f(u) = \langle u, v\rangle + \langle du, dv\rangle$. Thus, regarding $H^1_0(X)$ as a subspace of $H^1(\tilde{X})$, for an extension $\tilde{X}$ of X, as in Remark 3.1, we deduce that $f(u) = \langle u, (\Delta_{\tilde{g}} + 1)v\rangle$, so the identification of $H^{-1}(X)$ with $H^1_0(X)$ (regarded as its own dual) is given by $H^1_0(X) \ni v \mapsto (\Delta_{\tilde{g}} + 1)v \in H^{-1}(X)$.

Since $\Psi^0_{\mathrm{bc}}(X)$ is closed under taking adjoints, the following result is an immediate consequence of Lemma 3.2.

Corollary 3.4. *Any $A \in \Psi^0_{bc}(X)$ with compact support defines a continuous linear maps $A : H^{-1}(X) \to H^{-1}(X)$, $A : \dot{H}^{-1}(X) \to \dot{H}^{-1}(X)$, with norm bounded by a seminorm of A in $\Psi^0_{bc}(X)$.*

We now define subspaces of $H^1(X)$ which possess additional regularity with respect to $\Psi_{\mathrm{b}}(X)$.

Definition 3.5. For $m \geq 0$, we define $H^{1,m}_{b,c}(X)$ as the subspace of $H^1(X)$ consisting of $u \in H^1(X)$ with $\mathrm{supp}\, u$ compact and $Au \in H^1(X)$ for some (hence any, as shown below) $A \in \Psi^m_{\mathrm{b}}(X)$ (with compact support) which is elliptic over $\mathrm{supp}\, u$, i.e. A such that such that $\sigma_{b,m}(A)(q) \neq 0$ for any $q \in {}^{\mathrm{b}}T^*_{\mathrm{supp}\, u}X \setminus o$.

We let $H^{1,m}_{b,\mathrm{loc}}(X)$ be the subspace of $H^1_{\mathrm{loc}}(X)$ consisting of $u \in H^1_{\mathrm{loc}}(X)$ such that for any $\phi \in \mathcal{C}^\infty_c(X)$, $\phi u \in H^{1,m}_{b,c}(X)$.

We also let $H^{1,m}_{b,0,c}(X) = H^{1,m}_{b,c}(X) \cap H^1_0(X)$, and similarly for the local space $H^{1,m}_{b,0,\mathrm{loc}}(X)$.

Remark 3.6. The definition is independent of the choice of A, as can be seen by taking a parametrix $G \in \Psi^{-m}_{\mathrm{b}}(X)$ for A in a neighborhood of $\mathrm{supp}\, u$, so $GA - \mathrm{Id} = E \in \Psi^0_{\mathrm{b}}(X)$, and $\mathrm{WF}'_{\mathrm{b}}(E) \cap {}^{\mathrm{b}}T^*_{\mathrm{supp}\, u}X \setminus o = \emptyset$. Indeed, let $\rho \in \mathcal{C}^\infty_c(X)$ be identically 1 near $\mathrm{supp}\, u$, $\mathrm{WF}'_{\mathrm{b}}(E) \cap {}^{\mathrm{b}}T^*_{\mathrm{supp}\, \rho}X = \emptyset$. Then any A' with the properties of A

can be written as $A' = A'GA - A'E\rho - A'E(1-\rho)$, $A'G, A'E\rho \in \Psi^0_{\mathrm{b}}(X)$, while $(1-\rho)u = 0$, so by Lemma 3.2, $A'u \in H^1(X)$ provided that $u, Au \in H^1(X)$.

It is useful to note that if $Au \in H^1(X)$ and $u \in H^1_0(X)$, then in fact $Au \in H^1_0(X)$:

Lemma 3.7. *Suppose that $u \in H^1_0(X)$, $A \in \Psi^m_b(X)$ and $Au \in H^1(X)$. Then $Au \in H^1_0(X)$.*

Proof. Suppose that $u \in H^1_0(X)$, $A \in \Psi^m_{\mathrm{b}}(X)$ and $Au \in H^1(X)$. Let Λ_r, $r \in (0,1]$, be a uniformly bounded family in $\Psi^0_{\mathrm{bc}}(X)$ with $\Lambda_r \in \Psi^{-\infty}_{\mathrm{b}}(X)$ for $r > 0$, $\Lambda_r \to \mathrm{Id}$ in $\Psi^\epsilon_{\mathrm{b}}(X)$, $\epsilon > 0$, as $r \to 0$.

Then, for $r > 0$, $\Lambda_r A \in \Psi^{-\infty}_{\mathrm{b}}(X)$, so $u \in H^1_0(X)$ implies that $\Lambda_r Au \in H^1_0(X)$ by Lemma 3.2. As $Au \in H^1(X)$, and Λ_r is uniformly bounded as a family of operators on $H^1(X)$, we deduce that $\Lambda_r Au$ is uniformly bounded in $H^1(X)$. Thus, there is a weakly convergent sequence $\Lambda_{r_j} Au$, with $r_j \to 0$, in $H^1_0(X)$, as the latter is a closed subspace of $H^1(X)$; let v be the limit. But $\Lambda_r Au \to Au$ in $\mathcal{C}^{-\infty}(X)$ as $r \to 0$, since $\Lambda_r A \to A$ in $\Psi^{m+\epsilon}_{\mathrm{bc}}(X)$. As $\Lambda_{r_j} Au \to v$ in $\mathcal{C}^{-\infty}(X)$ as well, $Au = v \in H^1_0(X)$ as claimed. □

The following wave front set microlocalizes $H^{1,m}_{b,\mathrm{loc}}(X)$.

Definition 3.8. Suppose $u \in H^1_{\mathrm{loc}}(X)$, $m \geq 0$. We say that $q \in {}^{\mathrm{b}}T^*X \setminus o$ is not in $\mathrm{WF}^{1,m}_{\mathrm{b}}(u)$ if there exists $A \in \Psi^m_{\mathrm{b}}(X)$ such that $\sigma_{b,m}(A)(q) \neq 0$ and $Au \in H^1(X)$.

For $m = \infty$, we say that $q \in {}^{\mathrm{b}}T^*X \setminus o$ is not in $\mathrm{WF}^{1,m}_{\mathrm{b}}(u)$ if there exists $A \in \Psi^0_{\mathrm{b}}(X)$ such that $\sigma_{b,0}(A)(q) \neq 0$ and $LAu \in H^1(X)$ for all $L \in \mathrm{Diff}_{\mathrm{b}}(X)$, i.e. if $Au \in H^{1,\infty}_b(X)$.

We note that, by the preceeding lemma, if $u \in H^1_{0,\mathrm{loc}}(X)$ then $Au \in H^1_{0,\mathrm{loc}}(X)$, etc. (here $A \in \Psi^m_{\mathrm{b}}(X)$). Moreover, in the m infinite case we may equally allow $L \in \Psi_{\mathrm{b}}(X)$, and we can also rewrite the finite m definition analogously, i.e. to state that there exists $A \in \Psi^0_{\mathrm{b}}(X)$ such that $\sigma_{b,0}(A)(q) \neq 0$ and $LAu \in H^1(X)$ for all $L \in \Psi^m_{\mathrm{b}}(X)$ – this follows immediately from the next lemma. Although we do not need this here, so we do not comment on it any more, we could also allow $A \in \Psi^m_{\mathrm{bc}}(X)$ in the definition, provided we replace $\sigma_{b,m}(A)(q) \neq 0$ by the assumption that A is elliptic at q – this follows from the next results.

The following lemma shows that the action of elements of $\Psi_{\mathrm{b}}(X)$ is indeed microlocal.

Lemma 3.9. *Suppose that $u \in H^1_{loc}(X)$, $B \in \Psi^k_{bc}(X)$. Then $\mathrm{WF}^{1,m-k}_b(Bu) \subset \mathrm{WF}^{1,m}_b(u) \cap \mathrm{WF}'_b(B)$.*

Proof. We assume that m is finite; the proof for m infinite is similar.

Suppose $q \notin \mathrm{WF}'_{\mathrm{b}}(B)$. As $\mathrm{WF}'_{\mathrm{b}}(B)$ is closed, there is a neighborhood U of q such that $U \cap \mathrm{WF}'_{\mathrm{b}}(B) = \emptyset$. Let $A \in \Psi^{m-k}_{\mathrm{b}}(X)$ satisfy $\mathrm{WF}'_{\mathrm{b}}(A) \subset U$, $\sigma_{b,m-k}(A)(q) \neq 0$. Then $AB \in \Psi^{-\infty}_{\mathrm{b}}(X) \subset \Psi^0_{\mathrm{b}}(X)$, so $ABu \in H^1(X)$ by Lemma 3.2. Thus, $q \notin \mathrm{WF}^{1,m-k}_{\mathrm{b}}(Bu)$ by definition of the wave front set.

On the other hand, suppose that $q \notin \mathrm{WF}^{1,m}_{\mathrm{b}}(u)$. Then there is some $A \in \Psi^m_{\mathrm{b}}(X)$ such that $Au \in H^1(X)$ and $\sigma_{b,m}(A)(q) \neq 0$. Let $G \in \Psi^{-m}_{\mathrm{b}}(X)$ be a microlocal parametrix for A, so $GA = \mathrm{Id} + E$ with $E \in \Psi^0_{\mathrm{b}}(X)$, $q \notin \mathrm{WF}'_{\mathrm{b}}(E)$. Let $C \in \Psi^{m-k}_{\mathrm{b}}(X)$ be such that $\mathrm{WF}'_{\mathrm{b}}(C) \cap \mathrm{WF}'_{\mathrm{b}}(E) = \emptyset$ and $\sigma_{b,m-k}(C)(q) \neq 0$. Then $CBE \in \Psi^{-\infty}_{\mathrm{b}}(X)$, so $CBEu \in H^1(X)$ by Lemma 3.2. On the other hand,

$CBG \in \Psi^0_{\mathrm{bc}}(X)$ and $Au \in H^1(X)$, so $CBGAu \in H^1(X)$ also by Lemma 3.2. We thus deduce that $CBu = CBGAu - CBEu \in H^1(X)$, so $q \notin \mathrm{WF}_{\mathrm{b}}^{1,m-k}(u)$. $\square$

We will need a quantitative version of this lemma giving actual estimates, but first we state the precise sense in which this wave front set provides a refined version of the conormality of u.

Lemma 3.10. *Suppose $u \in H^1_{loc}(X)$, $m \geq 0$, $p \in X$. If ${}^{b}S^*_pX \cap \mathrm{WF}_b^{1,m}(u) = \emptyset$, then in a neighborhood of p, u lies in $H_b^{1,m}(X)$, i.e. there is $\phi \in \mathcal{C}^\infty_c(X)$ with $\phi \equiv 1$ near p such that $\phi u \in H_b^{1,m}(X)$.*

Proof. We assume that m is finite; the proof for m infinite is similar.

For each $q \in {}^{\mathrm{b}}S^*_pX$ there is $A_q \in \Psi^m_{\mathrm{b}}(X)$ such that $\sigma_{b,m}(A_q)(q) \neq 0$ and $A_qu \in H^1(X)$. Let U_q be the set on which $\sigma_{b,m}(A_q) \neq 0$; then U_q is an open set containing q. Thus, $\{U_q :\ q \in {}^{\mathrm{b}}S^*_pX\}$ is an open cover of the compact set ${}^{\mathrm{b}}S^*_pX$. Let U_{q_j}, $j = 1, \ldots, r$ be a finite subcover. Then $A_0 = \sum A^*_{q_j}A_{q_j}$ is elliptic on ${}^{\mathrm{b}}S^*_pX$ since $\sigma_{b,2m}(A_0) = \sum |\sigma_{b,m}(A_{q_j})|^2$, with each summand non-negative, and at any $q \in {}^{\mathrm{b}}S^*_pX$ at least one term is nonzero (namely one for which $q \in U_{q_j}$). Finally, we renormalize A_0 to make its order the same as that of A: this is achieved by taking any $Q \in \Psi^{-m}_{\mathrm{b}}(X)$ which is elliptic on ${}^{\mathrm{b}}S^*_pX$, and letting $A = QA_0 \in \Psi^m_{\mathrm{b}}(X)$. Thus, A is elliptic on ${}^{\mathrm{b}}S^*_pX$, and $Au \in H^1(X)$ as this holds for each summand $(QA^*_{q_j})(A_{q_j}u)$, for $QA^*_{q_j} \in \Psi^0_{\mathrm{b}}(X)$ and $A_{q_j}u \in H^1(X)$. Here we used Lemma 3.2.

Let $G \in \Psi^{-m}_{\mathrm{b}}(X)$ be a microlocal parametrix for A, so $GA = \mathrm{Id} + E$ and $\mathrm{WF}'_{\mathrm{b}}(E) \cap {}^{\mathrm{b}}S^*_pX = \emptyset$. Thus, p has a neighborhood O in X such that $\mathrm{WF}'_{\mathrm{b}}(E) \cap {}^{\mathrm{b}}S^*_OX = \emptyset$. Let $\phi \in \mathcal{C}^\infty_c(X)$ be supported in O, identically 1 near p, and let $T \in \Psi^m_{\mathrm{b}}(X)$ be elliptic on ${}^{\mathrm{b}}S^*_{\operatorname{supp}\phi}X$. Then $T\phi u = T\phi GAu - T\phi Eu$. Since $\mathrm{WF}'_{\mathrm{b}}(E) \cap \mathrm{WF}'_{\mathrm{b}}(\phi) = \emptyset$, we see that $T\phi E \in \Psi^{-\infty}_{\mathrm{b}}(X)$, and thus the last term is in $H^1(X)$ by Lemma 3.2. On the other hand, the first term is in $H^1(X)$ since $Au \in H^1(X)$ and $T\phi G \in \Psi^0_{\mathrm{b}}(X)$. Thus, $\phi u \in H_b^{1,m}(X)$ as claimed. $\square$

Corollary 3.11. *If $u \in H^1_{loc}(X)$ and $\mathrm{WF}_b^{1,m}(u) = \emptyset$, then $u \in H^{1,m}_{b,loc}(X)$.*

In particular, if $u \in H^1_{loc}(X)$ and $\mathrm{WF}_b^{1,m}(u) = \emptyset$ for all m, then $u \in H^{1,\infty}_{b,loc}(X)$, i.e. u is conormal in the sense that $Au \in H^1_{loc}(X)$ for all $A \in \mathrm{Diff}_b(X)$ (or indeed $A \in \Psi_b(X)$).

For the quantitative version of Lemma 3.9 we need a notion of the operator wave front set that is uniform in a family of operators:

Definition 3.12. Suppose that $\mathcal{B}$ is a bounded subset of $\Psi^k_{\mathrm{bc}}(X)$, and $q \in {}^{\mathrm{b}}S^*X$. We say that $q \notin \mathrm{WF}'_{\mathrm{b}}(\mathcal{B})$ if there is some $A \in \Psi_{\mathrm{b}}(X)$ which is elliptic at q such that $\{AB :\ B \in \mathcal{B}\}$ is a bounded subset of $\Psi^{-\infty}_{\mathrm{b}}(X)$.

Note that the wave front set of a family $\mathcal{B}$ is only defined for bounded families. It can be described directly in terms of quantization of (full) symbols, much like the operator wave front set of a single operator. All standard properties of the operator wave front set also hold for a family; e.g. if $E \in \Psi_{\mathrm{b}}(X)$ with $\mathrm{WF}'_{\mathrm{b}}(E) \cap \mathrm{WF}'_{\mathrm{b}}(\mathcal{B}) = \emptyset$ then $\{BE :\ B \in \mathcal{B}\}$ is bounded in $\Psi^{-\infty}_{\mathrm{b}}(X)$.

A quantitative version of Lemma 3.9 is the following result.

Lemma 3.13. *Suppose that $K \subset {}^{b}S^*X$ is compact, and U a neighborhood of K in ${}^{b}S^*X$. Let $\tilde{K} \subset X$ compact, and $\tilde{U}$ be a neighborhood of $\tilde{K}$ in X with compact* *closure. Let* $Q \in \Psi_b^k(X)$ *be elliptic on* K *with* $\mathrm{WF}_b'(Q) \subset U$*, with Schwartz kernel supported in* $\tilde{K} \times \tilde{K}$*. Let* $\mathcal{B}$ *be a bounded subset of* $\Psi_{bc}^k(X)$ *with* $\mathrm{WF}_b'(\mathcal{B}) \subset K$ *and Schwartz kernel supported in* $\tilde{K} \times \tilde{K}$*. Then there is a constant* $C > 0$ *such that for* $B \in \mathcal{B}$*,* $u \in H^1_{loc}(X)$ *with* $\mathrm{WF}_b^{1,k}(u) \cap U = \emptyset$*,*

$$\|Bu\|_{H^1(X)} \leq C(\|u\|_{H^1(\tilde{U})} + \|Qu\|_{H^1(X)}).$$

Proof. Let $\phi \in \mathcal{C}_c^\infty(\tilde{U})$ be identically 1 near $\tilde{K}$. We may replace u by ϕu in the estimate since $B\phi = B$, $Q\phi = Q$; then $\|\phi u\|_{H^1(\tilde{U})} = \|\phi u\|_{H^1(X)}$.

By Lemma 3.9 and Lemma 3.10, all terms in the estimate are finite, since e.g. $\mathrm{WF}_\mathrm{b}'(Q) \cap \mathrm{WF}_\mathrm{b}^{1,k}(u) = \emptyset$ so $\mathrm{WF}_\mathrm{b}^{1,0}(u) = \emptyset$, so $Qu \in H^{1,0}_{b,\mathrm{loc}}(X) = H^1_{\mathrm{loc}}(X)$, and indeed $Qu \in H^1_c(X)$, as the Schwartz kernel of Q has compact support.

Let G be a microlocal parametrix for Q, so $GQ = \mathrm{Id} + E$ with $E \in \Psi_\mathrm{b}^0(X)$, $\mathrm{WF}_\mathrm{b}'(E) \cap K = \emptyset$. Thus, $Bu = BGQu - BEu$. Now, $BE \in \Psi_\mathrm{b}^{-\infty}(X)$ since $\mathrm{WF}_\mathrm{b}'(E) \cap K = \emptyset$ and $\mathrm{WF}_\mathrm{b}'(B) \subset K$, and it lies in a bounded subset of $\Psi_\mathrm{b}^{-\infty}(X)$ for $B \in \mathcal{B}$. Thus, $\|BEu\|_{H^1(X)} \leq C_1 \|u\|_{H^1(X)}$ by Lemma 3.2. On the other hand, $BG \in \Psi_\mathrm{b}^0(X)$ and indeed in a bounded subset of $\Psi_\mathrm{bc}^0(X)$ for $B \in \mathcal{B}$, so Lemma 3.2 also gives that for some $C_2 > 0$ (independent of $B \in \mathcal{B}$), $\|BGQu\|_{H^1(X)} \leq C_2 \|Qu\|_{H^1(X)}$. Combining these proves the lemma. □

We can similarly microlocalize $H^{-1}_{\mathrm{loc}}(X)$:

Definition 3.14. Suppose $u \in H^{-1}_{\mathrm{loc}}(X)$, $m \geq 0$. We say that $q \in {}^\mathrm{b}T^*X \setminus o$ is not in $\mathrm{WF}_\mathrm{b}^{-1,m}(u)$ if there exists $A \in \Psi_\mathrm{b}^m(X)$ such that $\sigma_{b,m}(A)(q) \neq 0$ and $Au \in H^{-1}(X)$.

Then the analogues of Lemma 3.9-3.13 remain valid with $H^1(X)$ replaced by $H^{-1}(X)$ and $\mathrm{WF}_\mathrm{b}^{1,\cdot}$ replaced by $\mathrm{WF}_\mathrm{b}^{-1,\cdot}$, with analogous proofs using Corollary 3.4 in place of Lemma 3.2.

These results can be extended in another way, by considering Sobolev spaces with a negative order of regularity relative to $H^1(X)$.

Definition 3.15. Let k be an integer, $m < 0$, and $A \in \Psi_\mathrm{b}^{-m}(X)$ be elliptic on ${}^\mathrm{b}S^*X$ with proper support. We let $H^{k,m}_{\mathrm{b},c}(X)$ be the space of all $u \in \mathcal{C}^{-\infty}(X)$ of the form $u = u_1 + Au_2$ with $u_1, u_2 \in H^k_c(X)$. We let

$$\|u\|_{H^{k,m}_{\mathrm{b},c}(X)} = \inf\{\|u_1\|_{H^k(X)} + \|u_2\|_{H^k(X)} :\ u = u_1 + Au_2\}.$$

We also let $H^{k,m}_{\mathrm{b,loc}}(X)$ be the space of all $u \in \mathcal{C}^{-\infty}(X)$ such that $\phi u \in H^{k,m}_{\mathrm{b},c}(X)$ for all $\phi \in \mathcal{C}^\infty_c(X)$.

We also define $\dot{H}^{k,m}_{\mathrm{b},c}(X)$ and $\dot{H}^{k,m}_{\mathrm{b,loc}}(X)$ analogously, replacing $H^k(X)$ by $\dot{H}^k(X)$ throughout the above discussion. Here, for $k \geq 0$, $\dot{H}^k(X)$ stands for $H^k_0(X)$, see Remark 3.1, so we also write $\dot{H}^{k,m}_{\mathrm{b},c}(X) = H^{k,m}_{\mathrm{b},0,c}(X)$ for $k \geq 0$.

Remark 3.16. In this paper we are only concerned with the cases $k = \pm 1$. There is no difference between these two cases for the ensuing discussion, except for the boundary values considered in the next paragraph. For the sake of definiteness, we will use $k = 1$ throughout the discussion. We will also not consider $\dot{H}^k(X)$ explicitly for most of the discussion; there is no difference for the treatment of these spaces either.

We also remark that we can talk about the boundary values of $u \in H^{1,m}_{\mathrm{b},c}(X)$ at boundary hypersurfaces (codimension 1 boundary faces) H_j for $m < 0$, although we do not need this here. One way to do this is to define, for $u = u_1 + Au_2$, $u|_{H_j} = u_1|_{H_j} + \hat{N}_j(A)(0)(u_2|_{H_j})$, regarded e.g. as an element of $\mathcal{C}^{-\infty}(H_j)$ (note that $\hat{N}_j(A)(0) : \mathcal{C}^{-\infty}(H_j) \to \mathcal{C}^{-\infty}(H_j)$), and this is independent of the choices of u_1, u_2 and A. Of course, for $u \in H^{1,m}_{\mathrm{b},0,c}(X)$, in the sense just sketched, $u|_{H_j} = 0$ for all j. It is straightforward to see that for $u \in H^{1,m}_{\mathrm{b},c}$ with $u|_{H_j} = 0$ for all j, there exist $u_1, u_2 \in H^1_{0,c}(X)$ with $u = u_1 + Au_2$, so $u \in H^{1,m}_{\mathrm{b},0,c}(X)$.

We also remark that Lemma 3.7 still holds if one only assumes $u \in H^{1,m}_{\mathrm{b},0,c}(X)$.

First note that given any $K \subset X$ compact there is another $K' \subset X$ compact such that $u \in H^{1,m}_{\mathrm{b},c}(X)$ with $\operatorname{supp} u \subset K$ can be written as $u = u_1 + Au_2$ with $u_1, u_2 \in H^1_c(X)$ both supported in K'. Indeed, let $\phi \in \mathcal{C}^\infty_c(X)$ be identically 1 on a neighborhood of K, and let $G \in \Psi^m_{\mathrm{b}}(X)$ be a properly supported parametrix for A, so $AG = \mathrm{Id} + E$, $E \in \Psi^{-\infty}_{\mathrm{b}}(X)$, E also properly supported. By definition, if $u \in H^{1,m}_{\mathrm{b},c}(X)$ then there are $u_1', u_2' \in H^1_c(X)$ with $u = u_1' + Au_2'$, and as $\phi \equiv 1$ on a neighborhood of $\operatorname{supp} u$, $\phi u = u$. Thus,

$$\begin{gathered} u = \phi u = \phi u_1' - E\phi A u_2' + AG\phi A u_2' = u_1 + u_2, \\ u_1 = \phi u_1' - E\phi A u_2',\ u_2 = G\phi A u_2', \end{gathered}$$

so $u_1, u_2 \in H^1_c(X)$ as $E\phi A, G\phi A \in \Psi^0_{\mathrm{b}}(X)$, and $\operatorname{supp} u_j$, $j = 1, 2$, is bounded in terms of $\operatorname{supp} \phi$, $\operatorname{supp} E$ and $\operatorname{supp} G$. Namely,

$$\begin{aligned} &\operatorname{supp} u_j \subset K', \\ &K' = \operatorname{supp} \phi \cup \pi_L(\operatorname{supp} E \cap \pi_R^{-1}(\operatorname{supp} \phi)) \cup \pi_L(\operatorname{supp} G \cap \pi_R^{-1}(\operatorname{supp} \phi)), \end{aligned}$$

where $\pi_L, \pi_R : X \times X \to X$ are the projections to the left and right factors; K' is compact as E and G are properly supported, so $\operatorname{supp} E \cap \pi_R^{-1}(\operatorname{supp} \phi)$, $\operatorname{supp} G \cap \pi_R^{-1}(\operatorname{supp} \phi)$ are compact. Note also that, by Lemma 3.2, $\|u_1\|_{H^1(X)} + \|u_2\|_{H^1(X)} \leq C(\|u_1'\|_{H^1(X)} + \|u_2'\|_{H^1(X)})$. Since this holds for any u_1', u_2' with $u = u_1' + Au_2'$, we deduce that with this K', if we restrict $\operatorname{supp} u_j \subset K'$, and take inf just over these u_j, we get an equivalent norm on the subspace of $H^1_c(X)$ consisting of elements supported in K.

In fact, as $\operatorname{supp} G$, $\operatorname{supp} E$ can be made to lie in any neighborhood of the diagonal in $X \times X$, and $\operatorname{supp} \phi$ can be made to lie in any neighborhood of K, this argument shows that given any K compact and any U open with $K \subset U$, $\operatorname{supp} u_j$ may be assumed to lie in $K' = \overline{U}$, with the resulting norm equivalent to the $H^1_c(X)$ norm of the definition (with the equivalence constant of course depending on U!).

Moreover, Definition 3.15 is independent of the choice of A. Indeed, if $A' \in \Psi^{-m}_{\mathrm{b}}(X)$ is elliptic and has proper support, then it has a parametrix $G' \in \Psi^m_{\mathrm{b}}(X)$ with $E' = A'G' - \mathrm{Id} \in \Psi^{-\infty}_{\mathrm{b}}(X)$, all with proper support. Then $u = u_1 + Au_2 = u_1 - E'Au_2 + A'G'Au_2$, and $u_1' = u_1 - E'Au_2 \in H^1_c(X)$ since $E'A \in \Psi^{-\infty}_{\mathrm{b}}(X)$, and $u_2' = G'Au_2 \in H^1_c(X)$ since $G'A \in \Psi^0_{\mathrm{b}}(X)$. Moreover, if we fix $K \subset X$ compact, then for u with $\operatorname{supp} u \subset K$, the norms $\|u\|_{H^{1,m}_{\mathrm{b},c}(X)}$ are equivalent for different choices of A – this follows from Lemma 3.2 and the preceeding remark that we may take the support of u_1, u_2 lie in a compact set depending on K only.

Note also that for $F \in \Psi^m_{\mathrm{bc}}(X)$ with compactly supported Schwartz kernel, $F : H^{1,m}_{\mathrm{b},c}(X) \to H^1(X)$ is continuous. Indeed, $Fu = Fu_1 + FAu_2 \in H^1_c(X)$

by Lemma 3.2 since $F, FA \in \Psi^0_{\mathrm{bc}}(X)$ and $u_1, u_2 \in H^1_c(X)$, and this also gives a bound for $\|Fu\|_{H^1(X)}$ in terms of $\|u\|_{H^{1,m}_{\mathrm{b},c}(X)}$ and a seminorm of F in $\Psi^m_{\mathrm{bc}}(X)$. In particular, $\Psi^{-\infty}_{\mathrm{b}}(X)$ maps $H^{1,m}_{\mathrm{b},c}(X) \to H^1(X)$, and indeed into the conormal space $H^{1,\infty}_{\mathrm{b},c}(X)$.

Since any $A \in \Psi^m_{\mathrm{b}}(X)$ defines a map $A : \mathcal{C}^{-\infty}(X) \to \mathcal{C}^{-\infty}(X)$, our definition of the wave front set makes sense for $m < 0$ as well; it is independent of s if we take $u \in H^{1,s}_{\mathrm{loc}}(X)$ since the action of $\Psi_{\mathrm{b}}(X)$ is well-defined on the larger space $\mathcal{C}^{-\infty}(X)$ already.

Definition 3.17. Suppose $u \in H^{1,s}_{\mathrm{loc}}(X)$ for some $s \leq 0$, and suppose that $m \in \mathbb{R}$. We say that $q \in {}^{\mathrm{b}}T^*X \setminus o$ is not in $\mathrm{WF}^{1,m}_{\mathrm{b}}(u)$ if there exists $A \in \Psi^m_{\mathrm{b}}(X)$ such that $\sigma_{b,m}(A)(q) \neq 0$ and $Au \in H^1(X)$.

For $m = \infty$, we say that $q \in {}^{\mathrm{b}}T^*X \setminus o$ is not in $\mathrm{WF}^{1,m}_{\mathrm{b}}(u)$ if there exists $A \in \Psi^0_{\mathrm{b}}(X)$ such that $\sigma_{b,0}(A)(q) \neq 0$ and $LAu \in H^1(X)$ for all $L \in \mathrm{Diff}_{\mathrm{b}}(X)$, i.e. if $Au \in H^{1,\infty}_b(X)$.

Again, the analogues of Lemma 3.9-3.13 remain valid with $H^1(X)$ replaced by $H^{1,s}_{\mathrm{b},c}(X)$ for some s, and m allowed to be negative in $\mathrm{WF}^{1,m}_{\mathrm{b}}(u)$. In particular, Lemma 3.13 takes the form:

Lemma 3.18. *Suppose that $K \subset {}^bS^*X$ is compact, and U a neighborhood of K in ${}^bS^*X$. Let $\tilde{K} \subset X$ compact, and $\tilde{U}$ is a neighborhood of $\tilde{K}$ in X with compact closure. Let $Q \in \Psi^k_b(X)$ be elliptic on K with $\mathrm{WF}'_b(Q) \subset U$, with Schwartz kernel supported in $\tilde{K} \times \tilde{K}$. Let $\mathcal{B}$ be a bounded subset of $\Psi^k_{bc}(X)$ with $\mathrm{WF}'_b(\mathcal{B}) \subset K$ and Schwartz kernel supported in $\tilde{K} \times \tilde{K}$. Then for any $s < 0$ there is a constant $C > 0$ such that for $B \in \mathcal{B}$, $u \in H^{1,s}_{b,loc}(X)$ with $\mathrm{WF}^{1,k}_b(u) \cap U = \emptyset$,*

$$\|Bu\|_{H^1(X)} \leq C(\|u\|_{H^{1,s}_b(\tilde{U})} + \|Qu\|_{H^1(X)}),$$

where $\|u\|_{H^{1,s}_b(\tilde{U})}$ stands for $\|\phi u\|_{H^{1,s}_{b,c}(X)}$ for some fixed $\phi \in \mathcal{C}^\infty_c(X)$ with $\operatorname{supp} \phi \subset \tilde{U}$, $\phi \equiv 1$ on a neighborhood of $\tilde{K}$.

Finally, connecting $H^{k,m}_{\mathrm{b,loc}}(X)$ for $k = \pm 1$, we remark that any $P \in \mathrm{Diff}^2_{\mathrm{b}}(X)$ defines a continuous linear map $P : H^{1,m}_{\mathrm{b,loc}}(X) \to H^{-1,m}_{\mathrm{b,loc}}(X)$, as discussed before the statement of Corollary 3.4; now we need to use (2.3) as well to deduce this.

4. THE ELLIPTIC SET

We first prove an estimate that microlocally controls the Dirichlet form for microlocalized solutions $Pu = 0$, $u \in H^1_0(X)$, in terms of a *lower order* microlocal information and a global bound in $H^1_0(X)$. In fact, as it does not require much additional effort, we consider microlocal solutions, i.e. we make assumptions on $\mathrm{WF}^{-1,\infty}_{\mathrm{b}}(Pu)$, or indeed $\mathrm{WF}^{-1,s}_{\mathrm{b}}(Pu)$.

Remark 4.1. Since X is non-compact and our results are microlocal, we may always fix a compact set $\tilde{K} \subset X$ and assume that all ps.d.o's have Schwartz kernel supported in $\tilde{K} \times \tilde{K}$. We also let $\tilde{U}$ be a neighborhood of $\tilde{K}$ in X such that $\tilde{U}$ has compact closure, and use the $H^1(\tilde{U})$ norm in place of the $H^1(X)$ norm to accommodate $u \in H^1_{0,\mathrm{loc}}(X)$. (We may instead take $\phi \in \mathcal{C}^\infty_c(\tilde{U})$ identically 1 in a neighborhood of $\tilde{K}$, and use $\|\phi u\|_{H^1(X)}$.) Below we use the notation $\|.\|_{H^1_{\mathrm{loc}}(X)}$ for $\|.\|_{H^1(\tilde{U})}$ to avoid having to specify $\tilde{U}$. We also use $\|v\|_{H^{-1}_{\mathrm{loc}}(X)}$ for $\|\phi v\|_{H^{-1}(X)}$.

We give two versions of the Dirichlet estimates: the first one suffices for most purposes, but it does not give the optimal estimates in terms of the order m in $\mathrm{WF}_{\mathrm{b}}^{-1,m}(Pu)$. The second one takes care of this issue.

Lemma 4.2. *Suppose that $K \subset {}^bS^*X$ is compact, $U \subset {}^bS^*X$ is open, $K \subset U$. Suppose that $\mathcal{A} = \{A_r : \ r \in (0,1]\}$ is a bounded family of ps.d.o's in $\Psi^s_{bc}(X)$ with $\mathrm{WF}'_b(\mathcal{A}) \subset K$, and with $A_r \in \Psi_b^{s-1}(X)$ for $r \in (0,1]$. Then there are $G \in \Psi_b^{s-1/2}(X)$, $\tilde{G} \in \Psi_b^{s+1/2}(X)$ with $\mathrm{WF}'_b(G), \mathrm{WF}'_b(\tilde{G}) \subset U$ and $C_0 > 0$ such that for $r \in (0,1]$, $u \in H^1_{0,loc}(X)$ with $\mathrm{WF}_b^{1,s-1/2}(u) \cap U = \emptyset$, $\mathrm{WF}_b^{-1,s+1/2}(Pu) \cap U = \emptyset$, we have*

$$\begin{aligned}
&\Big|\int_X \big(|d_M A_r u|^2 - |D_t A_r u|^2\big)\Big| \\
&\qquad \le C_0(\|u\|^2_{H^1_{loc}(X)} + \|Gu\|^2_{H^1(X)} + \|Pu\|^2_{H^{-1}_{loc}(X)} + \|\tilde{G}Pu\|^2_{H^{-1}(X)}).
\end{aligned}$$

In particular, if the assumption on Pu is strengthened to $Pu = 0$, we have

$$\Big|\int_X \big(|d_M A_r u|^2 - |D_t A_r u|^2\big)\Big| \le C_0(\|u\|^2_{H^1_{loc}(X)} + \|Gu\|^2_{H^1(X)}).$$

The meaning of $\|u\|^2_{H^1_{\mathrm{loc}}(X)}$ and $\|Pu\|^2_{H^{-1}_{\mathrm{loc}}(X)}$ is stated above in Remark 4.1.

Remark 4.3. The point of this lemma is G is $1/2$ order lower ($s-1/2$ vs. s) than the *family* $\mathcal{A}$. We will later take a limit, $r \to 0$, which gives control of the Dirichlet form evaluated on $A_0 u$, $A_0 \in \Psi^s_{\mathrm{bc}}(X)$, in terms of lower order information.

The role of A_r, $r > 0$, is to regularize such an argument, i.e. to make sure various terms in a formal computation, in which one uses A_0 directly, actually make sense.

Proof. Then for $r \in (0,1]$, $A_r u \in H^1_0(X)$, so

$$\int_X (|d_M A_r u|^2 - |D_t A_r u|^2) = -\int_X P A_r u \, \overline{A_r u}.$$

Here the right hand side is the pairing of $H^{-1}(X)$ with $H^1_0(X)$. Writing $PA_r = A_r P + [P, A_r]$, and $\langle v, w\rangle = \int_X v\,\overline{w}$ for the L^2-pairing on X, we see that the right hand side can be estimated by

$$|\langle A_r Pu, A_r u\rangle| + |\langle [P, A_r]u, A_r u\rangle|. \tag{4.1}$$

The lemma is thus proved if we show that the first term of (4.1) is bounded by

$$C_0'(\|u\|^2_{H^1_{\mathrm{loc}}(X)} + \|Gu\|^2_{H^1(X)} + \|Pu\|^2_{H^{-1}_{\mathrm{loc}}(X)} + \|\tilde{G}Pu\|^2_{H^{-1}(X)}), \tag{4.2}$$

the second term is bounded by $C_0''(\|u\|^2_{H^1_{\mathrm{loc}}(X)} + \|Gu\|^2_{H^1(X)})$. *(Recall that the 'local' norms were defined in Remark 4.1.)*

The first term is straightforward to estimate. Let $\Lambda \in \Psi_{\mathrm{b}}^{-1/2}(X)$ be elliptic with $\Lambda^- \in \Psi_{\mathrm{b}}^{1/2}(X)$ a parametrix, so

$$E = \Lambda\Lambda^- - \mathrm{Id}, E' = \Lambda^-\Lambda - \mathrm{Id} \in \Psi_{\mathrm{b}}^{-\infty}(X).$$

Then

$$\begin{aligned}
\int_X A_r Pu\,\overline{A_r u} &= \int_X (\Lambda\Lambda^- - E) A_r Pu\,\overline{A_r u} \\
&= \int_X \Lambda^- A_r Pu\,\overline{\Lambda^* A_r u} - \int_X A_r Pu\,\overline{E^* A_r u}.
\end{aligned}$$

Since $\Lambda^- A_r$ is uniformly bounded in $\Psi_{\mathrm{bc}}^{s+1/2}(X)$, and $\Lambda^* A_r$ is uniformly bounded in $\Psi_{\mathrm{bc}}^{s-1/2}(X)$, $\int_X \Lambda^- A_r Pu\,\overline{\Lambda^* A_r u}$ is uniformly bounded, with a bound like (4.2) using Cauchy-Schwartz and Lemma 3.13. Indeed, by Lemma 3.13, choosing any $G \in \Psi_{\mathrm{b}}^{s-1/2}(X)$ which is elliptic on K, there is a constant $C_1 > 0$ such that

$$\|\Lambda^* A_r u\|^2_{H^1(X)} \leq C_1(\|u\|^2_{H^1_{\mathrm{loc}}(X)} + \|Gu\|^2_{H^1(X)}).$$

Similarly, by Lemma 3.13 and the remark following Definition 3.14, choosing any $\tilde{G} \in \Psi_{\mathrm{b}}^{s+1/2}(X)$ which is elliptic on K, there is a constant $C_1' > 0$ such that $\|\Lambda^- A_r Pu\|^2_{H^{-1}(X)} \leq C_1'(\|Pu\|^2_{H^{-1}_{\mathrm{loc}}(X)} + \|\tilde{G}Pu\|^2_{H^{-1}(X)})$. Combining these gives, with $C_0' = C_1 + C_1'$,

$$\begin{aligned}|\int_X \Lambda^- A_r Pu\,\overline{\Lambda^* A_r u}| &\leq \|\Lambda^- A_r Pu\|\,\|\Lambda^* A_r u\| \leq \|\Lambda^- A_r Pu\|^2 + \|\Lambda^* A_r u\|^2\\ &\leq C_0'(\|u\|^2_{H^1_{\mathrm{loc}}(X)} + \|Gu\|^2_{H^1(X)} + \|Pu\|^2_{H^{-1}_{\mathrm{loc}}(X)} + \|\tilde{G}Pu\|^2_{H^{-1}(X)}),\end{aligned}$$

as desired.

A similar argument, using that A_r is uniformly bounded in $\Psi_{\mathrm{bc}}^{s+1/2}(X)$ (in fact in $\Psi_{\mathrm{bc}}^{s}(X)$), and $E^* A_r$ is uniformly bounded in $\Psi_{\mathrm{bc}}^{s-1/2}(X)$ (in fact in $\Psi_{\mathrm{bc}}^{-\infty}(X)$), shows that $\int_X A_r Pu\,\overline{E^* A_r u}$ is uniformly bounded.

Now we turn to the second term in (4.1). Using (2.3) and Lemma 2.2,

$$[P, A_r] = \sum_{i,j} D_{x_i} D_{x_j} B_{ij,r} + \sum_j D_{x_j} B_{j,r} + B_r,$$

$B_r \in \Psi_{\mathrm{b}}^{s}(X)$, $B_{j,r} \in \Psi_{\mathrm{b}}^{s-1}(X)$, $B_{ij,r} \in \Psi_{\mathrm{b}}^{s-2}(X)$, uniformly bounded in $\Psi_{\mathrm{bc}}^{s+1}(X)$, resp. $\Psi_{\mathrm{bc}}^{s}(X)$, resp. $\Psi_{\mathrm{bc}}^{s-1}(X)$. With $\Lambda \in \Psi_{\mathrm{b}}^{-1/2}(X)$ as above, utilizing (2.3), we can write further

$$\Lambda^- \sum_{i,j} D_{x_i} D_{x_j} B_{ij,r} = \sum_{i,j} D_{x_i} D_{x_j} B'_{ij,r} + \sum D_{x_j} B'_{j,r} + B'_r,$$

with $B'_{ij,r}, B'_{j,r}, B'_r \in \Psi_{\mathrm{b}}^{s-3/2}(X)$, uniformly bounded in $\Psi_{\mathrm{bc}}^{s-1/2}(X)$. Thus,

$$\begin{aligned}&\langle [P, A_r]u, A_r u\rangle\\ &\quad= \sum_{ij} \langle \Lambda^- D_{x_i} D_{x_j} B_{ij,r} u, \Lambda^* A_r u\rangle - \sum_{ij} \langle D_{x_i} D_{x_j} B_{ij,r} u, E^* A_r u\rangle\\ &\quad+ \langle \Lambda^* (\sum_j D_{x_j} B_{j,r} + B_r) u, \Lambda^- A_r u\rangle - \langle E^* (\sum_j D_{x_j} B_{j,r} + B_r) u, \Lambda^- A_r u\rangle.\end{aligned} \tag{4.3}$$

Note that Λ^-, Λ^* and E^* are positioned differently for the first two, resp. last two terms; this is so that after integration by parts in the first two terms, moving D_{x_i} to $\Lambda^* A_r u$, resp. $E^* A_r u$, each of the two terms being paired involve operators of uniform order $s + 1/2$, when the derivatives D_{x_i}, etc., are included in the order count. (We need to integrate by parts so that at most one normal derivative falls on each of the two terms being paired, since we are working relative to $H^1(X)$.)

The first two terms on the right hand side of (4.3) can be expanded as

$$\begin{aligned}
&\sum_{ij}\int_X D_{x_i}D_{x_j}B'_{ij,r}u\,\overline{\Lambda^* A_r u}-\sum_{ij}\int_X D_{x_i}D_{x_j}B_{ij,r}u\,\overline{E^* A_r u}\\
&\qquad+\sum_j\int_X D_{x_j}B'_{j,r}u\,\overline{\Lambda^* A_r u}+\int_X B'_r u\,\overline{\Lambda^* A_r u}\\
&=\sum_{ij}\int_X D_{x_j}B'_{ij,r}u\,\overline{D^t_{x_i}\Lambda^* A_r u}-\sum_{ij}\int_X D_{x_j}B_{ij,r}u\,\overline{D^t_{x_i}E^* A_r u}\\
&\qquad+\sum_j\int_X D_{x_j}B'_{j,r}u\,\overline{\Lambda^* A_r u}+\int_X B'_r u\,\overline{\Lambda^* A_r u},
\end{aligned}\tag{4.4}$$

where $D^t_{x_i}$ is the formal adjoint of D_{x_i} with respect to dg, and where in the last step we used that

$$B'_{ij,r}u, B_{ij,r}u, \Lambda^* A_r u, E^* A_r u \in H^1_0(X).$$

Note that $D^t_{x_i} = J^{-1}D_{x_i}J$ if $dg = J dx_1 \ldots dx_k\, dy_1 \ldots dy_l$ is the Riemannian density, so $D^t_{x_i} = D_{x_i} + b$, $b \in \mathcal{C}^\infty(X)$. Thus,

$$\begin{aligned}
|\int_X D_{x_j}B'_{ij,r}u\,\overline{D^t_{x_i}\Lambda^* A_r u}| \le \|D_{x_j}B'_{ij,r}u\|_{L^2(X)}\|D_{x_i}\Lambda^* A_r u\|_{L^2(X)}\\
+\|D_{x_j}B'_{ij,r}u\|_{L^2(X)}\|\Lambda^* A_r u\|_{L^2(X)},
\end{aligned}$$

and both factors in both terms are uniformly bounded for $r \in (0,1]$ since $\Lambda^* A_r$, $B'_{ij,r}$ are uniformly bounded in $\Psi^{s-1/2}_{\mathrm{bc}}(X)$ with a uniform wave front bound disjoint from $\mathrm{WF}^{1,s-1/2}_{\mathrm{b}}(u)$. Indeed, as noted above, by Lemma 3.13, choosing any $G \in \Psi^{s-1/2}_{\mathrm{b}}(X)$ which is elliptic on K, there is a constant $C_1 > 0$ such that the right hand side is bounded by $C_1(\|u\|^2_{H^1_{\mathrm{loc}}(X)} + \|Gu\|^2_{H^1(X)})$. Similar estimates apply to the other terms on the right hand side of (4.4), and the last two terms on the right hand side of (4.3) can be treated similarly, showing that $\int_X [P, A_r]u\,\overline{A_r u}$ is uniformly bounded for $r \in (0,1]$, indeed is bounded by $C_0(\|u\|^2_{H^1_{\mathrm{loc}}(X)}+\|Gu\|^2_{H^1(X)})$, proving the lemma. □

The lemma which allows more precise estimates is the following.

Lemma 4.4. *Suppose that $K \subset {}^bS^*X$ is compact, $U \subset {}^bS^*X$ is open, $K \subset U$. Suppose that $\mathcal{A} = \{A_r : r \in (0,1]\}$ is a bounded family of ps.d.o's in $\Psi^s_{bc}(X)$ with $\mathrm{WF}'_b(\mathcal{A}) \subset K$, and with $A_r \in \Psi^{s-1}_b(X)$ for $r \in (0,1]$. Then there are $G \in \Psi^{s-1/2}_b(X)$, $\tilde G \in \Psi^s_b(X)$ with $\mathrm{WF}'_b(G), \mathrm{WF}'_b(\tilde G) \subset U$ and $C_0 > 0$ such that for $\epsilon > 0$, $r \in (0,1]$, $u \in H^1_{0,loc}(X)$ with $\mathrm{WF}^{1,s-1/2}_b(u) \cap U = \emptyset$, $\mathrm{WF}^{-1,s}_b(Pu) \cap U = \emptyset$, we have*

$$\begin{aligned}
|\int_X \big(|d_M A_r u|^2 - |D_t A_r u|^2\big)| \le \epsilon\|d_X A_r u\|^2_{L^2(X)} + C_0(\|u\|^2_{H^1_{loc}(X)} + \|Gu\|^2_{H^1(X)}\\
+\epsilon^{-1}\|Pu\|^2_{H^{-1}_{loc}(X)} + \epsilon^{-1}\|\tilde G P u\|^2_{H^{-1}(X)}).
\end{aligned}$$

Remark 4.5. The point of this lemma is that on the one hand the new term $\epsilon\|d_X A_r u\|^2$ can be absorbed in the left hand side in the elliptic region, hence is negligible, on the other hand, there is a gain in the order of $\tilde G$ (s, versus $s + 1/2$ in the previous lemma).

Proof. We only need to modify the previous proof slightly. Thus, we need to estimate the term $|\int_X A_r Pu\,\overline{A_r u}|$ in (4.1) differently, namely

$$|\int_X A_r Pu\,\overline{A_r u}| \leq \|A_r Pu\|_{H^{-1}(X)}\|A_r u\|_{H^1(X)} \leq \epsilon\|A_r u\|^2_{H^1(X)}+\epsilon^{-1}\|A_r Pu\|^2_{H^{-1}(X)}.$$

Now the lemma follows by using Lemma 3.13 and the remark following Definition 3.14, namely choosing any $\tilde{G} \in \Psi^s_{\mathrm{b}}(X)$ which is elliptic on K, there is a constant $C_1' > 0$ such that $\|A_r Pu\|^2_{H^{-1}(X)} \leq C_1'(\|Pu\|^2_{H^{-1}_{\mathrm{loc}}(X)} + \|\tilde{G}Pu\|^2_{H^{-1}(X)})$, and finishing the proof exactly as for Lemma 4.2. □

Using the microlocal positivity of the Dirichlet form, we now prove the elliptic estimates. Recall that $\pi : T^*X \to {}^{\mathrm{b}}T^*X$ is the natural 'inclusion' map, and ${}^{\mathrm{b}}\dot{T}^*X \subset {}^{\mathrm{b}}T^*X$ is its range.

Proposition 4.6. *(Microlocal elliptic regularity.) If* $u \in H^1_{0,loc}(X)$ *then*

$$\mathrm{WF}_b^{1,m}(u) \subset \mathrm{WF}_b^{-1,m}(Pu) \cup {}^b\dot{T}^*X, \text{ and } \mathrm{WF}_b^{1,m}(u) \cap \mathcal{E} \subset \mathrm{WF}_b^{-1,m}(Pu).$$

In particular, if $Pu = 0$, $u \in H^1_{0,loc}(X)$ *then*

$$\mathrm{WF}_b^{1,\infty}(u) \subset {}^b\dot{T}^*X, \text{ and } \mathrm{WF}_b^{1,\infty}(u) \cap \mathcal{E} = \emptyset.$$

Proof. We first prove a slightly weaker result in which $\mathrm{WF}_{\mathrm{b}}^{-1,m}(Pu)$ is replaced by $\mathrm{WF}_{\mathrm{b}}^{-1,m+1/2}(Pu)$ – we rely on Lemma 4.2. We then prove the original statement using Lemma 4.4.

Suppose that either $q \in {}^{\mathrm{b}}T^*X \setminus {}^{\mathrm{b}}\dot{T}^*X$ or $q \in \mathcal{E}$. We may assume iteratively that $q \notin \mathrm{WF}_{\mathrm{b}}^{1,s-1/2}(u)$; we need to prove then that $q \notin \mathrm{WF}_{\mathrm{b}}^{1,s}(u)$ (note that the inductive hypothesis holds for $s = 1/2$ since $u \in H^1_{\mathrm{loc}}(X)$). Let $A \in \Psi^s_{\mathrm{b}}(X)$ be such that $\mathrm{WF}'_{\mathrm{b}}(A) \cap \mathrm{WF}_{\mathrm{b}}^{1,s-1/2}(u) = \emptyset$, $\mathrm{WF}'_{\mathrm{b}}(A) \cap \mathrm{WF}_{\mathrm{b}}^{1,s+1/2}(Pu) = \emptyset$, and have $\mathrm{WF}'_{\mathrm{b}}(A)$ in a small conic neighborhood U of q so that for a suitable $C > 0$ or $\epsilon > 0$, in U

(i) $\tau^2 < C\sum_j \sigma_j^2$ if $q \in {}^{\mathrm{b}}T^*X \setminus {}^{\mathrm{b}}\dot{T}^*X$,
(ii) $|\sigma_j| < \epsilon(\tau^2 + |\zeta|^2)^{1/2}$ for all j, and $\frac{|\zeta|}{|\tau|} > 1 + \epsilon$, if $q \in \mathcal{E}$.

Let $\Lambda_r \in \Psi_{\mathrm{b}}^{-2}(X)$ for $r > 0$, such that $\mathcal{L} = \{\Lambda_r : r \in (0,1]\}$ is a bounded family in $\Psi^0_{\mathrm{b}}(X)$, and $\Lambda_r \to \mathrm{Id}$ as $r \to 0$ in $\Psi^{\tilde{\epsilon}}_{\mathrm{b}}(X)$, $\tilde{\epsilon} > 0$, e.g. the symbol of Λ_r could be taken as $(1 + r(\tau^2 + |\zeta|^2 + |\sigma|^2))^{-1}$. Let $A_r = \Lambda_r A$. Let a be the symbol of A, and let A_r have symbol $(1 + r(\tau^2 + |\zeta|^2 + |\sigma|^2))^{-1}a$, $r > 0$, so $A_r \in \Psi_{\mathrm{b}}^{s-2}(X)$ for $r > 0$, and A_r is uniformly bounded in $\Psi^s_{\mathrm{bc}}(X)$, $A_r \to A$ in $\Psi^{s+\tilde{\epsilon}}_{\mathrm{bc}}(X)$.

By Lemma 4.2,

$$\int_X \left(|d_M A_r u|^2 - |D_t A_r u|^2\right)$$

is uniformly bounded for $r \in (0,1]$. On the other hand,

$$\begin{aligned}\int_X |d_M A_r u|^2 = \int_X \sum A_{ij} D_{x_i} A_r u\,\overline{D_{x_j} A_r u} &+ \int_X \sum B_{ij} D_{y_i} A_r u\,\overline{D_{y_j} A_r u} \\ &+ \int_X \sum C_{ij} D_{x_i} A_r u\,\overline{D_{y_j} A_r u}.\end{aligned}$$

Using that $A_{ij}(x,y) = A_{ij}(0,y) + \sum x_k A'_{ijk}(x,y)$, we see that if A_r is supported in $x_k < \delta$ for all k,

$$|\int_X \sum x_k A'_{ijk} D_{x_i} A_r u \, \overline{D_{x_j} A_r u}| \leq C\delta \sum_{i',j'} \|D_{x_{i'}} A_r u\| \, \|D_{x_{j'}} A_r u\|, \tag{4.5}$$

with analogous estimates for $B_{ij}(x,y) - B_{ij}(0,y)$ and for $C_{ij}(x,y)$. Moreover, as the matrix A_{ij} is positive definite, for some $c > 0$,

$$c \int_X \sum_j |D_{x_j} A_r u|^2 \leq \frac{1}{2} \int_X \sum_{ij} A_{ij} D_{x_i} A_r u \overline{D_{x_j} A_r u}.$$

Thus, there exists $\tilde{C} > 0$ and $\delta_0 > 0$ such that if $\delta < \delta_0$ and A is supported in $|x| < \delta$ then

$$\begin{aligned} c \int_X \sum_j |D_{x_j} A_r u|^2 + \int_X ((1 - \tilde{C}\delta) \sum_j |D_{y_j} A_r u|_h^2 - |D_t A_r u|^2) \\ \leq \int_X (|d_M A_r u|^2 - |D_t A_r u|^2), \end{aligned} \tag{4.6}$$

where we used the notation

$$\sum_j |D_{y_j} A_r u|_h^2 = \sum_{ij} B_{ij}(0,y) D_{y_i} A_r u \, \overline{D_{y_j} A_r u},$$

i.e. h is the dual metric g restricted to the span of the dy_j, $j = 1, \ldots, l$.

Now we distinguish the cases $q \in \mathcal{E}$ and $q \in {}^{\mathrm{b}}T^*X \setminus {}^{\mathrm{b}}\dot{T}^*X$. If $q \in \mathcal{E}$, A is supported near $\mathcal{E}$, we choose $\delta \in (0, \frac{1}{2\tilde{C}})$ so that $(1 - \tilde{C}\delta)\frac{|\zeta|^2}{\tau^2} > 1 + \delta$ on a neighborhood of $\mathrm{WF}'_{\mathrm{b}}(A)$, which is possible in view of (ii) at the beginning of the proof. Then the second integral on the left hand side of (4.6) can be written as $\|BA_r u\|^2$, with the symbol of B given by $((1 - \tilde{C}\delta)|\zeta|^2 - \tau^2)^{1/2}$)(which is $\geq \delta\tau$), modulo a term

$$\int_X F A_r u \, \overline{A_r u}, \;\; F \in \Psi^1_{\mathrm{b}}(X).$$

But this expression is uniformly bounded as $r \to 0$ by the argument above. We thus deduce that

$$c \int_X (\sum_j |D_{x_j} A_r u|^2) + \|BA_r u\|^2$$

is uniformly bounded as $r \to 0$.

If $q \in {}^{\mathrm{b}}T^*X \setminus {}^{\mathrm{b}}\dot{T}^*X$, and A is supported in $|x| < \delta$,

$$\int_X \delta^{-2} |x_j D_{x_j} A_r u|^2 \leq \int_X |D_{x_j} A_r u|^2,$$

On the other hand, near ${}^{\mathrm{b}}T^*X \setminus {}^{\mathrm{b}}\dot{T}^*X$, for $\delta > 0$ sufficiently small,

$$\int_X \left(\frac{c}{2\delta^2} \sum_j |x_j D_{x_j} A_r u|^2 - |D_t A_r u|^2 \right) = \|BA_r u\|^2 + \int_X F A_r u \, \overline{A_r u},$$

with the symbol of B given by $(\frac{c}{2\delta^2} \sum \sigma_j^2 - \tau^2)^{1/2}$ (which does not vanish on U for $\delta > 0$ small), while $F \in \Psi^1_{\mathrm{b}}(X)$, so the second term on the right hand side is uniformly bounded as $r \to 0$. We thus deduce in this case that

$$\frac{c}{2}\int_X (\sum_j |D_{x_j} A_r u|^2) + \|BA_r u\|^2$$

is uniformly bounded as $r \to 0$.

We thus conclude that $D_{x_j} A_r u, BA_r u$ are uniformly bounded $L^2(X)$. Correspondingly there are sequences $D_{x_j} A_{r_k} u$, $BA_{r_k} u$, weakly convergent in $L^2(X)$, and such that $r_k \to 0$, as $k \to \infty$. Since they converge to $D_{x_j} Au$, BAu, respectively, in $\mathcal{C}^{-\infty}(X)$, we deduce that the weak limits are $D_{x_j} Au$, BAu, which therefore lie in $L^2(X)$. Consequently, $dAu \in L^2(X)$ proving the proposition with $\mathrm{WF}_{\mathrm{b}}^{-1,m}(Pu)$ replaced by $\mathrm{WF}_{\mathrm{b}}^{-1,m+1/2}(Pu)$.

To obtain the optimal result, we note that due to Lemma 4.4 we still have, for any $\epsilon > 0$, that

$$\begin{aligned}&\int_X \big(|d_M A_r u|^2 - |D_t A_r u|^2 - \epsilon |d_X A_r u|^2\big)\\ &\qquad = \int_X \big((1-\epsilon)|d_M A_r u|^2 - (1+\epsilon)|D_t A_r u|^2\big)\end{aligned}$$

is uniformly bounded above for $r \in (0,1]$. (Keep in mind that $d_X = (d_M, \partial_t)$ with respect to the product decomposition of X.) By arguing just as above, with B as above, for sufficiently small $\epsilon > 0$, the right hand side gives an upper bound for

$$\frac{c}{2}\int_X (\sum_j |D_{x_j} A_r u|^2) + \|BA_r u\|^2,$$

which is thus uniformly bounded as $r \to 0$. The proof is then finished exactly as above. □

A slightly different formulation of this argument is the following. Below $w = (x, y)$. Consider

$$\begin{aligned}&\|d_M A_r u\|^2 - \|D_t A_r u\|^2\\ &\qquad = \int_X \sum_{i,j} g^{ij} D_{w_i} A_r u\, \overline{D_{w_j} A_r u}\, J\, dw\, dt - \int_X D_t A_r u\, \overline{D_t A_r u}\, J\, dw\, dt.\end{aligned}$$

We move the A_r in the first factor of each term on the right hand side by first commuting it through $g^{ij} D_{w_i}$ (or D_t), then taking its adjoint with respect to $J\,dw\,dt$, and finally commuting it through D_{w_j}. Each of the commutator terms can be controlled by the inductive hypothesis as above. Modulo such terms the result is

$$\int_X \left(\sum_{i,j} g^{ij} D_{w_i} u\, \overline{D_{w_j} A_r^* A_r u} - D_t u\, \overline{D_t A_r^* A_r u}\right) J\, dw\, dt. \tag{4.7}$$

But by definition, a solution of the wave equation $Pu = f$ satisfying the Dirichlet boundary condition is $u \in H^1_{0,\mathrm{loc}}(X)$ with

$$\int_X \left(\sum_{i,j} g^{ij} D_{w_i} u\, \overline{D_{w_j} v} - D_t u\, \overline{D_t v}\right) J\, dw\, dt = -\int_X f\, \overline{v}\, J\, dw\, dt$$

for every $v \in H^1_{0,c}(X)$. In particular, as $A_r^* A_r$ preserves $H^1_{0,\mathrm{loc}}(X)$, this holds for $v = A_r^* A_r u$ when A_r has a compactly supported Schwartz kernel. If $f \in \dot{\mathcal{C}}^\infty(X)$, e.g. if $f = 0$, the right hand side now can also be estimated by the inductive hypothesis, showing that $\|d_M A_r u\|^2 - \|D_t A_r u\|^2$ is uniformly bounded as $r \to 0$. The rest of the arguments presented above apply then, so we can conclude that $q \notin \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u)$ as above.

This argument is immediately applicable for Neumann boundary conditions as well. Thus, we still get (4.7) modulo terms that can be estimated by the inductive hypothesis. Now, by definition, a solution of the wave equation $Pu = f$ satisfying the Neumann boundary condition is $u \in H^1_{\mathrm{loc}}(X)$ with

$$\int_X \left(\sum_{i,j} g^{ij} D_{w_i} u \, \overline{D_{w_j} v} - D_t u \, \overline{D_t v} \right) J \, dw \, dt = - \int_X f \, \overline{v} \, J \, dw \, dt \tag{4.8}$$

for every $v \in H^1_c(X)$. Here, for $f \in \dot{H}^{-1}_{\mathrm{loc}}(X)$, the right hand side is the pairing of $\dot{H}^{-1}_{\mathrm{loc}}(X)$ with $H^1_c(X)$ via duality. In particular, as $A_r^* A_r$ preserves $H^1_{\mathrm{loc}}(X)$, this holds for $v = A_r^* A_r u$, and the rest of the elliptic argument is as for the Dirichlet boundary condition.

We use this opportunity to remark that our methods also immediately give elliptic regularity for the Laplacian on M.

Theorem 4.7. *(Microlocal elliptic regularity for Δ.) Suppose that $u \in H^1_{0,loc}(M)$, and $\Delta u = f$, i.e.*

$$\langle du, dv \rangle_M = \langle f, v \rangle_M$$

for all $v \in H^1_{0,c}(M)$; here $\langle \cdot, \cdot \rangle_M$ is the L^2 inner product on M. Then $\mathrm{WF}_b^{1,m}(u) \subset \mathrm{WF}_b^{-1,m}(f)$.

In particular, if $f \in H^{-1,m}_{b,loc}(M)$ then $u \in H^{1,m}_{b,loc}(M)$.

The same conclusions hold for Neumann boundary conditions, i.e. with $H^1_0(M)$ replaced by $H^1(M)$.

Corollary 4.8. *Suppose that $u \in H^1_{0,loc}(M)$, and $(\Delta - \lambda)u = 0$. Then $u \in H^{1,\infty}_{b,loc}(M)$. The conclusion also holds if u satisfies Neumann boundary conditions.*

Proof. We have $\Delta u = f$ with $f = \lambda u \in H^1_{0,\mathrm{loc}}(M) \subset H^{-1,2}_{\mathrm{b,loc}}(M)$, so $u \in H^{1,2}_{\mathrm{b,loc}}(M)$. Iterating this, using $H^{1,m}_{\mathrm{b,loc}} \subset H^{-1,m+2}_{\mathrm{b,loc}}(M)$, completes the proof. □

5. Bicharacteristics

In this section we state the basic properties of generalized broken bicharacteristics that are instrumental in proving the propagation of singularities theorem in Section 8.1. The philosophy originating from the work of Melrose and Sjöstrand [13, 14] is that it is easier to analyze the bicharacteristics (i.e. the 'classical' system) precisely, and prove only rough propagation estimates for the 'quantum' system (in this case the wave equation), essentially merely getting the direction of the propagation correct, than to prove the precise propagation statements directly, for many different aspects (not only the classical geometry) interact in the latter setting. The precise propagation statement is thus a combination of the rough propagation statements with the detailed analysis of the bicharacteristics – this is the content of Section 8 here.

Turning to the generalized broken bicharacteristics, these have been described by Lebeau [11, Section III] in his setting, i.e. for domains M in real analytic manifolds

$\tilde{M}$, equipped with a real analytic metric g, with the boundary of M admitting a stratification. However, analyticity does *not* enter into the analysis of generalized broken bicharacteristic (called 'rayons' there), and manifolds with corners, by definition, admit the desired stratification (stratified by the boundary faces), in a $\mathcal{C}^\infty$ sense. Thus, all of Lebeau's results on generalized broken bicharacteristics apply in our setting, at least if one adopts his definitions.

Our definition differs from that of Lebeau in two ways. First, at boundary hypersurfaces (i.e. codimension 1 faces), Definition 1.1, part (iii), demands more than Lebeau's definition (from which (iii) is missing). Thus, our bicharacteristics are a subset of those of Lebeau's. However, since the analysis of bicharacteristics is local in X, the $\mathcal{C}^\infty$ boundary analysis of Melrose and Sjöstrand applies. As this only necessitates trivial changes, we point these out below after the statement of the propositions of this section.

The other difference is that we defined the topology of $\dot\Sigma$ as the subspace topology inherited from ${}^{\mathrm{b}}T^*X$, while Lebeau defines it by requiring that $\hat\pi$ be continuous, so we need to show that these are indeed the same, which we proceed to do now.

Lemma 5.1. *Define the topology of $\dot\Sigma$ as the subspace topology of ${}^{\mathrm{b}}T^*X$. Then $O \subset \dot\Sigma$ is open (resp. closed) if and only if $\hat\pi^{-1}(O)$ is open (resp. closed).*

Since the bundle inclusion map $\pi : T^*X \to {}^{\mathrm{b}}T^*X$ is $\mathcal{C}^\infty$, hence continuous, $\hat\pi$ is automatically continuous, so it only remains to show that if $\hat\pi^{-1}(O)$ is open, then O is open, which we do below.

First, however we remark that a basis of the subspace topology is given by

$$\begin{aligned} B_\delta(q_0) = \{q \in \dot\Sigma : |x(q)| < \delta,\ |y(q) - y_0(q)| < \delta,\ |t(q) - t(q_0)| < \delta,\\ |\tau(q) - \tau(q_0)| < \delta,\ |\zeta(q) - \zeta(q_0)| < \delta\}, \end{aligned} \tag{5.1}$$

as q_0 and $\delta > 0$ vary. Indeed, on $\dot\Sigma = \pi(\mathrm{Char}(P))$, $|\sigma(q)| \leq C|x(q)|\,|\tau(q)|$ over compact subsets of X. Assuming $\delta < 1$, $\delta < |\tau(q_0)|/2$, as we may, the above inequalities imply that $|\sigma(q)| < 2C\delta|\tau(q_0)|$. Given $\delta_0 > 0$, this set will thus be included in a δ_0-ball in ${}^{\mathrm{b}}T^*X$, centered at q_0, provided we choose $\delta < \delta_0/2C|\tau(q_0)|$, so every neighborhood of q_0 in $\dot\Sigma$ contains a set of the form (5.1).

Proof of Lemma 5.1. We now show that if $\hat\pi^{-1}(O)$ is open, then so is O. That is, we need to show for any set O with $\hat\pi^{-1}(O)$ open, and for any $q_0 \in O \cap T^*\mathcal{F}_{i,\mathrm{reg}}$, there is a $\delta > 0$ such that $B_\delta(q_0) \subset O$. But $\hat\pi^{-1}(\{q_0\})$ is the set of points $\tilde q_0 = (x, y, t, \xi, \zeta, \tau)$ in T^*X with $(x, y, t, \xi, \zeta, \tau) = (0, y(q_0), t(q_0), \xi, \zeta(q_0), \tau(q_0))$ and $\xi \cdot A(y(q_0))\xi = \tau(q_0)^2 - |\zeta(q_0)|^2_{y(q_0)}$. As A is positive definite, the last equation implies that ξ is bounded on $\hat\pi^{-1}(\{q_0\})$, and indeed $\hat\pi^{-1}(\{q_0\})$ is compact. So if $\hat\pi^{-1}(O)$ open, then for some $\delta > 0$ it contains the intersection of $\mathrm{Char}(P)$ with the set

$$\begin{aligned} \{\tilde q \in T^*X : |x(\tilde q)| < \delta,\ |y(\tilde q) - y(q_0)| < \delta,\ |t(\tilde q) - t(q_0)| < \delta,\\ |\tau(\tilde q) - \tau(q_0)| < \delta,\ |\zeta(\tilde q) - \zeta(q_0)| < \delta, |p(\tilde q)| < \delta\}, \end{aligned}$$

i.e. it contains the set

$$\begin{aligned} \tilde B_\delta(q_0) = \{\tilde q \in \mathrm{Char}(P) : |x(\tilde q)| < \delta,\ |y(\tilde q) - y(q_0)| < \delta,\ |t(\tilde q) - t(q_0)| < \delta,\\ |\tau(\tilde q) - \tau(q_0)| < \delta,\ |\zeta(\tilde q) - \zeta(q_0)| < \delta\}. \end{aligned}$$

Now $\hat\pi(\tilde B_\delta) = B_\delta(q_0)$, while $\hat\pi(\hat\pi^{-1}(O)) = O$, so we deduce that $B_\delta(q_0) \subset O$, and hence O is open as claimed. $\square$

Being a subset of ${}^{\mathrm{b}}T^*X$, $\dot{\Sigma}$ is a separable, locally compact metrizable space, although this follows also directly using the topology induced by $\hat{\pi}$ as in Lebeau's paper.

A stronger characterization of generalized broken bicharacteristics at $\mathcal{H}$ follows as in Lebeau's paper.

Proposition 5.2. *(Lebeau,* [11, Proposition 1]*) If γ is a generalized broken bicharacteristic, $t_0 \in I$, $q_0 = \gamma(t_0)$, then there exist unique $\tilde{q}_+, \tilde{q}_- \in \mathrm{Char}(P)$ satisfying $\pi(\tilde{q}_\pm) = q_0$ and having the property that if $f \in \mathcal{C}^\infty(T^*X)$ is π-invariant then $t \mapsto f_\pi(\gamma(t))$ is differentiable both from the left and from the right at t_0 and*

$$\left(\frac{d}{dt}\right)(f_\pi \circ \gamma)|_{t_0\pm} = H_p f(\tilde{q}_\pm). \tag{5.2}$$

Notice that if $\gamma : I \to \dot{\Sigma}$ is continuous and if in addition the conclusion of the following proposition holds, then (i) and (ii) of Definition 1.1 follow ((ii) follows as x_j are π-invariant), so the proposition indeed provides an alternative to (i)-(ii) of our definition. Note that (iii) is not required for this proposition, and conversely, it does not imply (iii). (We also remark paranthetically that there is yet another way of phrasing (i) and (ii) in the definition of generalized broken bicharacteristics, which is important in N-body scattering in the presence of bound states, see [25, Definition 2.1].)

Corollary 5.3. *(Lebeau,* [11, Corollaire 2]*) Suppose that K is a compact subset of $\dot{\Sigma}$. Then there is a constant $C > 0$ such that for all generalized broken bicharacteristics $\gamma : I \to K$, and for all π-invariant functions f on a neighborhood of $\pi^{-1}(K)$ in T^*X, one has the uniform Lipschitz estimate*

$$|f_\pi \circ \gamma(s_1) - f_\pi \circ \gamma(s_2)| \leq M \|f\|_{C^1} |s_1 - s_2|, \ s_1, s_2 \in I.$$

In particular, (locally) the functions x, $\bar{y}$ and $\bar{\zeta}$ are Lipschitz on generalized broken bicharacteristics.

We also need to analyze the uniform behavior of generalized broken bicharacteristics. Here we quote Lebeau's results.

Proposition 5.4. *(Lebeau,* [11, Proposition 5]*) Suppose that K is a compact subset of $\dot{\Sigma}$, $\gamma_n : [a, b] \to K$ is a sequence of generalized broken bicharacteristics which converge uniformly to γ. Then γ is a generalized broken bicharacteristic.*

Proof. By Lebeau's result, γ is a 'rayon', i.e. it satisfies (i)-(ii) of Definition 1.1. Thus, we only need to show that it satisfies (iii) in order to prove that it is a generalized broken bicharacteristic. But if $\gamma(t_0) \in \mathcal{G} \cap T^*\mathcal{F}_{i,\mathrm{reg}}$, $\mathcal{F}_i$ a boundary hypersurface, then, using that the projection of γ to X is Lipschitz by Corollary 5.3, we see that for $\delta > 0$ sufficiently small, $\tilde{\gamma}_n = \gamma_n|_{[t_0-\delta, t_0+\delta]}$ lie in $T^*X^\circ \cup T^*\mathcal{F}_{i,\mathrm{reg}}$ for all n, as does $\tilde{\gamma} = \gamma|_{[t_0-\delta, t_0+\delta]}$. Thus, $\tilde{\gamma}$ is a generalized broken bicharacteristic by the results of [14], which implies that γ satisfies (iii), finishing the proof. □

Proposition 5.5. *(Lebeau,* [11, Proposition 6]*) Suppose that K is a compact subset of $\dot{\Sigma}$, $[a, b] \subset \mathbb{R}$ and*

$$\mathcal{R} = \{\text{generalized broken bicharacteristics } \gamma : [a, b] \to K\}. \tag{5.3}$$

If $\mathcal{R}$ is not empty then it is compact in the topology of uniform convergence.

Proof. $\mathcal{R}$ is equicontinuous, as in Lebeau's proof (since every generalized broken bicharacteristic is a rayon), so the proposition follows from the theorem of Ascoli-Arzelà and Proposition 5.4. □

Corollary 5.6. *(Lebeau,* [11, Corollaire 7]*) If $\gamma : (a, b) \to \mathbb{R}$ is a generalized broken bicharacteristic then γ extends to $[a, b]$.*

6. THE HYPERBOLIC SET

In $\mathcal{H} \cup \mathcal{G}$ the Dirichlet form is not positive, but we can use it to estimate $d_M u$ microlocally in terms of $D_t u$ and Pu. This follows immediately from Lemma 4.2 for it implies, with the notation of that lemma, that

$$\begin{aligned}\|d_M A_r u\|^2 \leq &\|D_t A_r u\|^2 \\ &+ C_0(\|u\|^2_{H^1_{\mathrm{loc}}(X)} + \|Gu\|^2_{H^1(X)} + \|Pu\|^2_{H^{-1}_{\mathrm{loc}}(X)} + \|\tilde{G}Pu\|^2_{H^{-1}(X)})).\end{aligned} \tag{6.1}$$

In particular, if the assumption on Pu is strengthened to $Pu = 0$, we have

$$\|d_M A_r u\|^2 \leq \|D_t A_r u\|^2 + C_0(\|u\|^2_{H^1_{\mathrm{loc}}(X)} + \|Gu\|^2_{H^1(X)}). \tag{6.2}$$

Recall here that the meaning of $\|u\|^2_{H^1_{\mathrm{loc}}(X)}$ and $\|Pu\|^2_{H^{-1}_{\mathrm{loc}}(X)}$ was stated in Remark 4.1. (As an aside, we do not need the sharp elliptic version, as in Lemma 4.4, since Lemma 4.2 is only 1/2 derivative weaker than Lemma 4.4, and at $\mathcal{H} \cup \mathcal{G}$, u loses a whole derivative as compared to the elliptic estimates.)

The estimate (6.1) roughly says that $D_{x_i} A_r u$ (and also $D_{y_i} A_r u$, but the latter follows more directly from general properties of the b-ps.d.o's near $\mathcal{H} \cup \mathcal{G}$) is bounded by $D_t A_r u$, modulo lower order error terms. This allows us to estimate various error terms in the positive commutator argument below, and it shows that we only need to find a uniform bound on $\|D_t A_r u\|^2$ in terms of other terms on the right hand side in order to get a bound on $\|d_M A_r u\|^2$, hence conclude that points at which $\sigma_{b,s}(A) \neq 0$ do not lie in $\mathrm{WF}^{1,s}_{\mathrm{b}}(u)$. (Here $A_r \to A$ in a suitable sense.)

A related consequence of this estimate is that for microlocal solutions of $Pu = 0$, $u \in H^1_0(X)$, $\mathrm{WF}^{1,m}_{\mathrm{b}}(u)$ agrees with the b-wave front set of u defined with respect to the more traditional L^2 space.

Lemma 6.1. *Suppose $u \in H^1_{0,loc}(X)$, $\mathrm{WF}^{-1,\infty}_b(Pu) = \emptyset$. Then*

$$\mathrm{WF}^{1,m}_b(u)^c = \{q \in {}^bT^*X \setminus o : \ \exists A \in \Psi^{m+1}_b(X),\ \sigma_{b,m+1}(A)(q) \neq 0,\ Au \in L^2(X)\}.$$

More generally, for $u \in H^1_{0,loc}(X)$,

$$\begin{aligned}&\mathrm{WF}^{1,m}_b(u)^c \cap \mathrm{WF}^{-1,\infty}_b(Pu)^c \\ &\quad = \{q \in \mathrm{WF}^{-1,\infty}_b(Pu)^c : \ \exists A \in \Psi^{m+1}_b(X),\ \sigma_{b,m+1}(A)(q) \neq 0,\ Au \in L^2(X)\}.\end{aligned}$$

Proof. In T^*X°, both sides are the standard wave front set, $\mathrm{WF}^{m+1}(u)$, so it suffices to consider the case when q lies over ∂X.

First we show that the left hand side is a subset of the right hand side, which is the 'easy direction', and does not use any condition on Pu. Now, if $q \in \mathrm{WF}^{1,m}_{\mathrm{b}}(u)^c$, then there is some $B \in \Psi^m_{\mathrm{b}}(X)$ with $\sigma_{b,m}(B)(q) \neq 0$ and $Bu \in H^1_0(X)$. We may assume that B is supported near the projection of q to X, so in particular we can use local coordinates in the rest of the argument. If $\zeta_j(q) \neq 0$, then $A = D_{y_j} B \in \Psi^{m+1}_{\mathrm{b}}(X)$ with non-vanishing principal symbol at q and $D_{y_j} Bu \in L^2(X)$ since $Bu \in H^1_0(X)$, so q indeed lies in the right hand side. A similar argument works of $\tau(q) \neq 0$. If $\sigma_j(q) \neq 0$, then $A = x_j D_{x_j} B \in \Psi_{\mathrm{b}}^{m+1}(X)$ with non-vanishing principal symbol at q and $D_{x_j} Bu \in L^2(X)$ since $Bu \in H_0^1(X)$, so $x_j D_{x_j} Bu \in L^2(X)$ as well – thus, again, q lies in the right hand side. Therefore the left hand side is indeed a subset of the right hand side.

To see the converse direction, i.e. that the right hand side is a subset of the left hand side, we note that as $u \in H_{0,\mathrm{loc}}^1(X)$, $\mathrm{WF}_{\mathrm{b}}^{1,m}(u)^c \supset (({}^{\mathrm{b}}\dot{T}^*X)^c \cup \mathcal{E}) \setminus \mathrm{WF}_{\mathrm{b}}^{-1,\infty}(Pu)$ by Proposition 4.6, so it suffices to consider $q \in \mathcal{G} \cup \mathcal{H}$. We use induction on m to prove that if q is in the right hand side then it is also in the left hand side – with the case $m = 0$ being trivial as we are assuming $u \in H_{0,\mathrm{loc}}^1(X)$. In general, suppose that the inclusion has been proved for m replaced by $m - 1/2$. Suppose that $q \in \mathcal{G} \cup \mathcal{H}$ is in the right hand side, so there is $A \in \Psi_{\mathrm{b}}^{m+1}(X)$, A elliptic at q, $Au \in L^2(X)$, and $q \notin \mathrm{WF}_{\mathrm{b}}^{1,m-1/2}(u)$ by the inductive hypothesis. Note that $\tau(q) \neq 0$, i.e. D_t is elliptic at q. We may assume that $\mathrm{WF}_{\mathrm{b}}'(A)$ lies close to q, hence that τ is elliptic on $\mathrm{WF}_{\mathrm{b}}'(A)$, and in addition $\mathrm{WF}_{\mathrm{b}}^{1,m-1/2}(u) \cap \mathrm{WF}_{\mathrm{b}}'(A) = \emptyset$. Then we can write $A = D_t B + R$, $B \in \Psi_{\mathrm{b}}^m(X)$ elliptic at q and $R \in \Psi_{\mathrm{b}}^{-\infty}(X)$. Thus, (as $u \in L^2(X)$) $Ru \in L^2(X)$, so $D_t Bu \in L^2(X)$. Taking $B_r \in \Psi_{\mathrm{b}}^{m-1}(X)$ uniformly bounded with $B_r \to B$ in $\Psi_{\mathrm{bc}}^{m+\epsilon}(X)$ ($\epsilon > 0$), and using Lemma 4.2 (in the form of (6.1)) gives that $d_M B_r u$ is uniformly bounded in L^2. Since it converges to $d_M Bu$ in $\mathcal{C}^{-\infty}(X)$ on the one hand, and there must be a weakly convergent sequence $d_M B_{r_k} u$ in $L^2(X)$, $r_k \to 0$ as $k \to \infty$, by the uniform bound, we deduce that $d_M Bu \in L^2(X)$ as well, so $d_X Bu \in L^2(X)$, hence $Bu \in H_0^1(X)$. □

After these preliminary discussions, we turn to the propagation estimate at $q \in \mathcal{H}$. As usual, the key ingredient is to find a $\mathcal{C}^\infty$ function f on ${}^{\mathrm{b}}T^*X$ such that, at least near q, $H_p \pi^* f$ has a fixed sign. We usually drop the pull-back π^* below; recall that $\pi : T^*X \to {}^{\mathrm{b}}T^*X$ is the 'inclusion'. In our setting, we can take $f = \eta$ where $\eta = -\frac{x \cdot \xi}{|\tau|} = -\frac{\sum \sigma_j}{|\tau|}$. Indeed, the Hamilton vector field H_p of p is given by

$$
\begin{aligned}
H_p = 2\tau\partial_t - H_g = 2\tau\partial_t - 2A\xi \cdot \partial_x - 2B\zeta \cdot \partial_y - 2\sum C_{ij}\zeta_j\partial_{x_i} - 2\sum C_{ij}\xi_i\partial_{y_j} \\
+ 2\sum(\partial_{x_k}A_{ij})\xi_i\xi_j\partial_{\xi_k} + 2\sum(\partial_{x_k}C_{ij})\xi_i\zeta_j\partial_{\xi_k} \\
+ 2\sum(\partial_{x_k}B_{ij})\zeta_i\zeta_j\partial_{\xi_k} \\
+ 2\sum(\partial_{y_k}A_{ij})\xi_i\xi_j\partial_{\zeta_k} + 2\sum(\partial_{y_k}C_{ij})\xi_i\zeta_j\partial_{\zeta_k} \\
+ 2\sum(\partial_{y_k}B_{ij})\zeta_i\zeta_j\partial_{\zeta_k}.
\end{aligned}
\tag{6.3}
$$

Thus,

$$
\begin{aligned}
|\tau| H_p \eta = 2\xi \cdot A\xi + 2\sum C_{ij}\xi_i\zeta_j - 2\sum(\partial_{x_k}A_{ij})\xi_i\xi_j x_k \\
- 2\sum(\partial_{x_k}C_{ij})\xi_i\zeta_j x_k - 2\sum(\partial_{x_k}B_{ij})\zeta_i\zeta_j x_k,
\end{aligned}
$$

so at $x = 0$, where C vanishes,

$$
|\tau| H_p \eta = 2\xi \cdot A\xi = 2\tau^2 - 2\zeta \cdot B\zeta - 2p = 2\tau^2 - 2|\zeta|_y^2 - 2p. \tag{6.4}
$$

Thus, $H_p \eta > 0$ at $\pi^{-1}(\mathcal{H}) \cap \mathrm{Char}(P) = \hat{\pi}^{-1}(H)$.

We only state the following propagation result for propagation in the forward direction along the generalized broken bicharacteristics. A similar result holds in the backward direction, i.e. if we replace $\eta(\xi) < 0$ by $\eta(\xi) > 0$ in (6.5); the proof in this case only requires changes in some signs in the argument given below. The construction of a positive commutator below closely mirrors that of [24] in the N-body setting.

Proposition 6.2. *Let* $q_0 = (y_0, t_0, \zeta_0, \tau_0) \in \mathcal{H} \cap T^*\mathcal{F}_{reg}$ *and let* $\eta = -\frac{x\cdot\xi}{|\tau|}$ *be the* π*-invariant function defined in the local coordinates discussed above, and suppose that* $u \in H^1_{0,loc}(X)$*,* $q_0 \notin \mathrm{WF}_b^{-1,\infty}(Pu)$*. If there exists a conic neighborhood* U *of* q_0 *in* ${}^b\dot{T}^*X$ *such that*

$$q \in U \text{ and } \eta(q) < 0 \Rightarrow q \notin \mathrm{WF}_b^{1,\infty}(u) \tag{6.5}$$

then $q_0 \notin \mathrm{WF}_b^{1,\infty}(u)$*.*

In fact, if the wave front set assumptions are relaxed to $q_0 \notin \mathrm{WF}_b^{-1,s+1}(Pu)$ *and the existence of a conic neighborhood* U *of* q_0 *in* ${}^b\dot{T}^*X$ *such that*

$$q \in U \text{ and } \eta(q) < 0 \Rightarrow q \notin \mathrm{WF}_b^{1,s}(u), \tag{6.6}$$

then we can still conclude that $q_0 \notin \mathrm{WF}_b^{1,s}(u)$*.*

Remark 6.3. Note that $\eta(q) < 0$ implies $x \neq 0$, so $q \notin T^*\mathcal{F}$.

Remark 6.4. We recall that every conic neighborhood U of $q_0 = (y_0, t_0, \zeta_0, \tau_0) \in \mathcal{H} \cap T^*\mathcal{F}_{\mathrm{reg}}$ in $\dot{\Sigma}$ contains an open set of the form

$$\{q : \ |x(q)|^2 + |y(q) - y_0|^2 + |t(q) - t_0|^2 + |\hat{\zeta}(q) - \hat{\zeta}_0|^2 < \delta\}, \tag{6.7}$$

$\hat{\zeta} = \frac{\zeta}{\tau}$. Note also that (6.5) implies the same statement with U replaced by any smaller neighborhood of q_0; in particular, for the set (6.7), provided that δ is sufficiently small. We can also assume that $\mathrm{WF}_{\mathrm{b}}^{-1,\infty}(Pu) \cap U = \emptyset$.

Proof. As in Proposition 4.6 we use an inductive argument to show that $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{1,s}(u)$, provided that $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{1,s-1/2}(u)$; again the inductive hypothesis holds for $s = 1/2$ since $u \in H^1_{\mathrm{loc}}(X)$. Because of Lemma 6.1, we only need to show that for some $B \in \Psi_{\mathrm{b}}^{s+1}(X)$ with $\sigma_{b,s+1}(B)(q_0) \neq 0$, $Bu \in L^2(X)$.

Below we fix a small neighborhood U_0 of q_0 such that U_0 is inside a coordinate neighborhood of q_0 and $\mathrm{WF}_{\mathrm{b}}^{-1,\infty}(Pu) \cap U_0 = \emptyset$.

The key is to construct an operator A with $\mathrm{WF}'_{\mathrm{b}}(A) \subset U$ and $i[A^*A, P]$ positive, modulo terms that we can estimate by the a priori assumptions, namely those on Pu and those on $\mathrm{WF}_{\mathrm{b}}(u)$, summarized in (6.5) above. Thus, we do not need to make the commutator positive in $\eta < 0$, and also 'away from $\mathrm{Char}(P)$', although the latter is a moral statement as the locus of the microlocalization is ${}^{\mathrm{b}}T^*X \setminus o$, not $T^*X \setminus o$. Our A will in fact be formally self-adjoint modulo lower order operators, and we only take A^*A to avoid having to comment on the subprincipal terms.

The main technical problem below is that P does not lie in $\Psi_{\mathrm{b}}(X)$, so we cannot simply use the symbol calculus on $\Psi_{\mathrm{b}}(X)$ – we need to write out various expressions semi-explicitly as elements of $\mathrm{Diff}\,\Psi_{\mathrm{b}}(X)$. On the other hand, while $\Psi_{\mathrm{b}}(X)$ is the locus of the microlocalization, at the level of the symbol calculus one can rely on standard ps.d.o's on an extension $\tilde{X}$ of X, i.e. work with symbols on T^*X. This has the advantage that p is a symbol on T^*X, as is the pull-back of symbols on

${}^{\mathrm{b}}T^*X$ via π, so one can calculate their Poisson bracket, etc. However, it is not trivial to make this into a technically useful computation, since we need to control various expression in Diff $\Psi_{\mathrm{b}}(X)$. In order to make the argument more digestable, we start with a symbol construction, and do a formal commutator computation in $\Psi(\tilde{X})$ (in fact, we will ignore that we need an extension $\tilde{X}$ here and write '$\Psi(X)$' at times) to show why the constructed symbol *should* be useful, and then give the actual proof.

We construct the symbol of A in a few steps. The two main ingredients are a homogeneous degree zero function that is increasing along the Hamilton flow, which will be η, and a homogeneous degree zero function ω on a conic neighborhood of q_0 in ${}^{\mathrm{b}}T^*X \setminus o$ that roughly measures the square of the distance from q_0 in ${}^{\mathrm{b}}\dot{T}^*X$. Note that ω can also be regarded as a function on a subset of ${}^{\mathrm{b}}S^*X$, if desired. Thus, we let

$$\omega(q) = |x(q)|^2 + |y(q) - y_0|^2 + |t(q) - t_0|^2 + |\hat{\zeta}(q) - \hat{\zeta}_0|^2, \tag{6.8}$$

$|.|$ denoting the Euclidean norm, and $\hat{\zeta} = \frac{\zeta}{\tau}$ as above. Then ω vanishes quadratically at q_0, in fact is a sum of squares, so $|d\omega| \le C_1'\omega^{1/2}$, and in particular

$$|\tau^{-1}H_p\omega| \le C_1''\omega^{1/2}. \tag{6.9}$$

Were we merely using the symbol calculus for $\Psi_{\mathrm{b}}(X)$ or '$\Psi(X)$', this is all that would matter. Since this is not the case, we need that more explicitly,

$$\begin{gathered}\tau^{-1}H_p\omega = f_0 + \sum_i f_i\tau^{-1}\xi_i + \sum_{i,j} f_{ij}\tau^{-2}\xi_i\xi_j,\\ f_i, f_{ij} \in \mathcal{C}^\infty({}^{\mathrm{b}}T^*X),\ |f_i|, |f_{ij}| \le C_1\omega^{1/2},\end{gathered} \tag{6.10}$$

f_i, f_{ij} homogeneous of degree 0, which follows from (6.3).

Next, we use the variable $\eta = -\frac{x\cdot\xi}{|\tau|}$ to measure propagation. Since

$$\eta = -\frac{x\cdot\xi}{|\tau|} = -\sum_j \sigma_j|\tau|^{-1},$$

η is a homogeneous degree zero $\mathcal{C}^\infty$ function on a conic neighborhood of q_0 in ${}^{\mathrm{b}}T^*X\setminus o$, hence it (or more precisely its pullback by π) is a $\mathcal{C}^\infty$, π-invariant function on T^*X. This function indeed measures the flow along bicharacteristics near q_0 since at points $\tilde{q}_0$ in $\hat{\pi}^{-1}(\{q_0\})$, where thus $p = 0$,

$$|\tau|H_p\eta(\tilde{q}_0) = \tau_0^2 - |\zeta_0|^2_{y_0} = c_0\tau_0^2 > 0, \tag{6.11}$$

due to (6.4), where we used that $q_0 \in \mathcal{H}$. Again, if we could use '$\Psi(X)$', all we would need is that $|\tau|H_p\eta > c_0\tau^2/2 > 0$ on U_0, which is automatic if the neighborhood U_0 is small enough. Now, however, we need the more explicit expression

$$\begin{aligned}|\tau|^{-1}H_p\eta =& \tau^{-2}(2\tau^2 - 2|\zeta|^2 - 2p) + g_0 + \sum_i \xi_i\tau^{-1}g_i + \sum_{i,j} g_{ij}\tau^{-2}\xi_i\xi_j,\\ & g_i, g_{ij} \in \mathcal{C}^\infty({}^{\mathrm{b}}T^*X),\ |g_i|, |g_{ij}| \le C_1\omega^{1/2},\end{aligned}$$

g_i, g_{ij} homogeneous of degree 0, which again follows from (6.3).

We are now ready to define the symbol a of A. For $\epsilon > 0$, $\delta > 0$, with other restrictions to be imposed later on, let

$$\phi = \eta + \frac{1}{\epsilon^2\delta}\omega, \tag{6.12}$$

so ϕ is a homogeneous degree zero $\mathcal{C}^\infty$ function on a conic neighborhood of q_0 in ${}^{\mathrm{b}}T^*X \setminus o$ – we can again regard it as a π-invariant function on $T^*X \setminus o$. (Here ϵ^{-2} plays the role of β in the analogous – normal – propagation estimate of [24].)

Let $\chi_0 \in \mathcal{C}^\infty(\mathbb{R})$ be equal to 0 on $(-\infty, 0]$ and $\chi_0(t) = \exp(-1/t)$ for $t > 0$. Thus, $\chi_0'(t) = t^{-2}\chi_0(t)$. Let $\chi_1 \in \mathcal{C}^\infty(\mathbb{R})$ be 0 on $(-\infty, 0]$, 1 on $[1, \infty)$, with $\chi_1' \geq 0$ satisfying $\chi_1' \in \mathcal{C}_c^\infty((0,1))$. Finally, let $\chi_2 \in \mathcal{C}_c^\infty(\mathbb{R})$ be supported in $[-2c_1, 2c_1]$, identically 1 on $[-c_1, c_1]$, where c_1 is such that if $|\sigma|^2/\tau^2 < c_1/2$ in $\dot\Sigma \cap U_0$. Thus, $\chi_2(|\sigma|^2/\tau^2)$ is a cutoff in $|\sigma|/|\tau|$, with its support properties ensuring that $d\chi_2(|\sigma|^2/\tau^2)$ is supported in $|\sigma|^2/\tau^2 \in [c_1, 2c_1]$ hence outside $\dot\Sigma$ – it should be thought of as a factor that microlocalizes near the characteristic set but effectively commutes with P. Then, for $A_0 > 0$ large, to be determined, let

$$a = \chi_0(A_0^{-1}(2 - \phi/\delta))\chi_1(\eta/\delta + 2)\chi_2(|\sigma|^2/\tau^2); \tag{6.13}$$

so a is a homogeneous degree zero $\mathcal{C}^\infty$ function on a conic neighborhood of q_0 in ${}^{\mathrm{b}}T^*X$. Indeed, as we see momentarily, for any $\epsilon > 0$, a has compact support inside this neighborhood (regarded as a subset of ${}^{\mathrm{b}}S^*X$, i.e. quotienting out by the $\mathbb{R}^+$-action) for δ sufficiently small, so in fact it is globally well-defined. In fact, on $\operatorname{supp} a$ we have $\phi \leq 2\delta$ and $\eta \geq -2\delta$. Since $\omega \geq 0$, the first of these inequalities implies that $\eta \leq 2\delta$, so on $\operatorname{supp} a$

$$|\eta| \leq 2\delta. \tag{6.14}$$

Hence,

$$\omega \leq \epsilon^2\delta(2\delta - \eta) \leq 4\delta^2\epsilon^2. \tag{6.15}$$

In view of (6.8) and (6.7), this shows that for any $\epsilon > 0$, a is supported in U, provided $\delta > 0$ is sufficiently small. The role that A_0 large plays is that it increases the size of the first derivatives of a relative to the size of a, hence it allows us to give a bound for a in terms of a small multiple of its derivative along the Hamilton vector field. This is crucial as we need to deal with weight factors, such as $|\tau|^{s+1/2}$ in the next paragraph, if the weight factors do not commute with P. In this case, they can be arranged to commute (at least microlocally, which suffices), so we could eliminate A_0, but its presence is helpful if one is to weaken the assumptions on the structure of P.

This is the point where the technical argument needs significantly more details than the motivational one. So we start with the motivation. Thus, using (6.9), (6.15),

$$\begin{aligned}|\tau|^{-1}H_p\phi = |\tau|^{-1}H_p\eta + |\tau|^{-1}\frac{1}{\epsilon^2\delta}H_p\omega &\geq c_0/2 - \frac{1}{\epsilon^2\delta}C_1''\omega^{1/2}\\ &\geq c_0/2 - 2C_1''\epsilon^{-1} \geq c_0/4 > 0\end{aligned}$$

provided that $\epsilon > \frac{8C_1''}{c_0}$, i.e. that ϵ is not too small. We fix some such ϵ for the rest of the arguments *in this paragraph*, and then we will take $\delta > 0$ sufficiently small. With this,

$$H_pa^2 = -b^2 + e,\ b = |\tau|^{1/2}(2|\tau|^{-1}H_p\phi)^{1/2}(A_0\delta)^{-1/2}(\chi_0\chi_0')^{1/2}\chi_1\chi_2,$$

with e arising from the derivative of $\chi_1\chi_2$. Here χ_0 stands for $\chi_0(A_0^{-1}(2 - \frac{\phi}{\delta}))$, etc. Since $\eta < 0$ on $\operatorname{supp} d\chi_1$ while $\operatorname{supp} d\chi_2$ is disjoint from the characteristic set, both being regions disjoint from $\mathrm{WF}_{\mathrm{b}}(u)$, $i[A^*A, P]$ is positive modulo terms that we can a priori control, so the standard positive commutator argument gives an estimate for Bu, where B has symbol b. Replacing a by $a|\tau|^{s+1/2}$, we still have a positive commutator (in this case τ, or rather D_t, actually commutes with P, but in any case we could use A_0 to bound the additional commutator term), which now gives (with the new B) that $Bu \in L^2(X)$, which means in particular that $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{1,s}(u)$.

This argument is of course *very* imprecise. The technically correct version is the following. First, for $\epsilon, \delta > 0$ still to be determined (i.e. ϵ is not yet fixed; the previous paragraph was motivational only)

(6.16)
$$\begin{aligned}|\tau|^{-1}H_p\phi &= |\tau|^{-1}H_p\eta + \frac{1}{\epsilon^2\delta}|\tau|^{-1}H_p\omega\\ &= -2p\tau^{-2} + \tau^{-2}(2\tau^2 - 2|\zeta|_y^2) + g_0 + \sum_i \tau^{-1}\xi_i g_i + \sum_{ij}\tau^{-2}\xi_i\xi_j g_{ij}\\ &\qquad + \frac{1}{\epsilon^2\delta}(f_0 + \sum \xi_i\tau^{-1}f_i + \sum \tau^{-2}\xi_i\xi_j g_{ij})\end{aligned}$$

Let $\tilde{B} \in \Psi_{\mathrm{b}}^{1/2}(X)$ with

$$\tilde{b} = \sigma_{b,1/2}(\tilde{B}) = |\tau|^{1/2}(A_0\delta)^{-1/2}(\chi_0\chi_0')^{1/2}\chi_1\chi_2 \in \mathcal{C}^\infty({}^{\mathrm{b}}T^*X \setminus o), \tag{6.17}$$

and let $A \in \Psi_{\mathrm{b}}^0(X)$ with $\sigma_{b,0}(A) = a$. Again, χ_0 stands for $\chi_0(A_0^{-1}(2 - \frac{\phi}{\delta}))$, etc. Also, let $C \in \Psi_{\mathrm{b}}^0(X)$ have symbol $\sigma_{b,0}(C) = |\tau|^{-1}(2\tau^2 - 2|\zeta|_y^2)^{1/2}\psi$ where $\psi \in S^0({}^{\mathrm{b}}T^*X)$ is identically 1 on U considered as a subset of ${}^{\mathrm{b}}T^*X$. Then an explicit calculation using Lemma 2.8 and $P = D_t^2 - \Delta$,

$$\Delta = \sum_{i,j} A_{ij}(x,y)D_{x_i}D_{x_j} + \sum_{i,j} 2C_{ij}(x,y)D_{x_i}D_{y_j} + \sum_{i,j} B_{ij}(x,y)D_{y_i}D_{y_j} + P_1,$$

$P_1 \in \mathrm{Diff}^1(X)$, gives, in accordance with (6.16),

(6.18)
$$\begin{aligned}&i[A^*A, P]\\ &\quad = R'P + \tilde{B}^*(C^*C + R_0 + \sum_i D_{x_i}R_i + \sum_{ij} D_{x_i}R_{ij}D_{x_j})\tilde{B} + R'' + E + E'\end{aligned}$$

with

$$\begin{aligned}&R_0 \in \Psi_{\mathrm{b}}^0(X),\ R_i \in \Psi_{\mathrm{b}}^{-1}(X),\ R_{ij} \in \Psi_{\mathrm{b}}^{-2}(X),\\ &R' \in \Psi_{\mathrm{b}}^{-1}(X),\ R'' \in \mathrm{Diff}^2\,\Psi_{\mathrm{b}}^{-2}(X),\ E, E' \in \mathrm{Diff}^2\,\Psi_{\mathrm{b}}^{-1}(X),\end{aligned}$$

with $\mathrm{WF}_{\mathrm{b}}'(E) \subset \eta^{-1}((-\infty, -\delta]) \cap U$, $\mathrm{WF}_{\mathrm{b}}'(E') \cap \dot{\Sigma} = \emptyset$ (E arises from the commutator of P with an operator with symbol $\chi_1(\eta/\delta + 2)$, while E' from the commutator of P with an operator with symbol $\chi_2(|\sigma|^2/\tau^2)$) and with $r_0 = \sigma_{b,0}(R_0)$, $r_i = \sigma_{b,-1}(R_i)$, $r_{ij} \in \sigma_{b,-2}(R_{ij})$,

$$|r_0| \leq C_2(1 + \frac{1}{\epsilon^2\delta})\omega^{1/2},\ |\tau r_i| \leq C_2(1 + \frac{1}{\epsilon^2\delta})\omega^{1/2},\ |\tau^2 r_{ij}| \leq C_2(1 + \frac{1}{\epsilon^2\delta})\omega^{1/2},$$

and $\operatorname{supp} r_j$ lying in $\omega \leq 9\delta^2\epsilon^2$. Thus,

$$|r_0| \leq 3C_2(\delta\epsilon + \epsilon^{-1}),\ |\tau r_i| \leq 3C_2(\delta\epsilon + \epsilon^{-1}),\ |\tau^2 r_{ij}| \leq 3C_2(\delta\epsilon + \epsilon^{-1}).$$

Having calculated the commutator, we proceed to estimate the 'error terms' R_0, R_i, R_{ij} as operators. We start with R_0. As follows from the standard square root construction to prove the boundedness of ps.d.o's on L^2, there exists $R'_0 \in \Psi_{\mathrm{b}}^{-1}(X)$ such that

$$\|R_0 v\| \leq 2 \sup |r_0| \, \|v\| + \|R'_0 v\|$$

for all $v \in L^2(X)$. Here $\|\cdot\|$ is the $L^2(X)$-norm, as usual. Thus, we can estimate, for any $\gamma > 0$,

$$\begin{aligned} |\langle R_0 v, v\rangle| &\leq \|R_0 v\| \, \|v\| \leq 2 \sup |r_0| \, \|v\|^2 + \|R'_0 v\| \, \|v\| \\ &\leq 6C_2(\delta\epsilon + \epsilon^{-1})\|v\|^2 + \gamma^{-1}\|R'_0 v\|^2 + \gamma\|v\|^2. \end{aligned}$$

Now we turn to R_i. Let $T \in \Psi_{\mathrm{b}}^{-1}(X)$ be elliptic (which we use to keep track of the orders of ps.d.o's), $T^- \in \Psi_{\mathrm{b}}^{1}(X)$ a parametrix, so $T^-T = \mathrm{Id} + F$, $F \in \Psi_{\mathrm{b}}^{-\infty}(X)$. Then there exist $R'_i \in \Psi_{\mathrm{b}}^{-1}(X)$ such that

$$\begin{aligned} \|R_i w\| = \|R_i(T^-T - F)w\| &\leq \|(R_i T^-)(Tw)\| + \|R_i F w\| \\ &\leq 6C_2(\delta\epsilon + \epsilon^{-1})\|Tw\| + \|R'_i T w\| + \|R_i F w\| \end{aligned}$$

for all w with $Tw \in L^2(X)$. Similarly, there exist $R'_{ij} \in \Psi_{\mathrm{b}}^{-1}(X)$ such that

$$\|(T^-)^* R_{ij} w\| \leq 6C_2(\delta\epsilon + \epsilon^{-1})\|Tw\| + \|R'_{ij} T w\| + \|(T^-)^* R_{ij} F w\|$$

for all w with $Tw \in L^2(X)$. Thus,

$$\begin{aligned} |\langle R_i D_{x_i} v, v\rangle| \leq & 6C_2(\delta\epsilon + \epsilon^{-1})\|T D_{x_i} v\| \, \|v\| \\ & + 2\gamma\|v\|^2 + \gamma^{-1}\|R'_i T D_{x_i} v\|^2 + \gamma^{-1}\|F_i D_{x_i} v\|^2, \end{aligned}$$

and, writing $D_{x_j} v = T^-Tv - Fv$ in the right factor, and taking the adjoint of T^-,

$$\begin{aligned} |\langle R_{ij} D_{x_i} v, D_{x_j} v\rangle| \leq & 6C_2(\delta\epsilon + \epsilon^{-1})\|T D_{x_i} v\| \, \|T D_{x_j} v\| \\ & + 2\gamma\|T D_{x_j} v\|^2 + \gamma^{-1}\|R'_{ij} T D_{x_i} v\|^2 + \gamma^{-1}\|F_{ij} D_{x_i} v\|^2 \\ & + \|R_{ij} D_{x_i} v\| \, \|F D_{x_j} v\|, \end{aligned}$$

with $F_i, F_{ij} \in \Psi_{\mathrm{b}}^{-\infty}(X)$.

Let Λ_r have symbol

$$|\tau|^{s+1/2}(1 + r|\tau|^2)^{-s}, \quad r \in [0,1), \tag{6.19}$$

so $A_r = A\Lambda_r \in \Psi_{\mathrm{b}}^0(X)$ for $r > 0$ and it is uniformly bounded in $\Psi_{\mathrm{bc}}^{s+1/2}(X)$. In similar constructions in general, the commutator $[P, \Lambda_r]$ can be controlled by the other terms using A_0, for A_0 large – in the present setting $[P, \Lambda_r] = 0$.

Now, by (6.18),

$$\begin{aligned} \langle i[A_r^* A_r, P]u, u\rangle = & \|C\tilde{B}\Lambda_r u\|^2 + \langle R' P \Lambda_r u, \Lambda_r u\rangle + \langle R_0 \tilde{B}\Lambda_r u, \tilde{B}\Lambda_r u\rangle \\ & + \sum \langle R_i D_{x_i} \tilde{B}\Lambda_r u, \tilde{B}\Lambda_r u\rangle + \sum \langle R_{ij} D_{x_i} \tilde{B}\Lambda_r u, D_{x_j} \tilde{B}\Lambda_r u\rangle \\ & + \langle R'' \Lambda_r u, \Lambda_r u\rangle + \langle (E + E')\Lambda_r u, \Lambda_r u\rangle \end{aligned} \tag{6.20}$$

On the other hand, as $A_r \in \Psi_{\mathrm{b}}^0(X)$ for $r > 0$ and $u \in H_0^1(X)$, so $A_r^* A_r u \in H_0^1(X)$,

$$\begin{aligned} \langle [A_r^* A_r, P]u, u\rangle &= \langle A_r^* A_r P u, u\rangle - \langle P A_r^* A_r u, u\rangle \\ &= \langle A_r P u, A_r u\rangle - \langle A_r u, A_r P u\rangle = 2i \operatorname{Im}\langle A_r P u, A_r u\rangle; \end{aligned} \tag{6.21}$$

the pairing makes sense for $r > 0$ since $A_r \in \Psi_{\mathrm{b}}^0(X)$ then.

Assume for the moment that $\mathrm{WF}_{\mathrm{b}}^{-1,s+3/2}(Pu) \cap U = \emptyset$ – this is certainly the case in our setup if $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{-1,\infty}(Pu)$, but this assumption is a little stronger than

$q_0 \notin \mathrm{WF}_{\mathrm{b}}^{-1,s+1}(Pu)$, which is what we need to assume for the second paragraph in the statement of the proposition. We deal with the weakened hypothesis $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{-1,s+1}(Pu)$ at the end of the proof. Returning to (6.21), the utility of the commutator calculation is that we have good information about Pu (this is where we use that we have a microlocal solution of the PDE!). Namely, we estimate the right hand side as

$$\begin{aligned} |\langle A_r Pu, A_r u\rangle| &\leq |\langle (T^-)^* A_r Pu, TA_r u\rangle| + |\langle A_r Pu, FA_r u\rangle| \\ &\leq \|(T^-)^* A_r Pu\|_{H^{-1}(X)} \|TA_r u\|_{H^1(X)} \\ &\qquad + \|A_r Pu\|_{H^{-1}(X)} \|FA_r u\|_{H^1(X)}. \end{aligned} \tag{6.22}$$

Since $(T^-)^* A_r$ is uniformly bounded in $\Psi_{\mathrm{bc}}^{s+3/2}(X)$, TA_r is uniformly bounded in $\Psi_{\mathrm{bc}}^{s-1/2}(X)$, both with $\mathrm{WF}_{\mathrm{b}}'$ in U, with $\mathrm{WF}_{\mathrm{b}}^{-1,s+3/2}(Pu)$, resp. $\mathrm{WF}_{\mathrm{b}}^{1,s-1/2}(u)$ disjoint from them, we deduce (using Lemma 3.13 and its H^{-1} analogue) that $|\langle (T^-)^* A_r Pu, TA_r u\rangle|$ is uniformly bounded. Similarly, taking into account that FA_r is uniformly bounded in $\Psi_{\mathrm{b}}^{-\infty}(X)$, we see that $|\langle A_r Pu, FA_r u\rangle|$ is also uniformly bounded, so $|\langle A_r Pu, A_r u\rangle|$ is uniformly bounded for $r \in (0,1]$.

Thus, for some $C_3 > 0$ depending only on the dimension of X,

$$\begin{aligned} \|C\tilde{B}\Lambda_r u\|^2 \leq & 2|\langle A_r Pu, A_r u\rangle| + |\langle (E+E')\Lambda_r u, \Lambda_r u\rangle| \\ & + \big(6C_2(\delta\epsilon + \epsilon^{-1}) + C_3\gamma\big) \|\tilde{B}\Lambda_r u\|^2 + \gamma^{-1}\|R_0'\tilde{B}\Lambda_r u\|^2 \\ & + 6C_2(\delta\epsilon + \epsilon^{-1})\|\tilde{B}\Lambda_r u\| \sum_i \|TD_{x_i}\tilde{B}\Lambda_r u\| \\ & + \gamma^{-1}\sum_i \|TR_i' D_{x_i}\tilde{B}\Lambda_r u\|^2 + \gamma\|\tilde{B}\Lambda_r u\|^2 \\ & + \big(6C_2(\delta\epsilon + \epsilon^{-1}) + C_3\gamma\big) \sum_i \|TD_{x_i}\tilde{B}\Lambda_r u\|^2 \\ & + \gamma^{-1}\sum_{ij} \|R_{ij}' TD_{x_i}\tilde{B}\Lambda_r u\|^2 \\ & + \gamma^{-1}\sum_i \|F_i D_{x_i}\tilde{B}\Lambda_r u\|^2 + \gamma^{-1}\sum_{ij}\|F_{ij}D_{x_i}\tilde{B}\Lambda_r u\|^2 \\ & + \sum_{ij} \|R_{ij}D_{x_i}\tilde{B}\Lambda_r u\|\, \|FD_{x_j}\tilde{B}\Lambda_r u\|. \end{aligned} \tag{6.23}$$

All terms but the ones involving C_2 or γ (not γ^{-1}) remain bounded as $r \to 0$. The C_2 and γ terms can be estimated by writing $TD_{x_i} = D_{x_i}T_i' + T_i''$ for some $T_i', T_i'' \in \Psi_{\mathrm{b}}^{-1}(X)$, and using Lemma 4.2 (in the form (6.1)) where necessary, to conclude that there exist $\gamma > 0$, $\epsilon > 0$, $\delta_0 > 0$ and $C_4 > 0$, $C_5 > 0$ such that for $\delta \in (0, \delta_0)$,

$$\begin{aligned} C_4\|\tilde{B}\Lambda_r u\|^2 \leq & 2|\operatorname{Im}\langle A_r Pu, A_r u\rangle| + |\langle (E+E')\Lambda_r u, \Lambda_r u\rangle| \\ & + \gamma^{-1}\|R_0'\tilde{B}\Lambda_r u\|^2 + C_5\gamma^{-1}\|d_X T^2\tilde{B}\Lambda_r u\|^2. \end{aligned}$$

Letting $r \to 0$ now keeps the right hand side bounded, proving that $\|\tilde{B}\Lambda_r u\|$ is uniformly bounded as $r \to 0$, hence $\tilde{B}\Lambda_0 u \in L^2(X)$ (cf. the proof of Proposition 4.6). In view of Lemma 4.2 (in the form (6.1)) this proves that $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{1,s}(u)$, and hence proves the first statement of the proposition.

In fact, recalling that we needed $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{-1,s+3/2}(Pu)$ for the uniform boundedness in (6.22), this proves a slightly weaker version of the second statement of the proposition with $\mathrm{WF}_{\mathrm{b}}^{-1,s+1}(Pu)$ replaced by $\mathrm{WF}_{\mathrm{b}}^{-1,s+3/2}(Pu)$. For the more precise statement we modify (6.22) – this is the only term in (6.23) that needs modification to prove the optimal statement. Let $\tilde{T} \in \Psi_{\mathrm{b}}^{-1/2}(X)$ be elliptic, $\tilde{T}^- \in \Psi_{\mathrm{b}}^{1/2}(X)$ a parametrix, $\tilde{F} = \tilde{T}^-\tilde{T} - \mathrm{Id} \in \Psi_{\mathrm{b}}^{-\infty}(X)$. Then, similarly to (6.22), we have for any $\gamma > 0$,

$$\begin{aligned}|\langle A_r Pu, A_r u\rangle| &\leq |\langle (\tilde{T}^-)^* A_r Pu, \tilde{T} A_r u\rangle| + |\langle A_r Pu, \tilde{F} A_r u\rangle| \\ &\leq \gamma^{-1}\|(\tilde{T}^-)^* A_r Pu\|^2_{H^{-1}(X)} + \gamma\|\tilde{T} A_r u\|^2_{H^1(X)} \\ &\qquad + \|A_r Pu\|_{H^{-1}(X)}\|\tilde{F} A_r u\|_{H^1(X)}.\end{aligned} \tag{6.24}$$

The last term on the right hand side can be estimated as before. As $(\tilde{T}^-)^* A_r$ is bounded in $\Psi_{\mathrm{bc}}^{s+1}(X)$ with $\mathrm{WF}_{\mathrm{b}}'$ disjoint from U, we see that $\|(\tilde{T}^-)^* A_r Pu\|_{H^{-1}(X)}$ is uniformly bounded. Moreover, $\|d_X \tilde{T} A \Lambda_r u\|^2$ can be estimated, using Lemma 4.2 (in the form (6.1)), by $\|D_t \tilde{T} A \Lambda_r u\|^2$ modulo terms that are uniformly bounded as $r \to 0$. The principal symbol of $D_t \tilde{T} A$ is $\tau \sigma_{b,-1/2}(\tilde{T}) a$, with $a = \chi_0\chi_1\chi_2$, where χ_0 stands for $\chi_0(A_0^{-1}(2 - \frac{\phi}{\delta}))$, etc., while the principal symbol $\tilde{b}$ of $\tilde{B}$ is given by (6.17), so we can write:

$$|\tau|^{1/2} a = |\tau|^{1/2}\chi_0\chi_1\chi_2 = A_0^{-1}(2-\phi/\delta)|\tau|^{1/2}(\chi_0\chi_0')^{1/2}\chi_1\chi_2 = A_0^{-1/2}\delta^{1/2}(2-\phi/\delta)\tilde{b},$$

where we used that

$$\chi_0'(A_0^{-1}(2-\phi/\delta)) = A_0^2(2-\phi/\delta)^{-2}\chi_0(A_0^{-1}(2-\phi/\delta))$$

when $2 - \phi/\delta > 0$, while a, $\tilde{b}$ vanish otherwise. Correspondingly, as $|\tau|^{1/2}\sigma_{b,-1/2}(\tilde{T})$ is $\mathcal{C}^\infty$, homogeneous degree zero, near the support of a in ${}^{\mathrm{b}}T^*X \setminus o$, we can write $D_t \tilde{T} A = G\tilde{B} + F$, $G \in \Psi_{\mathrm{b}}^0(X)$, $F \in \Psi_{\mathrm{b}}^{-1/2}(X)$. Correspondingly, modulo terms that are bounded as $r \to 0$, $\|D_t \tilde{T} A \Lambda_r u\|^2$ (hence $\|d_X \tilde{T} A \Lambda_r u\|^2$) can be estimated from above by $C_6\|\tilde{B}\Lambda_r u\|^2$. Thus, modulo terms that are bounded as $r \to 0$, for $\gamma > 0$ sufficiently small, $\gamma\|\tilde{T} A_r u\|^2_{H^1(X)}$ can be absorbed into $\|C\tilde{B}\Lambda_r u\|^2$. As the treatment of the other terms on the right hand side of (6.23) requires no change, we deduce as above that $\tilde{B}\Lambda_0 u \in L^2(X)$, which (in view of Lemma 4.2 and (6.1)) proves that $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{1,s}(u)$, completing the proof of the iterative step.

We need to make one more remark to prove the proposition for $\mathrm{WF}_{\mathrm{b}}^{1,\infty}(u)$, namely we need to show that the neighborhoods of q_0 which are disjoint from $\mathrm{WF}_{\mathrm{b}}^{1,s}(u)$ do not shrink uncontrollably to $\{q_0\}$ as $s \to \infty$. This argument parallels to last paragraph of the proof of [4, Proposition 24.5.1]. In fact, note that above we have proved that the elliptic set of $\tilde{B} = \tilde{B}_s$ is disjoint from $\mathrm{WF}_{\mathrm{b}}^{1,s}(u)$. In the next step, when we are proving $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{1,s+1/2}(u)$, we decrease $\delta > 0$ slightly (by an arbitrary small amount), thus decreasing the support of $a = a_{s+1/2}$ in (6.13), to make sure that $\operatorname{supp} a_{s+1/2}$ is a subset of the elliptic set of the union of $\tilde{B}_s$ with the region $\eta < 0$, and hence that $\mathrm{WF}_{\mathrm{b}}^{1,s}(u) \cap \operatorname{supp} a_{s+1/2} = \emptyset$. Each iterative step thus shrinks the elliptic set of $\tilde{B}_s$ by an arbitrarily small amount, which allows us to conclude that q_0 has a neighborhood U' such that $\mathrm{WF}_{\mathrm{b}}^{1,s}(u) \cap U' = \emptyset$ for all s. This proves that $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u)$, and indeed that $\mathrm{WF}_{\mathrm{b}}^{1,\infty}(u) \cap U' = \emptyset$, for if $A \in \Psi_{\mathrm{b}}^m(X)$ with $\mathrm{WF}_{\mathrm{b}}'(A) \subset U'$ then $Au \in H^1(X)$ by Lemma 3.9 and Corollary 3.11. □

Again, this can be modified to allow Neumann boundary conditions. Namely, rather than consider $[A_r^*A_r, P]$, we work directly with the quadratic form, see (4.8). Thus, writing $w = (x, y, t)$ and $\tilde{g}$ for the semi-Riemannian metric $g - dt^2$, while $J\,dw$ is the volume form of $g + dt^2$, and $\langle\cdot,\cdot\rangle$ is the corresponding inner product on $L^2(X)$, (4.8) shows that

$$\begin{aligned}&\langle A_r^*A_r u, f\rangle - \langle f, A_r^*A_r u\rangle\\ &\qquad = \sum_{ij}\langle \tilde{g}^{ij} D_{w_i} u, D_{w_j} A_r^*A_r u\rangle - \sum_{ij}\langle \tilde{g}^{ij} D_{w_i} A_r^*A_r u, D_{w_j} u\rangle.\end{aligned}\tag{6.25}$$

Then the replacement of (6.21) is achieved by expanding the right hand side:

$$\begin{aligned}&\sum_{ij}\langle \tilde{g}^{ij} D_{w_i} u, D_{w_j} A_r^*A_r u\rangle - \sum_{ij}\langle \tilde{g}^{ij} D_{w_i} A_r^*A_r u, D_{w_j} u\rangle\\ &= \sum_{ij}\langle \tilde{g}^{ij} D_{w_i} u, [D_{w_j}, A_r^*A_r] u\rangle + \sum_{ij}\langle \tilde{g}^{ij} D_{w_i} u, A_r^*A_r D_{w_j} u\rangle\\ &\qquad - \sum_{ij}\langle [\tilde{g}^{ij} D_{w_i}, A_r^*A_r] u, D_{w_j} u\rangle - \sum_{ij}\langle A_r^*A_r \tilde{g}^{ij} D_{w_i} u, D_{w_j} u\rangle\\ &= \sum_{ij}\langle \tilde{g}^{ij} D_{w_i} u, [D_{w_j}, A_r^*A_r] u\rangle - \sum_{ij}\langle [\tilde{g}^{ij} D_{w_i}, A_r^*A_r] u, D_{w_j} u\rangle;\end{aligned}\tag{6.26}$$

the second and fourth terms in the middle cancel as $A_r^*A_r$ is symmetric. If there were no boundary present, i.e. if $\partial X = \emptyset$, we could of course write the right hand side as

$$\begin{aligned}&-\sum_{ij}\langle ([D_{w_j}^*, A_r^*A_r]\tilde{g}^{ij} D_{w_i} + D_{w_j}^*[\tilde{g}^{ij} D_{w_i}, A_r^*A_r]) u, u\rangle\\ &\qquad = \langle [D_t^2 - \Delta, A_r^*A_r] u, u\rangle,\end{aligned}$$

so formally this is indeed the same commutator as the one considered in (6.21). The actual expression, the right hand side of (6.26), can be analyzed much as in the Dirichlet problem, using Lemma 2.8 to compute the commutators.

To illustrate the form that (6.25) takes, replace $A_r^*A_r$ by A^*A temporarily, now $\sigma_{\mathrm{b},0}(A^*A) = a^2$. Thus, by Lemma 2.8, up to terms of similar form with vanishing symbol at $x = 0$, $y = y_0$, $t = t_0$, the right hand side of (6.25) is, $\frac{1}{i}$ times,

$$\int \sum_{ij} g^{ij} D_{x_i} u\, \overline{\tilde{C} D_{x_j} u}\, J\,dw + \int \sum_{ij} g^{ij} \tilde{C} D_{x_i} u\, \overline{D_{x_j} u}\, J\,dw,$$

where the summation is only over the coordinates vanishing at the corner (i.e. $x_1, \ldots, x_k$), and $\tilde{C} \in \Psi_{\mathrm{b}}^{-1}(X)$ with $\sigma_{\mathrm{b},-1}(\tilde{C}) = |\tau|^{-1}(A_0\delta)^{-1}\chi_0\chi_0'\chi_1^2\chi_2^2$, cf. (6.17) and the sentence afterwards. We can subtract this from the PDE (which corresponds to restricting to the characteristic set of P, or allowing the term $R'P$ in (6.18)), considered in the form

$$\int \sum_{ij} \tilde{g}^{ij} D_{w_i} u\, \overline{D_{w_j} \tilde{C} u}\, J\,dw + \int \sum_{ij} \tilde{g}^{ij} D_{w_i} \tilde{C} u\, \overline{D_{w_j} u}\, J\,dw,$$

plus terms involving f, commute the C through the D_{w_i}, D_{w_j} (the commutators are lower order in terms of b-differential order, so we ignore them), to obtain an expression for

$$\int \sum_{ij} g^{ij} D_{\bar{y}_i} u \, \overline{\tilde{C} D_{\bar{y}_j} u} \, J \, dw + \int \sum_{ij} g^{ij} \tilde{C} D_{\bar{y}_i} u \, \overline{D_{\bar{y}_j} u} \, J \, dw,$$

$\bar{y} = (y, t)$ as usual. Shifting the tangential derivatives $D_{\bar{y}_i}$ over and rearranging this gives (modulo lower order terms), with $\tilde{B}$ as in (6.17), and C also as there,

$$\int C\tilde{B}u \, \overline{C\tilde{B}u} \, J \, dw = \|C\tilde{B}u\|^2.$$

The neglected error terms can be treated much as in the Dirichlet problem, giving the desired positivity estimate.

7. Glancing points

We again need a technical lemma, roughly stating that when applied to solutions of $Pu = 0$, $u \in H_0^1(X)$, microlocally near $\mathcal{G}$, D_{x_i} is not merely bounded by D_t, but it is small compared to it. Such an estimate is natural since $p|_{x=0} = \tau^2 - |\xi|_y^2 - |\zeta|_y^2$ gives $\tau^{-2}|\xi|^2 \leq C(\tau^{-2}|p| + |x| + |1 - \tau^{-2}|\zeta|_y^2|)$, and $1 - \tau^{-2}|\zeta|_y^2$ is homogeneous of degree zero and vanishes at $\mathcal{G}$, so the right hand size is small near $\mathcal{G}$. Below a δ-neighborhood refers to a δ-neighborhood with respect to the metric associated to any Riemannian metric on the manifold ${}^{\mathrm{b}}T^*X$, and we identify ${}^{\mathrm{b}}S^*X$ as the unit ball bundle with respect to some fibre metric on ${}^{\mathrm{b}}T^*X$.

Lemma 7.1. *Suppose $u \in H^1_{0,loc}(X)$, and suppose that we are given $K \subset {}^bS^*X$ compact satisfying*

$$K \subset \mathcal{G} \cap T^*\mathcal{F}_{k,reg} \setminus \mathrm{WF}_b^{-1,s+1/2}(Pu).$$

Then there exist $\delta_0 > 0$ and $C_0 > 0$ with the following property. Let $\delta < \delta_0$, $U \subset {}^bS^*X$ open in a δ-neighborhood of K, and $\mathcal{A} = \{A_r : r \in (0,1]\}$ be a bounded family of ps.d.o's in $\Psi^s_{bc}(X)$ with $\mathrm{WF}'_b(\mathcal{A}) \subset U$, and with $A_r \in \Psi_b^{s-1}(X)$ for $r \in (0,1]$.

Then there exist $G \in \Psi_b^{s-1/2}(X)$, $\tilde{G} \in \Psi_b^{s+1/2}(X)$ with $\mathrm{WF}'_b(G), \mathrm{WF}'_b(\tilde{G}) \subset U$ and $\tilde{C}_0 = \tilde{C}_0(\delta) > 0$ such that for all $r > 0$,

$$\sum_i \|D_{x_i} A_r u\|^2 \leq C_0 \delta \|D_t A_r u\|^2 + \tilde{C}_0(\|u\|^2_{H^1_{loc}(X)} + \|Gu\|^2_{H^1(X)} + \|Pu\|^2_{H^{-1}_{loc}(X)} + \|\tilde{G}Pu\|^2_{H^{-1}(X)}).$$

The meaning of $\|u\|_{H^1_{loc}(X)}$ and $\|Pu\|^2_{H^{-1}_{loc}(X)}$ is stated in Remark 4.1.

Remark 7.2. As K is compact, this is essentially a local result. In particular, we may assume that K is a subset of ${}^{\mathrm{b}}T^*X$ over a suitable local coordinate patch. Moreover, we may assume that $\delta_0 > 0$ is sufficiently small so that D_t is elliptic on U.

Proof. By Lemma 4.2 and (6.1), applied with K replaced by $\mathrm{WF}'_{\mathrm{b}}(\mathcal{A})$ in the hypothesis (note that the latter is compact), we already know that

(7.1)
$$\|d_X A_r u\|^2 \leq \|D_t A_r u\|^2 + C'_0(\|u\|^2_{H^1_{\mathrm{loc}}(X)} + \|Gu\|^2_{H^1(X)} + \|Pu\|^2_{H^{-1}_{\mathrm{loc}}(X)} + \|\tilde{G}Pu\|^2_{H^{-1}(X)}).$$

for some $C_0' > 0$ and for some G, $\tilde{G}$ as in the statement of the lemma. Thus, we only need to show that if we replace the left hand side by $\sum_i \|D_{x_i}A_r u\|^2$ (i.e. we drop the tangential derivatives, at least roughly speaking), the constant in front of $\|D_t A_r u\|^2$ can be made small.

As a first step, we freeze the coefficients at $\mathcal{F}_k$, i.e. replace $A_{ij}(x,y)$, etc., by $A_{ij}(0,y)$. Writing $A_{ij}(x,y) = A_{ij}(0,y) + \sum x_l A'_{ijl}(x,y)$ as in the proof of Proposition 4.6, we deduce that if the operators A_r are supported in $|x| < \delta$, then (4.5) holds, i.e.

$$|\int_X \sum x_l A'_{ijl} D_{x_i} A_r u\, \overline{D_{x_j} A_r u}| \leq C\delta \sum_{i',j'} \|D_{x_{i'}} A_r u\|\, \|D_{x_{j'}} A_r u\|,$$

with analogous estimates with $A_{ij}(x,y) - A_{ij}(0,y)$ replaced by $B_{ij}(x,y) - B_{ij}(0,y)$ or $C_{ij}(x,y)$. Combined with (7.1) above, this gives that

$$\begin{aligned}\int_X \Big(\sum_{ij} A_{ij}(0,y) D_{x_i} A_r u\, \overline{D_{x_j} A_r u} + \sum_{ij} B_{ij}(0,y) D_{y_i} A_r u\, \overline{D_{y_j} A_r u}\Big)&\\ \leq (1 + C_1\delta)\|D_t A_r u\|^2&\\ + C_0''(\|u\|^2_{H^1_{\mathrm{loc}}(X)} + \|Gu\|^2_{H^1(X)} + \|Pu\|^2_{H^{-1}_{\mathrm{loc}}(X)} + \|\tilde{G}Pu\|^2_{H^{-1}(X)}),&\end{aligned}$$

and hence, after rearrangement, that

$$\begin{aligned}\int_X \sum_{ij} A_{ij}(0,y) D_{x_i} A_r u\, \overline{D_{x_j} A_r u}&\\ \leq \int_X \Big((D_t^2 - \sum B_{ij}(0,y) D_{y_i} D_{y_j}) A_r u\, \overline{A_r u}\Big) + C_1\delta \|D_t A_r u\|^2&\\ + C_0''(\|u\|^2_{H^1_{\mathrm{loc}}(X)} + \|Gu\|^2_{H^1(X)} + \|Pu\|^2_{H^{-1}_{\mathrm{loc}}(X)} + \|\tilde{G}Pu\|^2_{H^{-1}(X)}).&\end{aligned}$$

It thus suffices to prove that

$$\begin{aligned}&\Big|\int_X \Big((D_t^2 - \sum B_{ij}(0,y) D_{y_i} D_{y_j}) A_r u\, \overline{A_r u}\Big)\Big|\\ &\qquad \leq C_2\delta \|D_t A_r u\|^2 + \tilde{C}_2(\delta)(\|u\|^2_{H^1_{\mathrm{loc}}(X)} + \|Gu\|^2_{H^1(X)}),\end{aligned} \tag{7.2}$$

which we proceed to do.

Let $\psi \in \mathcal{C}^\infty({}^{\mathrm{b}}S^*X)$ (which can thus be identified with a homogeneous degree zero function on ${}^{\mathrm{b}}T^*X \setminus o$) with $\psi \equiv 1$ near $\mathrm{WF}'_{\mathrm{b}}(\mathcal{A})$, $\operatorname{supp}\psi \subset U$, $|\psi| \leq 1$, and let $F \in \Psi^0_{\mathrm{b}}(X)$ be such that

$$\begin{aligned}&\mathrm{WF}'_{\mathrm{b}}(F) \subset U,\ \mathrm{WF}'_{\mathrm{b}}\Big(D_t F D_t - (D_t^2 - \sum B_{ij} D_{y_i} D_{y_j})\Big) \cap \mathrm{WF}'_{\mathrm{b}}(\mathcal{A}) = \emptyset\\ &f = \sigma_{b,0}(F) = \psi(1 - \tau^{-2} \sum B_{ij}\zeta_i\zeta_j).\end{aligned} \tag{7.3}$$

Such ψ and F exist, since D_t is elliptic on $\mathrm{WF}'_{\mathrm{b}}(\mathcal{A})$. Now,

$$\Big|\int_X \Big((D_t F D_t - (D_t^2 - \sum B_{ij}(0,y) D_{y_i} D_{y_j})) A_r u\, \overline{A_r u}\Big)\Big| \leq C_2' \|u\|^2_{H^1_{\mathrm{loc}}(X)}$$

since $(D_t F D_t - (D_t^2 - \sum B_{ij} D_{y_i} D_{y_j}))A_r$ is uniformly bounded in $\Psi^{-\infty}_{\mathrm{b}}(X)$, by the first line of (7.3). Moreover,

$$\sup |f| \leq C_3\delta$$

since $|1-\tau^{-2}\sum B_{ij}\zeta_i\zeta_j| < C_3\delta$ on a δ-neighborhood of K. Indeed, $1-\tau^{-2}\sum B_{ij}\zeta_i\zeta_j$ is a homogeneous degree zero $\mathcal{C}^\infty$ function on a neighborhood of K in ${}^{\mathrm{b}}T^*X$ (hence $\mathcal{C}^\infty$ near K in ${}^{\mathrm{b}}S^*X$) which vanishes at $\mathcal{G}\cap T^*\mathcal{F}_k$. Since there exists $F' \in \Psi_{\mathrm{b}}^{-1}(X)$ with $\mathrm{WF}_{\mathrm{b}}'(F') \subset U$ satisfying

$$\|Fv\| \leq 2\sup|f|\,\|v\| + \|F'v\|$$

for all $v \in L^2(X)$, we deduce that $\|Fv\| \leq 2C_3\delta\|v\| + \|F'v\|$ for all $v \in L^2(X)$. Applying this with $v = D_tA_ru$, and estimating $\|F'v\|$ using Lemma 3.13, (7.2) follows, which in turn completes the proof of the lemma. □

We are now ready to state and prove the tangential propagation estimate. First, local coordinates (x,y,t) near $p \in \mathcal{F}_{i,\mathrm{reg}}$ give a product decomposition of a neighborhood of $p \in \mathcal{F}_{i,\mathrm{reg}}$ in X of the form $U \times V$, $U \subset [0,\infty)^k$, $V \subset \mathbb{R}^{l+1}$ (where k is the codimension of $\mathcal{F}_i$ in X), hence of T^*X as $T^*U \times T^*V$. We denote the projection $T^*X \to T^*V$ by π_i^e. Explicitly, in local coordinates (x,y,t,ξ,ζ,τ) on T^*X,

$$\pi_i^e(x,y,t,\xi,\zeta,\tau) = (y,t,\zeta,\tau).$$

With $\pi_i : T^*_{\mathcal{F}_{i,\mathrm{reg}}}X \to {}^{\mathrm{b}}\dot{T}^*X$ being the restriction of π to $T^*_{\mathcal{F}_{i,\mathrm{reg}}}X$, π_i^e is an extension of π_i in the sense that $\pi_i^e|_{T^*_{\mathcal{F}_{i,\mathrm{reg}}}X\cap(T^*U\times T^*V)} = \pi_i$. The tangential propagation estimate is then the following:

Proposition 7.3. *Let* $u \in H^1_{0,loc}(X)$. *Given* $K \subset {}^{b}S^*X$ *compact with*

$$K \subset (\mathcal{G}\cap T^*\mathcal{F}_{i,reg}) \setminus \mathrm{WF}_b^{-1,\infty}(Pu), \tag{7.4}$$

there exist constants $C_0 > 0$, $\delta_0 > 0$ *such that the following holds. If* $q_0 = (y_0,t_0,\zeta_0,\tau_0) \in K$ *and for some* $0 < \delta < \delta_0$, $C_0\delta \leq \epsilon < 1$ *and for all* $\alpha = (x,y,t,\xi,\zeta,\tau) \in \mathrm{Char}(P)$

$$\begin{gathered}\alpha \in T^*\mathcal{F}_{j,reg} \text{ and } |\pi_i^e(\alpha - \exp(-\delta H_p)(\hat\pi^{-1}(q_0)))| \leq \epsilon\delta \text{ and } |x(\alpha)| \leq \epsilon\delta \\ \Rightarrow \pi_j(\alpha) \notin \mathrm{WF}_b(u),\end{gathered} \tag{7.5}$$

then $q_0 \notin \mathrm{WF}_b(u)$. *Here recall that* $\hat\pi = \pi|_{\mathrm{Char}(P)}$.

Remark 7.4. In the estimate (7.5), H_p can be replaced by any $\mathcal{C}^\infty$ vector field which agrees with H_p at the point $\hat\pi^{-1}(q_0)$, since flow to distance δ along a vector field only depends on the vector field evaluated at the initial point of the flow, up to committing an error $\mathcal{O}(\delta^2)$. In particular, it can be replaced by the vector field $W^\flat$ defined below. Similarly, changing the initial point of the flow by $\mathcal{O}(\delta^2)$ will not affect the endpoint up to an error $\mathcal{O}(\delta^2)$. Thus, estimate (7.5) can be further rewritten, at the cost of changing C_0 again, as

$$\begin{gathered}\alpha \in T^*\mathcal{F}_{j,\mathrm{reg}} \text{ and } |\pi_i^e(\exp(\delta W^\flat)(\alpha)) - \xi_0| \leq \epsilon\delta \text{ and } |x(\exp(\delta W^\flat)(\alpha))| \leq \epsilon\delta \\ \Rightarrow \pi_j(\alpha) \notin \mathrm{WF}_{\mathrm{b}}(u);\end{gathered} \tag{7.6}$$

here we also interchanged the roles of the intial and final points of the flow.

Proof. The proof is very similar to the previous one and now the positive commutator construction follows that of Melrose and Sjöstrand [13], as well as [24] in N-body scattering without bound states. Thus, we take local coordinates as above, i.e. of the form (x,y,t) with the $\mathcal{F}_j$ intersecting the coordinate neighborhood defined by the vanishing of components of x. We can use $t-t_0$ now to measure propagation, since $\tau^{-1}H_p(t-t_0)=2>0$. More precisely, to allow for both signs of τ and yet keep the sign of the derivative along H_p fixed, we need to take

$$\tilde{\eta}=(\operatorname{sign}\tau)(t-t_0)$$

as the propagation variable, so $|\tau|^{-1}H_p\tilde{\eta}=2$. However, for the sake of notational simplicity and clarity, we take $\tau_0>0$, and make all symbols below supported in $\tau>0$ – the general setting only requires replacing $t-t_0$ by $\tilde{\eta}$ in (7.11) below.

Then we could construct $\omega_0\in\mathcal{C}^\infty(T^*\mathcal{F}_i)$ (defined near q_0) to measure the squared distance from the integral curve of

$$W^\flat=2\tau\partial_t-H_h,\ h(y,\zeta)=\zeta\cdot B(y)\zeta \tag{7.7}$$

through q_0; this can be achieved by solving a Cauchy problem as in [13], [24]. In fact, this does not need to be done precisely – after all, $W^\flat$ is only an approximation to H_p in the very first place. Thus, all we need is that ω_0 is the sum of squares of $2l$ homogeneous degree zero functions ρ_j:

$$\omega_0=\sum_{j=1}^{2l}\rho_j^2,\ W^\flat\rho_j(q_0)=0,\ \rho_j(q_0)=0,$$

$d\rho_j(q_0)$, $j=1,\ldots,2l$ linearly independent at q_0. Since $\dim\mathcal{F}_j=l+1$, $d\rho_j(q_0)$, $j=1,\ldots,2l$, together with dt (t is also homogeneous degree zero), span the cotangent space of the quotient of $T^*\mathcal{F}_i$ by the $\mathbb{R}^+$-action, for dimensional reasons (note that $W^\flat t(q_0)\neq 0$). In particular,

$$|\tau^{-1}W^\flat\omega_0|\leq C_1'\omega_0^{1/2}(\omega_0^{1/2}+|t-t_0|)$$

Then we extend ω_0 to a function on ${}^{\mathrm{b}}T^*X$ (using the coordinates $(x,y,t,\sigma,\zeta,\tau)$), let

$$\omega=\omega_0+|x|^2. \tag{7.8}$$

Then the 'naive' estimate, playing an analogous role to (6.9) in the hyperbolic region, is

$$\begin{aligned}|\tau^{-1}H_p\omega|&\leq\tilde{C}_1''\omega^{1/2}(\omega^{1/2}+|t-t_0|+\tau^{-2}|\xi|^2)\\&\leq C_1''\omega^{1/2}(\omega^{1/2}+|t-t_0|+\tau^{-2}|p|),\end{aligned} \tag{7.9}$$

where we used that $p|_{x=0}=\tau^2-|\xi|_y^2-|\zeta|_y^2$ lets us estimate

$$\tau^{-2}|\xi|^2\leq C(\tau^{-2}|p|+|x|+\omega_0^{1/2}+|t-t_0|),$$

for $1-\tau^{-2}|\zeta|_y^2$ is homogeneous degree zero and vanishes at $\mathcal{G}$ (recall from the beginning of the section that this last estimate motivates Lemma 7.1). Note that (7.9) is much more precise than (6.9): we have a factor of $\omega^{1/2}+|t-t_0|+\tau^{-2}|p|$ in addition to $\omega^{1/2}$ – this is crucial since we need to get the direction of propagation right. Again, we in fact need a more explicit version of this:

$$\begin{gathered}\tau^{-1}H_p\omega=f_0+\sum_i f_i\tau^{-1}\xi_i+\sum_{i,j}f_{ij}\tau^{-2}\xi_i\xi_j,\\ f_i,f_{ij}\in\mathcal{C}^\infty({}^{\mathrm{b}}T^*X),\ |f_i|\leq C_1\omega^{1/2}(\omega^{1/2}+|t-t_0|),\ |f_{ij}|\leq C_1\omega^{1/2}\end{gathered} \tag{7.10}$$

f_i, f_{ij} homogeneous of degree 0. Note that the estimates on f_{ij} are weaker than the estimates on f_i. In fact, f_{ij} arises from the $2\sum(\partial_{y_k}A_{ij})\xi_i\xi_j\partial_{\zeta_k}$ term of H_p in (6.3) – when applied to ρ_j^2, it gives a result of the stated form. The reason for the sufficiency of this weaker estimate is that at $\hat{\pi}^{-1}(q_0)$, $\xi = 0$, so the f_{ij} term can be estimated using P (as will be done below), as was already done at a formal level in (7.9).

Finally, we let

$$\phi = t - t_0 + \frac{1}{\epsilon^2\delta}\omega, \tag{7.11}$$

and define a almost as in (6.13), with η replaced by $t - t_0$, namely

$$a = \chi_0(A_0^{-1}(2 - \phi/\delta))\chi_1((t - t_0 + \delta)/\epsilon\delta + 1)\chi_2(|\sigma|^2/\tau^2). \tag{7.12}$$

The slight difference is in the argument of χ_1, in order to microlocalize more precisely in the 'hypothesis region', i.e. where u is a priori assumed to have no wave front set. This is natural, since for the hyperbolic points we only needed to prove that singularities cannot stay at the given boundary face $\mathcal{F}_{i,\mathrm{reg}}$, while for glancing points we need to get the correct direction of propagation. We always assume $\epsilon < 1$, so on $\operatorname{supp} a$ we have

$$\phi \leq 2\delta \text{ and } t - t_0 \geq -\epsilon\delta - \delta \geq -2\delta.$$

Since $\omega \geq 0$, the first of these inequalities implies that $t - t_0 \leq 2\delta$, so on $\operatorname{supp} a$

$$|t - t_0| \leq 2\delta. \tag{7.13}$$

Hence,

$$\omega \leq \epsilon^2\delta(2\delta - (t - t_0)) \leq 4\delta^2\epsilon^2. \tag{7.14}$$

Moreover, on $\operatorname{supp} d\chi_1$,

$$t - t_0 \in [-\delta - \epsilon\delta, -\delta],\ \omega^{1/2} \leq 2\epsilon\delta, \tag{7.15}$$

so this region lies in (7.6) after ϵ and δ are both replaced by appropriate constant multiples, namely the present δ should be replaced by $\delta/2\tau_0$.

We again start with the imprecise motivational argument. Thus, using (7.9), (7.14), $\tau^{-1}H_p(t - t_0) = 2 = c_0 > 0$, we deduce that at $p = 0$,

$$\begin{aligned}\tau^{-1}H_p\phi &= H_p(t - t_0) + \frac{1}{\epsilon^2\delta}H_p\omega\\ &\geq c_0/2 - \frac{1}{\epsilon^2\delta}C_1''\omega^{1/2}(\omega^{1/2} + |t - t_0|)\\ &\geq c_0/2 - 2C_1''(\delta + \frac{\delta}{\epsilon}) \geq c_0/4 > 0\end{aligned}$$

provided that $\delta < \frac{c_0}{16C_1''}$, $\frac{\epsilon}{\delta} > \frac{16C_1''}{c_0}$, i.e. that δ is small, but ϵ/δ is not too small – roughly, ϵ can go to 0 at most proportionally to δ (with an appropriate constant) as $\delta \to 0$. (Recall also that $\epsilon < 1$, so there is an upper bound as well for ϵ, but this is of no significance as we let $\delta \to 0$. It is also worth remembering that in the hyperbolic region, ϵ roughly played the same role as here, but was bounded below by an absolute constant, rather than by a suitable multiple of δ, hence could not go to 0 as $\delta \to 0$.) With this, we can proceed exactly as in the hyperbolic region, so (recall that $\tau > 0$ on $\operatorname{supp} a$!)

$$H_pa^2 = -b^2 + e,\ b = \tau^{1/2}(2\tau^{-1}H_p\phi)^{1/2}(A_0\delta)^{-1/2}(\chi_0\chi_0')^{1/2}\chi_1\chi_2,$$

with e arising from the derivative of $\chi_1\chi_2$. Again, χ_0 stands for $\chi_0(A_0^{-1}(2 - \frac{\phi}{\delta}))$, etc. In view of (7.15) and (7.6) on the one hand, and that $d\chi_2$ is disjoint from the characteristic set on the other, both $\operatorname{supp} d\chi_1$ and $\operatorname{supp} d\chi_2$ are disjoint from

$\mathrm{WF}_{\mathrm{b}}(u)$. Thus, $i[A^*A, P]$ is positive modulo terms that we can a priori control, so the standard positive commutator argument gives an estimate for Bu, where B has symbol b. Replacing a by $a\tau^{s+1/2}$, we still have a positive commutator (again, D_t actually commutes with P, but in any case we could use A_0 to bound the additional commutator term), which now gives (with the new B) that $Bu \in L^2(X)$, which means in particular that $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{1,s}(u)$.

The detailed proof is analogous to the hyperbolic case, with the biggest difference being the treatment of the f_{ij} term in $\tau^{-1}H_p\omega$. First,

$$\begin{aligned}\tau^{-1}H_p\phi &= \tau^{-1}H_p(t-t_0) + \frac{1}{\epsilon^2\delta}\tau^{-1}H_p\omega\\ &= 2 + \frac{1}{\epsilon^2\delta}(f_0 + \sum_i f_i\tau^{-1}\xi_i + \sum_{i,j} f_{ij}\tau^{-2}\xi_i\xi_j).\end{aligned}\tag{7.16}$$

Let $\tilde{B} \in \Psi_{\mathrm{b}}^{1/2}(X)$ with

$$\tilde{b} = \sigma_{b,0}(\tilde{B}) = \tau^{1/2}(A_0\delta)^{-1/2}(\chi_0\chi_0')^{1/2}\chi_1\chi_2 \in \mathcal{C}^\infty({}^{\mathrm{b}}T^*X \setminus o),$$

and let $A \in \Psi_{\mathrm{b}}^0(X)$ with $\sigma_{b,0}(A) = a$. Again, χ_0 stands for $\chi_0(A_0^{-1}(2 - \frac{\phi}{\delta}))$, etc. Also, let $C \in \Psi_{\mathrm{b}}^0(X)$ have symbol $\sigma_{b,0}(C) = \sqrt{2}\,\psi$ where $\psi \in S^0({}^{\mathrm{b}}T^*X)$ is identically 1 on U considered as a subset of ${}^{\mathrm{b}}T^*X$. Then an explicit calculation using Lemma 2.8 gives, in accordance with (7.16),

$$\begin{aligned}&i[A^*A, P]\\ &\quad = R'P + \tilde{B}^*(C^*C + R_0 + \sum_i D_{x_i}R_i + \sum_{ij} D_{x_i}R_{ij}D_{x_j})\tilde{B} + R'' + E + E'\end{aligned}$$

with

$$\begin{aligned}&R_0 \in \Psi_{\mathrm{b}}^0(X),\ R_i \in \Psi_{\mathrm{b}}^{-1}(X),\ R_{ij} \in \Psi_{\mathrm{b}}^{-2}(X),\\ &R' \in \Psi_{\mathrm{b}}^{-1}(X),\ R'' \in \mathrm{Diff}^2\Psi_{\mathrm{b}}^{-2}(X),\ E, E' \in \mathrm{Diff}^2\Psi_{\mathrm{b}}^{-1}(X),\end{aligned}$$

with $\mathrm{WF}_{\mathrm{b}}'(E) \subset \eta^{-1}((-\infty, -\delta]) \cap U$, $\mathrm{WF}_{\mathrm{b}}'(E') \cap \dot{\Sigma} = \emptyset$ (E arises from the commutator of P with an operator with symbol $\chi_1(\eta/\delta + 2)$, while E' from the commutator of P with an operator with symbol $\chi_2(|\sigma|^2/\tau^2)$) and with $r_0 = \sigma_{b,0}(R_0)$, $r_i = \sigma_{b,-1}(R_i)$, $r_{ij} \in \sigma_{b,-2}(R_{ij})$,

$$|r_0| \leq \frac{C_2}{\epsilon^2\delta}\omega^{1/2}(|t-t_0| + \omega^{1/2}),\ |\tau r_i| \leq \frac{C_2}{\epsilon^2\delta}\omega^{1/2}(|t-t_0| + \omega^{1/2}),\ |\tau^2 r_{ij}| \leq \frac{C_2}{\epsilon^2\delta}\omega^{1/2},$$

and $\operatorname{supp} r_j$ lying in $\omega^{1/2} \leq 3\epsilon\delta$, $|t-t_0| < 3\delta$. Thus,

$$|r_0| \leq 3C_2(\delta + \frac{\delta}{\epsilon}),\ |\tau r_i| \leq 3C_2(\delta + \frac{\delta}{\epsilon}),\ |\tau^2 r_{ij}| \leq 3C_2\epsilon^{-1}.$$

Thus, the R_0 and R_i terms can be treated exactly as in the hyperbolic case, i.e. as in the proof of Proposition 6.2. That is, as in the hyperbolic setting, let $T \in \Psi_{\mathrm{b}}^{-1}(X)$ be elliptic, $T^- \in \Psi_{\mathrm{b}}^1(X)$ a parametrix, so $T^-T = \mathrm{Id} + F$, $F \in \Psi_{\mathrm{b}}^{-\infty}(X)$. Then there exist $R_0', R_i' \in \Psi_{\mathrm{b}}^{-1}(X)$ such that for any $\gamma > 0$,

$$\begin{aligned}|\langle R_0 v, v\rangle| &\leq \|R_0 v\|\,\|v\| \leq 2\sup|r_0|\,\|v\|^2 + \|R_0'v\|\,\|v\|\\ &\leq 6C_2(\frac{\delta}{\epsilon} + \delta)\|v\|^2 + \gamma^{-1}\|R_0'v\|^2 + \gamma\|v\|^2,\end{aligned}$$

$$\begin{aligned}\|R_i w\| = \|R_i(T^- T - F)w\| &\leq \|(R_i T^-)(Tw)\| + \|R_i F w\| \\ &\leq 6C_2(\frac{\delta}{\epsilon} + \delta)\|Tw\| + \|R_i' Tw\| + \|R_i F w\|\end{aligned}$$

for all w with $Tw \in L^2(X)$, hence

$$\begin{aligned}|\langle R_i D_{x_i} v, v\rangle| \leq &6C_2(\frac{\delta}{\epsilon} + \delta)\|TD_{x_i} v\|\,\|v\| \\ &+ 2\gamma\|v\|^2 + \gamma^{-1}\|R_i' TD_{x_i} v\|^2 + \gamma^{-1}\|F_i D_{x_i} v\|^2,\end{aligned}$$

with

However, the R_{ij} term needs to be treated separately, since we need that microlocally $\tau^{-1}D_{x_i}$ is small (bounded by a constant multiple of δ), and not merely bounded, which is all we needed both in the proof of Proposition 6.2 and here for the R_0 and R_i terms. This is accomplished by the use of Lemma 7.1. Namely, as in the hyperbolic setting, there exist $R_{ij}' \in \Psi_{\mathrm{b}}^{-1}(X)$ such that

$$\|(T^-)^* R_{ij} w\| \leq 6C_2\epsilon^{-1}\|Tw\| + \|R_{ij}' Tw\| + \|(T^-)^* R_{ij} F w\|$$

for all w with $Tw \in L^2(X)$. Thus,

$$\begin{aligned}|\langle R_{ij} D_{x_i} v, D_{x_j} v\rangle| \leq 6C_2\epsilon^{-1}&\|TD_{x_i} v\|\,\|TD_{x_j} v\| \\ &+ \gamma\|TD_{x_j} v\|^2 + \gamma^{-1}\|R_{ij}' TD_{x_i} v\|^2 + \gamma^{-1}\|F_{ij} D_{x_i} v\|^2 \\ &+ \|R_{ij} D_{x_i} v\|\,\|FD_{x_j} v\|,\end{aligned}$$

with $F_{ij} \in \Psi_{\mathrm{b}}^{-\infty}(X)$. For $v = \tilde{B}_r u$, $\tilde{B}_r = \tilde{B}\Lambda_r$, Lemma 7.1 thus gives

$$\begin{aligned}|\langle R_{ij} D_{x_i} \tilde{B}_r u, D_{x_j} \tilde{B}_r u\rangle| \leq 6C_2'\frac{\delta}{\epsilon}&\|\tilde{B}_r u\|^2 + \gamma\|\tilde{B}_r u\|^2 \\ &+ \gamma^{-1}\|R_{ij}' TD_{x_i} \tilde{B}_r u\|^2 + \gamma^{-1}\|F_{ij} D_{x_i} \tilde{B}_r u\|^2 \\ &+ \|R_{ij} D_{x_i} \tilde{B}_r u\|\,\|FD_{x_j} \tilde{B}_r u\|.\end{aligned}$$

For $\delta < \delta_0$, $\frac{\delta}{\epsilon} < C_0'$ sufficiently small, we finish the proof as in the hyperbolic setting, showing that $\tilde{B}\Lambda_0 u \in L^2(X)$, and hence that $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{1,s}(u)$.

Again, (7.12) needs to be modified slightly to show $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u)$. Now we take, with $\nu \leq 1$,

$$a = \chi_0(A_0^{-1}(1 + \nu - \phi/\delta))\chi_1((t - t_0 + \delta)/\epsilon\delta + \nu)\chi_2(|\sigma|^2/\tau^2),$$

i.e. we replace 2 by $1 + \nu$ in in the argument of χ_0, and we replace 1 by ν in the argument of χ_1. In the iterative step we decrease ν by an arbitrarily small amount, which suffices to prove $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u)$; see also the proof of Proposition 6.2 here, and the proof of [4, Proposition 24.5.1]. □

The results of this section can be adapted to Neumann boundary conditions, using the argument presented at the end of the previous section.

8. Propagation of singularities

An argument of Melrose and Sjöstrand [13, 14], see also [4, Chapter XXIV] and [11] allows us to conclude our main result concerning the singularities of solutions of the wave equation. The proof presented below essentially follows Lebeau's paper [11, Proposition VII.1]. Correspondingly, we only give the proof at $\mathcal{H}$ in full detail; at $\mathcal{G}$ the arguments are sketched, but the details are *precisely* as in Lebeau's case. We mostly discuss the Dirichlet boundary condition – the results are also valid for Neumann boundary conditions, see Theorem 8.5, and the arguments presented need no modification at all in that case. We thus have the following theorem.

Theorem 8.1. *Suppose that* $u \in H^1_{0,loc}(X)$. *Then* $\mathrm{WF}_b^{1,\infty}(u) \setminus \mathrm{WF}_b^{-1,\infty}(Pu) \subset \dot{\Sigma}$, *and it is a union of maximally extended generalized broken bicharacteristics of* P *in* $\dot{\Sigma} \setminus \mathrm{WF}_b^{-1,\infty}(Pu)$.

In fact, if $u \in H^{1,m}_{0,loc}(X)$ *for some* $m \leq 0$, *then for all* $s \in \mathbb{R} \cup \{\infty\}$, $\mathrm{WF}_b^{1,s}(u) \setminus \mathrm{WF}_b^{-1,s+1}(Pu) \subset \dot{\Sigma}$, *and it is a union of maximally extended generalized broken bicharacteristics of* P *in* $\dot{\Sigma} \setminus \mathrm{WF}_b^{-1,s+1}(Pu)$.

Remark 8.2. Suppose that for each boundary hypersurface H_j, we are given Dirichlet data $g_j \in \mathcal{C}^\infty(H_j)$, which are compatible, so at $H_i \cap H_j$, $g_i|_{H_i \cap H_j} = g_j|_{H_i \cap H_j}$ for all i, j. Then there is $g \in \mathcal{C}^\infty(X)$ with $g|_{H_j} = g_j$. Now, if $u \in H^1_{\mathrm{loc}}(X)$ and $u|_{H_j} = g_j$, then $v = u - g \in H^1_{0,\mathrm{loc}}(X)$. Thus, the theorem is applicable to v. Since $Pv = Pu - Pg$ and $Pg \in \mathcal{C}^\infty(X)$, $\mathrm{WF}_{\mathrm{b}}^{-1,\infty}(Pu) = \mathrm{WF}_{\mathrm{b}}^{-1,\infty}(Pv)$, and similarly $\mathrm{WF}_{\mathrm{b}}^{1,\infty}(u) = \mathrm{WF}_{\mathrm{b}}^{1,\infty}(v)$, we deduce that $\mathrm{WF}_{\mathrm{b}}^{1,\infty}(u) \setminus \mathrm{WF}_{\mathrm{b}}^{-1,\infty}(Pu)$ is a union of maximally extended generalized broken bicharacteristics of P in $\dot{\Sigma} \setminus \mathrm{WF}_{\mathrm{b}}^{-1,\infty}(Pu)$.

Remark 8.3. As already expained in the introduction, we can relax the hypothesis $u \in H^1_{0,\mathrm{loc}}(X)$ in the results of Sections 4-7 to $u \in H^{1,m}_{\mathrm{b},0,\mathrm{loc}}(X)$, $m \leq 0$ without changing the arguments, except replacing the $H^1_{\mathrm{loc}}(X)$ norms by the $H^{1,m}_{\mathrm{b,loc}}$ norms for the 'background terms', such as $\|u\|_{H^1_{\mathrm{loc}}(X)}$ in Lemma 4.2 (and (6.1)), and analogously for $\|Pu\|_{H^{-1}_{\mathrm{loc}}(X)}$. The microlocal norms, in which we are gaining regularity, such as those of Gu and $\tilde{G}Pu$ in Lemma 4.2 and (6.1) are *unchanged!* Indeed, now we merely need to apply Lemma 3.18 in place of Lemma 3.13.

The point of this generalization is to allow more singular (approximate) solutions of the wave equation, such as its fundamental solution. An alternative way to deal with these solutions is to regularize them in time (which one can do without destroying, say, $Pu = 0$), and use the $H^1_{0,\mathrm{loc}}(X)$ results – but stating (and proving) the result for $u \in H^{1,m}_{\mathrm{b},0,\mathrm{loc}}(X)$ is the neater way to proceed.

Corollary 8.4. *Suppose that* $Pu = 0$, $u \in H^1_{0,loc}(X)$. *Then* $\mathrm{WF}_b(u) \subset \dot{\Sigma}$, *and it is a union of maximally extended generalized broken bicharacteristics of* P *in* $\dot{\Sigma}$.

The theorem for Neumann boundary conditions takes the following form.

Theorem 8.5. *Suppose that* $u \in H^1_{loc}(X)$ *and* $f \in \dot{H}^{-1}_{loc}(X)$. *Suppose also that for all* $v \in H^1_c(X)$,

$$\langle D_t u, D_t v\rangle - \langle d_M u, d_M v\rangle = \langle f, v\rangle. \tag{8.1}$$

Then $\mathrm{WF}_b^{1,s}(u) \setminus \mathrm{WF}_b^{-1,s+1}(f) \subset \dot{\Sigma}$, *and it is a union of maximally extended generalized broken bicharacteristics of* P *in* $\dot{\Sigma} \setminus \mathrm{WF}_b^{-1,s+1}(f)$.

In fact, if $u \in H^{1,m}_{loc}(X)$ *for some* $m \leq 0$, *and* (8.1) *holds for all* $v \in H^{1,-m}_c(X)$ *then for all* $s \in \mathbb{R} \cup \{\infty\}$, $\mathrm{WF}_b^{1,s}(u) \setminus \mathrm{WF}_b^{-1,s+1}(f) \subset \dot{\Sigma}$, *and it is a union of maximally extended generalized broken bicharacteristics of* P *in* $\dot{\Sigma} \setminus \mathrm{WF}_b^{-1,s+1}(f)$.

Proof. (Proof of Theorem 8.1.) For notational simplicity, we state the proof for $\mathrm{WF}_{\mathrm{b}}^{1,\infty}(u)$. The case of general s only requires notational changes. Note that $\mathrm{WF}_{\mathrm{b}}^{1,\infty}(u) \setminus \mathrm{WF}_{\mathrm{b}}^{-1,\infty}(Pu) \subset \dot{\Sigma}$ by Proposition 4.6, so we only need to prove that it is a union of maximally extended generalized broken bicharacteristics of P in $\dot{\Sigma} \setminus \mathrm{WF}_{\mathrm{b}}^{-1,\infty}(Pu)$.

We start by remarking that for every $V \subset \dot{\Sigma}$ and $q \in V$, the set $\mathcal{R}$ of generalized broken bicharacteristics γ defined on open intervals including 0, satisfying $\gamma(0) = q$, and with image in V, has a natural partial order, namely if $\gamma : (\alpha, \beta) \to V$, $\gamma' : (\alpha', \beta') \to V$, then $\gamma \leq \gamma'$ if the domains satisfy $(\alpha, \beta) \subset (\alpha', \beta')$ and $\gamma = \gamma'|_{(\alpha,\beta)}$. Moreover, any non-empty totally ordered subset has an upper bound: one can take the generalized broken bicharacteristic with domain given by the union of the domains of those in the totally ordered subset, and which extends these, as an upper bound. Hence, by Zorn's lemma, if $\mathcal{R}$ is not empty, it has a maximal element. Note that we can also work with intervals of the form $(\alpha, 0]$, $\alpha < 0$, instead of open intervals.

We only need to prove that for every $q_0 \in \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u) \setminus \mathrm{WF}_{\mathrm{b}}^{-1,\infty}(Pu)$ there exists a generalized broken bicharacteristic $\gamma : [-\epsilon_0, \epsilon_0] \to \dot{\Sigma}$, $\epsilon_0 > 0$, with $\gamma(0) = q_0$ and such that $\gamma(t) \in \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u) \setminus \mathrm{WF}_{\mathrm{b}}^{-1,\infty}(Pu)$ for $t \in [-\epsilon_0, \epsilon_0]$. In fact, once this statement is shown, taking $V = \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u) \setminus \mathrm{WF}_{\mathrm{b}}^{-1,\infty}(Pu)$, $q = q_0$, in the argument of the previous paragraph, we see that $\mathcal{R}$ is non-empty, hence has a maximal element. We need to show that such an element, $\gamma : (\alpha, \beta) \to \dot{\Sigma}$, is maximal in $\dot{\Sigma} \setminus \mathrm{WF}_{\mathrm{b}}^{-1,\infty}(Pu)$ as well, i.e. if we take $V = \dot{\Sigma} \setminus \mathrm{WF}_{\mathrm{b}}^{-1,\infty}(Pu)$, $q = q_0$ in the first paragraph. But if $\gamma' : (\alpha', \beta') \to \dot{\Sigma}$ is any proper extension of γ, with say $\alpha' < \alpha$, with image in $\dot{\Sigma} \setminus \mathrm{WF}_{\mathrm{b}}^{-1,\infty}(Pu)$, then $\gamma'(\alpha) \in \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u)$ since $\mathrm{WF}_{\mathrm{b}}^{1,\infty}(u)$ is closed, and γ maps into it, hence by our assumption there is a generalized broken bicharacteristic $\tilde{\gamma} : (\alpha - \epsilon', \alpha + \epsilon') \to \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u) \setminus \mathrm{WF}_{\mathrm{b}}^{-1,\infty}(Pu)$, $\epsilon' > 0$, $\tilde{\gamma}(\alpha) = \gamma'(\alpha)$; piecing together $\tilde{\gamma}|_{(\alpha-\epsilon',\alpha]}$ and γ, directly from Definition 1.1, gives a generalized broken bicharacteristic which is a proper extension of γ, with image in $\mathrm{WF}_{\mathrm{b}}^{1,\infty}(u) \setminus \mathrm{WF}_{\mathrm{b}}^{-1,\infty}(Pu)$, contradicting the maximality of γ.

Indeed, it suffices to show that for any i, if

$$q_0 \in \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u) \setminus \mathrm{WF}_{\mathrm{b}}^{-1,\infty}(Pu) \text{ and } q_0 \in T^*\mathcal{F}_{i,\mathrm{reg}} \tag{8.2}$$

then

$$\begin{gathered} \text{there exists a generalized broken bicharacteristic } \gamma : [-\epsilon_0, 0] \to \dot{\Sigma},\ \epsilon_0 > 0, \\ \gamma(0) = q_0,\ \gamma(t) \in \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u) \setminus \mathrm{WF}_{\mathrm{b}}^{-1,\infty}(Pu),\ t \in [-\epsilon_0, 0], \end{gathered} \tag{8.3}$$

for the existence of a generalized broken bicharacteristic on $[0, \epsilon_0]$ can be demonstrated similarly by replacing the forward propagation estimates by backward ones, and, directly from Definition 1.1, piecing together the two generalized broken bicharacteristics gives one defined on $[-\epsilon_0, \epsilon_0]$.

We proceed to prove that (8.2) implies (8.3) by induction on i. For $i = 0$, this is certainly true by Hörmander's theorem on propagation of singularities, and if $\operatorname{codim} \mathcal{F}_i = 1$, it follows from the Melrose-Sjöstrand theorem.

So suppose that (8.2)⇒(8.3) has been proved for all j with $\mathcal{F}_i \subsetneq \mathcal{F}_j$ and that $q_0 \in \mathcal{H} \cap T^*\mathcal{F}_{i,\mathrm{reg}}$ satisfies (8.2). We use the notation of the proof of Proposition 6.2 below. Let $U \subset \cup_{\mathcal{F}_i \subset \mathcal{F}_j} T^*\mathcal{F}_{j,\mathrm{reg}}$ be a neighborhood of $q_0 = (0, y_0, t_0, \zeta_0, \tau_0)$ in $\dot{\Sigma}$ which is given by equations of the form $|x| < \delta'$, $|y - y_0| < \delta'$, $|t - t_0| < \delta'$, $|\tau - \tau_0| < \delta'$, $|\zeta - \zeta_0| < \delta'$, $\delta' > 0$, such that $H_p \eta > 0$ on $\hat{\pi}^{-1}(U)$ and $U \cap \mathrm{WF}_{\mathrm{b}}^{-1,\infty}(Pu) = \emptyset$. (Recall that $\hat{\pi} = \pi|_{\mathrm{Char}(P)}$.) Such a neighborhood exists since $q_0 \notin \mathrm{WF}_{\mathrm{b}}^{-1,\infty}(Pu)$ and $H_p \eta(\tilde{q}_0) = \tau_0^2 - |\zeta|^2 > 0$ for every $\tilde{q}_0 \in \hat{\pi}^{-1}(q_0)$. Also let U' be a subset of U defined by replacing δ' by a smaller $\delta'' > 0$, and let $\epsilon_0 > 0$ be such that for any generalized broken bicharacteristic γ with $\gamma(0) \in U'$, $\gamma|_{[-\epsilon_0,\epsilon_0]} \in U$. By Proposition 6.2, there is a sequence of points $q_n \in \dot\Sigma$ such that $q_n \in \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u)$, $q_n \to q_0$ as $n \to \infty$, and $\eta(q_n) < 0$ for all n, so we may assume that $q_n \in U'$ for all n. By the inductive hypothesis, for each n, there exists a generalized broken bicharcteristic

$$\tilde\gamma_n : (-\epsilon_n', 0] \to (\mathrm{WF}_{\mathrm{b}}^{1,\infty}(u) \setminus \mathrm{WF}_{\mathrm{b}}^{-1,\infty}(Pu)) \cap \bigcup_{\mathcal{F}_i \subsetneq \mathcal{F}_j} T^*\mathcal{F}_{j,\mathrm{reg}} \tag{8.4}$$

with $\tilde\gamma_n(0) = q_n$. We now use the argument of the first paragraph of the proof (after the introductory remark about s) with $V = (\mathrm{WF}_{\mathrm{b}}^{1,\infty}(u) \setminus \mathrm{WF}_{\mathrm{b}}^{-1,\infty}(Pu)) \cap \bigcup_{\mathcal{F}_i \subsetneq \mathcal{F}_j} T^*\mathcal{F}_{j,\mathrm{reg}}$, and $q = q_n$. Thus, $\tilde\gamma_n \in \mathcal{R}$, which is hence non-empty, hence has a maximal element. We let

$$\gamma_n : (-\epsilon_n, 0] \to (\mathrm{WF}_{\mathrm{b}}^{1,\infty}(u) \setminus \mathrm{WF}_{\mathrm{b}}^{-1,\infty}(Pu)) \cap \bigcup_{\mathcal{F}_i \subsetneq \mathcal{F}_j} T^*\mathcal{F}_{j,\mathrm{reg}} \tag{8.5}$$

be a maximal element of $\mathcal{R}$; it may happen that $-\epsilon_n = -\infty$.

We claim that $\epsilon_n \geq \epsilon_0$. For suppose that $\epsilon_n < \epsilon_0$. By Corollary 5.6, γ_n extends to a generalized broken bicharacteristic on $[-\epsilon_n, 0]$, we continue to denote this by γ_n. Since $\epsilon_n < \epsilon_0$, γ_n is a generalized broken bicharacteristic with image in U; indeed the closure of the image is still in U. Taking into account that η is increasing on generalized broken bicharacteristics in U since $H_p\eta > 0$ there, we conclude that

$$-|\tau(\gamma_n(t))|^{-1}(x(\gamma_n(t)) \cdot \xi(\gamma_n(t))) = \eta(\gamma_n(t)) \leq \eta(\gamma_n(0)) < 0$$

for $t \in [-\epsilon_n, 0]$, hence $x(\gamma_n(t)) \neq 0$. Thus, $\gamma_n(-\epsilon_n) \in \cup_{\mathcal{F}_i \subsetneq \mathcal{F}_j} T^*\mathcal{F}_{j,\mathrm{reg}}$. Moreover, $\gamma_n(-\epsilon_n) \in \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u)$ since $\mathrm{WF}_{\mathrm{b}}^{1,\infty}(u)$ is closed, and $\gamma_n|_{(-\epsilon_n,0]}$ maps into it. Thus, by the inductive hypothesis, there is a generalized broken bicharacteristic,

$$\tilde\gamma_n : (\alpha, -\epsilon_n] \to (\mathrm{WF}_{\mathrm{b}}^{1,\infty}(u) \setminus \mathrm{WF}_{\mathrm{b}}^{-1,\infty}(Pu)) \cap \bigcup_{\mathcal{F}_i \subsetneq \mathcal{F}_j} T^*\mathcal{F}_{j,\mathrm{reg}} \tag{8.6}$$

with $\alpha < -\epsilon_n$, $\tilde\gamma_n(-\epsilon_n) = \gamma_n(-\epsilon_n)$. Hence, piecing together $\tilde\gamma_n$ and γ_n gives a generalized broken bicharacteristic mapping into $(\mathrm{WF}_{\mathrm{b}}^{1,\infty}(u) \setminus \mathrm{WF}_{\mathrm{b}}^{-1,\infty}(Pu)) \cap \bigcup_{\mathcal{F}_i \subsetneq \mathcal{F}_j} T^*\mathcal{F}_{j,\mathrm{reg}}$ and extending γ_n, which contradicts the maximal property of γ_n. Thus, $\epsilon_n \geq \epsilon_0$ as claimed.

By Proposition 5.5, applied with $K = \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u)$, there is a subsequence of $\gamma_n|_{[-\epsilon_0,0]}$ converging uniformly to a generalized broken bicharacteristic

$$\gamma : [-\epsilon_0, 0] \to \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u).$$

In particular, $\gamma(0) = q_0$ and $\gamma(t) \in \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u)$ for all $t \in [-\epsilon_0, 0]$, providing the inductive step.

We now turn to $q_0 \in \mathcal{G} \cap T^*\mathcal{F}_{i,\mathrm{reg}}$. We repeat the argument of Melrose-Sjöstrand, as presented in Lebeau's paper [11, Proposition VII.1]. We very briefly outline the proof below; the detailed version follows Lebeau's closely, with some changes in the notation. Let $U \subset \cup_{\mathcal{F}_i \subset \mathcal{F}_j} T^*\mathcal{F}_{j,\mathrm{reg}} \setminus \mathrm{WF}_{\mathrm{b}}^{-1,\infty}(Pu)$ be a neighborhood of q_0, U_0 a smaller neighborhood, as above. We take $\epsilon_0 > 0$ small. Suppose that $0 < \epsilon < \epsilon_0$,

$q \in U_0$. Let

(8.7)

$$\begin{aligned}\mathcal{R}^1_{q,\epsilon} = \{&\text{generalized broken bicharacteristics } \gamma : [-\epsilon, 0] \to \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u),\\ &\gamma(0) = q,\ \gamma(t) \notin \mathcal{G} \cap T^*\mathcal{F}_{i,\mathrm{reg}} \text{ for } t \in (-\epsilon, 0]\},\\ \mathcal{R}^2_{q,\epsilon} = \{&\text{generalized broken bicharacteristics } \gamma : [-\epsilon', 0] \to \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u),\ \epsilon' \in (0, \epsilon),\\ &\gamma(0) = q,\ \gamma(t) \notin \mathcal{G} \cap T^*\mathcal{F}_{i,\mathrm{reg}} \text{ for } t \in (-\epsilon', 0],\\ &\gamma(-\epsilon') \in \mathcal{G} \cap T^*\mathcal{F}_{i,\mathrm{reg}}\}.\end{aligned}$$

Moreover, reflecting the inequalities in (7.5), let

$$B(q, \epsilon) = \{q' \in \dot{\Sigma} :\ \max\{|\pi_i^e(q') - q|, |x(q')|\} \le \epsilon\}. \tag{8.8}$$

Let $C_0 > 0$ be as in Proposition 7.3. For $q \in \mathcal{G} \cap T^*\mathcal{F}_{i,\mathrm{reg}}$, let

$$D(q, \epsilon) = B(\exp(-\epsilon H_p)(\hat{\pi}^{-1}(q)), C_0\epsilon^2) \cap \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u), \tag{8.9}$$

and for $q \notin \mathcal{G} \cap T^*\mathcal{F}_{i,\mathrm{reg}}$, let

(8.10)

$$\begin{aligned}&D(q, \epsilon) = \{\gamma(-\epsilon) :\ \gamma \in \mathcal{R}^1_{q,\epsilon}\}\\ &\qquad\cup \{B(\exp(-(\epsilon - \epsilon')H_p)(\hat{\pi}^{-1}(\gamma(\epsilon')), C_0(\epsilon - \epsilon')^2) \cap \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u) :\ \gamma \in \mathcal{R}^2_{q,\epsilon}\}.\end{aligned}$$

The reason for introducing $D(q, \epsilon)$ is that it is a good candidate for the beginning point of a generalized broken bicharacteristic segment in $\mathrm{WF}_{\mathrm{b}}^{1,\infty}(u)$, defined over an interval of length ϵ, and ending in q.

Indeed, for $q \in \mathcal{G} \cap T^*\mathcal{F}_{i,\mathrm{reg}} \cap \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u)$, we deduce from Proposition 7.3 that $D(q, \epsilon) \neq \emptyset$. For $q \in \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u) \setminus (\mathcal{G} \cap T^*\mathcal{F}_{i,\mathrm{reg}})$, by the inductive hypothesis, the previous part of the proof concerning $\mathcal{H} \cap T^*\mathcal{F}_{i,\mathrm{reg}}$, and the first two paragraphs (after the introductory remark about s) with $V = \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u) \setminus ((\mathcal{G} \cap T^*\mathcal{F}_{i,\mathrm{reg}}) \cup \mathrm{WF}_{\mathrm{b}}^{-1,\infty}(Pu)$, $q = q_0$, there is a maximally extended generalized broken bicharacteristic γ with image in V. By the argument of the second paragraph, this is either defined on all of $[-\epsilon, 0]$, or only on $(-\epsilon', 0]$ with $0 < \epsilon' < \epsilon$, in which case $\gamma(-\epsilon') \in \mathcal{G} \cap T^*\mathcal{F}_{i,\mathrm{reg}}$, hence again by Proposition 7.3 we conclude that $D(q, \epsilon) \neq \emptyset$. Thus, for all $q \in U \cap \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u)$ we have deduced $D(q, \epsilon) \neq \emptyset$.

For each integer $N \ge 1$ now we define a sequence of $2^N + 1$ points $q_{j,N}$, $j \in \mathbb{N}$, $0 \le j \le 2^N$, which will be used to construct points $\gamma(-j2^{-N}\epsilon_0)$ on the desired generalized broken bicharacteristic $\gamma : [-\epsilon_0, 0] \to \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u)$ through q_0. Namely, let $\epsilon = 2^{-N}\epsilon_0$, $q_{0,N} = q_0$, and choose $q_{j+1,N} \in D(q_{j,N}, \epsilon)$. Let $\mathcal{J}_N = \{-j2^{-N}\epsilon_0 :\ 0 \le j \le 2^N\} \subset [-\epsilon_0, 0]$, $\mathcal{J} = \cup_{N=1}^{\infty}\mathcal{J}_N$. We write $\gamma_N(t) = q_{j,N}$ for $t = -j2^{-N}\epsilon_0$. For each $t \in \mathcal{J}$, the sequence $\gamma_N(t)$ (defined for large N) stays in a compact set. Hence there exists a subsequence γ_{N_k} such that for all $t \in \mathcal{J}$, $\gamma_{N_k}(t)$ converges to some $\gamma(t)$.

This defines $\gamma : [-\epsilon_0, 0] \to \mathrm{WF}_{\mathrm{b}}^{1,\infty}(u)$ at elements of $\mathcal{J}$. One can check exactly as in Lebeau's proof (which we have been following very closely) that γ extends to a continuous map defined on $[-\epsilon_0, 0]$, and that it is a generalized broken bicharacteristic. This completes the inductive step for tangential points $q_0 \in \mathcal{G} \cap T^*\mathcal{F}_{i,\mathrm{reg}}$, hence the proof of the theorem. □

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