Small Representations for Odd Orthogonal Groups

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Abstract. If $2n+1 \geq 9$ Vogan has shown that there is no representation of SO_{2n+1} or a metaplectic cover attached to the minimal coadjoint orbit. Thus there are no small representations in the standard sense. In this paper, representations of the metaplectic double cover of SO_{2n+1} are constructed both locally and globally, and it is shown that these representations are small in the sense that they are attached to one of the next smallest coadjoint orbits with respect to the standard partial ordering. The global automorphic representations are shown to be square integrable and the vanishing properties of their Fourier coefficients are established.

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In recent years, the importance of the minimal representations of reductive Lie groups in both representation theory over local fields and in the theory of automorphic forms has been widely recognized. These minimal representations play a role similar to that of the theta representation of the metaplectic double cover in the classical work of Siegel, Weil and Howe.

Two classes of applications exist for minimal representations: theta liftings when the representation is restricted to a dual reductive pair, giving either a local correspondence (Howe duality) or a global theta correspondence; and Rankin-Selberg constructions of automorphic L-functions. What makes these constructions work is the vanishing of many Jacquet modules locally, or the vanishing of many theta Fourier coefficients globally. The method of coadjoint orbits gives a framework for describing these vanishings. In this language the term "minimal representation" means a representation attached to the minimal nontrivial coadjoint orbit with respect to the partial order.

One may seek such a representation over a local field, and one may also seek a global version which is an automorphic representation. Minimal representations for most groups have already been constructed, beginning with the construction of minimal representations for simply laced groups over local fields by Kazhdan and Savin [KS] and the construction of the corresponding global automorphic representations by Ginzburg, Rallis and Soudry [GRS1]. In many cases the minimal representation is actually a representation of a metaplectic double cover, or in the case of G_2, a metaplectic three-fold cover. For odd orthogonal groups, the local minimal representation of SO_7 has been constructed and studied by Roskies [R], Sabourin [Sa] and Torasso [T] (the paper of Torasso is more general but includes this case), and Bump, Friedberg and Ginzburg [BFG] have shown that there is an automorphic minimal module which moreover may be applied in a Rankin-Selberg construction. However, Vogan has shown that for odd orthogonal groups, there cannot exist even locally a representation attached to the minimal coadjoint orbit of SO_{2n+1} if $2n+1 \geq 9$.

In this paper, we will obtain and study a representation Θ_{2n+1} of the metaplectic double cover of SO_{2n+1} which agrees with the minimal representation when $2n+1=5$ or 7. When $2n+1 \geq 9$ it is not attached to the minimal coadjoint orbit but to one of the next smallest possibilities. In fact we will show that Θ_{2n+1} is attached to the unipotent orbit $\mathcal{O} = (2^{2m}1)$ if $n = 2m$ and to $\mathcal{O} = (2^{2m}1^3)$ if $n = 2m+1$. By this we mean that Θ_{2n+1} does not have any nontrivial Fourier coefficient corresponding to any unipotent orbit which is bigger than or not comparable with $\mathcal{O}$—see Theorem 14. The results of [HL] suggest that there is no automorphic representation on SO_{2n+1} or any of its covering groups which is attached to a smaller orbit than $\mathcal{O}$ in the sense we explained.

We will construct this representation locally using intertwining operators and globally as the residue of a metaplectic Eisenstein series on the metaplectic double cover of SO_{2n+1}. We will use the unipotent classes to describe Jacquet modules of the local representations and the Fourier coefficients of the global automorphic representation. Our main global result is Theorem 14, where we show that every nondegenerate Fourier coefficient attached to a unipotent class is zero, unless that unipotent class is dominated by a particular small one. Thus though this representation is not minimal, it possesses remarkable properties which are close to minimality.

The automorphic forms constructed in this paper can be applied to produce theta-like liftings. In another paper we will show that the restriction of Θ_{2n+1} to the dual pair $\widetilde{SO}_n \times \widetilde{SO}_{n+1}$ gives a metaplectic functorial lifting, much as in the classical case of a dual reductive pair in $\widetilde{Sp}_{2n}$.

In this paper we prove Hypothesis 1.6 in [BFG], thus completing the results of that paper.

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1. Nonarchimedean local theory. Let SO_{2n+1} denote the split special orthogonal group consisting of $g \in SL_{2n+1}$ such that $gJ\,{}^tg = J$, where

$$J = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}.$$

Let T be the maximal torus of diagonal matrices in SO_{2n+1}. If F is a field of characteristic not equal to two, then following [BFG] let $SO_{2n+1}(F)'$ be the kernel of the spinor norm $\mathrm{SN} : SO_{2n+1}(F) \to F^\times/(F^{\times\,2})$. The intersection $T'(F) = T(F) \cap SO_{2n+1}(F)'$ consists of matrices

$$D(t_1, \cdots, t_n) = \mathrm{diag}(t_1, \cdots, t_n, 1, t_n^{-1}, \cdots, t_1^{-1})$$

with $\prod t_i$ a square.

We assume that F is a local field of characteristic not equal to 2 which contains the subgroup μ_4 of all four 4-th roots of unity. Let $\widetilde{SL}_{2n+1}(F)$ denote the metaplectic four-fold cover as in Matsumoto [M] or Kazhdan and Patterson [KP]. For computational purposes we follow these authors in using the cocycle whose value on diagonal elements is

$$\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_{2n+1} \end{pmatrix}, \begin{pmatrix} u_1 & & \\ & \ddots & \\ & & u_{2n+1} \end{pmatrix} \mapsto \prod_{i<j} (t_i, u_j)_4.$$

Let $\sigma \in H^2(SO_{2n+1}(F), \mu_4)$ be the restriction of this cocycle to $SO_{2n+1}(F)$. We have

$$\sigma(D(t_1, \cdots, t_n), D(u_1, \cdots, u_n)) = \prod_{i=1}^{n} (t_i, u_i)_4^{-1}. \tag{1}$$

The cover $\widetilde{SO}_{2n+1}(F)$ of $SO_{2n+1}(F)$ corresponding to this cocycle is morally a double cover in the sense that the square of the cocycle is almost trivial. More precisely we have

$$\sigma(g, h)^2 = (\mathrm{SN}(g), \mathrm{SN}(h))_2$$

(up to a coboundary). To see this, we may work on the torus since the restriction map $H^2(SO_{2n+1}(F), \mu_4) \to H^2(T(F), \mu_4)$ is injective, and there it can be checked by direct computation. The square of σ is therefore trivial on the large subgroup $SO_{2n+1}(F)'$.

The restriction of the metaplectic 4-fold cover of $SL_{2n+1}(F)$ (as constructed by Matsumoto [M]) to $SO_{2n+1}(F)$ is then the cover which we call the metaplectic double cover of $SO_{2n+1}(F)$. We denote this group as $\widetilde{SO}_{2n+1}(F)$. It has μ_4 as a central subgroup.

We recall the construction of the unramified principal series of $\widetilde{SO}_{2n+1}(F)$. Generally if H is a subgroup of $SO_{2n+1}(F)$ we will denote by $\widetilde{H}$ its preimage in $\widetilde{SO}_{2n+1}(F)$. Let $B = TU$ be the standard Borel subgroup of upper triangular matrices in SO_{2n+1}, where U is the unipotent radical. The cover splits canonically over $U(F)$ and we will identify $U(F)$ with its image under this canonical splitting. Any irreducible representation of $\widetilde{T}(F)$ may be extended to $\widetilde{B}(F)$ by letting $U(F)$ act trivially and then parabolically induced to $\widetilde{SO}_{2n+1}(F)$.

However $\widetilde{T}(F)$ is not commutative, so its irreducible representations are not 1-dimensional. It is a 2-step nilpotent group and its irreducible representations are themselves induced representations. We call a representation ρ of any subgroup of $\widetilde{SO}_{2n+1}(F)$ containing μ_4 *genuine* if for $\epsilon \in \mu_4$ we have $\rho(\epsilon g) = \epsilon\, \rho(g)$, where we have fixed an injection $\mu_4 \to \mathbb{C}^\times$, and by abuse of notation identify ϵ with its image in $\mathbb{C}^\times$. If ρ is genuine, then, as explained in Section 0.3 of Kazhdan and Patterson [KP] or in Bump and Ginzburg [BG], it is induced from a one-dimensional quasicharacter ρ_0 of any maximal abelian subgroup of $\widetilde{T}(F)$. Two genuine quasicharacters ρ_0 and ρ_0' give isomorphic inductions to $\widetilde{T}(F)$ if and only if they agree on the center $Z(\widetilde{T}(F))$. Here $Z(\widetilde{T}(F)) \cong T_Z(F) \times \mu_4$, where $T_Z(F)$ is the subgroup of $T(F)$ consisting of elements $D(t_1, \cdots, t_n)$ with each t_i a fourth power, since the canonical section $SO_{2n+1}(F) \to \widetilde{SO}_{2n+1}(F)$ restricted to $T_Z(F)$ is a homomorphism. Thus any quasicharacter of $T_Z(F)$ can be extended uniquely to a genuine quasicharacter of $Z(\widetilde{T}(F))$, and there is a unique genuine irreducible representation of $\widetilde{T}(F)$ having this central character. Therefore the genuine irreducible representations of $\widetilde{T}(F)$ are parametrized by the quasicharacters of $T_Z(F)$. This parametrization is an analog of the Stone-Von Neumann theorem. If η is a quasicharacter of $T_Z(F)$ let $\rho(\eta)$ be the irreducible genuine representation of $\widetilde{T}(F)$ parametrized this way.

Let $s = (s_1, \cdots, s_n)$ be complex parameters. Let $\chi_s : T(F) \to \mathbb{C}^\times$ be the quasicharacter mapping $D(t_1, \cdots, t_n)$ to $\prod_{i=1}^{n} |t_i|^{s_i}$. Let $\delta = \delta_B$ be the modular quasicharacter of B. Thus

$$\delta(D(t_1, \cdots, t_n)) = |t_1|^{2n-1} |t_2|^{2n-3} \cdots |t_n|.$$

Let $\mathrm{Ind}(\delta^{1/2}\chi_s)$ denote the representation of $\widetilde{SO}_{2n+1}(F)$ parabolically induced from the irreducible representation $\rho(\chi_s\delta^{1/2})$ of $\widetilde{T}(F)$. (This is the same as the representation parabolically induced from $\delta^{1/2}\,\rho(\chi_s)$.) If s is in general position this representation is irreducible. These are the unramified principal series representations.

The Weyl group W of SO_{2n+1} acts by conjugation on $T(F)$, and therefore on the parameters s. Let $w \in W$. As in Casselman [C] and Kazhdan and Patterson [KP], we have intertwining integrals $M_w = M_w(s) : \mathrm{Ind}(\delta^{1/2}\chi_s) \to \mathrm{Ind}(\delta^{1/2}\chi_{w(s)})$, defined by

$$M_w(s)f(g) = \int_{(U(F)\cap wU(F)w^{-1})\backslash U(F)} f(w^{-1}ug)\,du. \tag{2}$$

This integral is convergent if $\mathrm{re}(s_i - s_{i+1})$ and $\mathrm{re}(s_n)$ are sufficiently large, but has meromorphic continuation to all s as in Casselman [C].

This definition does depend on the choice of a representative w of the Weyl group element, but only up to a nonzero constant. In the special case where the residue characteristic of F is odd, we can make this choice more canonically. Assume temporarily therefore that the residue characteristic of F is odd. Let K be the standard maximal compact subgroup of $SO_{2n+1}(F)$. In (2) we choose the representative for the Weyl group element w to be an element of K. This determines the entries of w up to units, and since χ_s is nonramified the intertwining operator is independent of this choice. The metaplectic cover splits over K, and we will identify K with its image in $\widetilde{SO}_{2n+1}(F)$ under this splitting. Strictly speaking, elements of $\mathrm{Ind}(\delta^{1/2}\chi_s)$ are functions $\widetilde{SO}_{2n+1}(F) \to \rho(\delta^{1/2}\chi_s)$, but choosing a fixed maximal abelian subgroup $\widetilde{T}(F)'$ of $\widetilde{T}(F)$, $\mathrm{Ind}(\delta^{1/2}\chi_s)$ may be identified with a representation induced from a fixed genuine extension of $\delta^{1/2}\chi_s$ to $\widetilde{B}(F)'$, where $\widetilde{B}(F)' = \widetilde{T}(F)'U(F)$. Elements of this space are functions $\widetilde{SO}_{2n+1}(F) \to \mathbb{C}$. Taking this point of view for the purpose of normalizing the intertwining operators, let $f_s^\circ : \widetilde{SO}_{2n+1}(F) \to \mathbb{C}$ be the standard K-fixed vector in $\mathrm{Ind}(\delta^{1/2}\chi_s)$, which as a function is equal to 1 on K. Then

$$M_w(s)\, f_s^\circ(g) = c(w,\chi_s) f_{w(s)}^\circ(g), \tag{3}$$

where $c(w,\chi)$ may be described as follows. If α is a root of SO_{2n+1}, let $i_\alpha : SL_2 \to SO_{2n+1}$ be the standard embedding of SL_2 along α. We call a root *metaplectic* if the pullback of the metaplectic double cover of SO_{2n+1} along i_α to SL_2 is nontrivial. The metaplectic roots are the long ones. If α is a root of SO_{2n+1}, we let $n(\alpha) = 1$ if α is a nonmetaplectic root, and $n(\alpha) = 2$ if it is metaplectic. As in Casselman [C] let a_α be the image of $\left(\begin{smallmatrix} p & 0 \\ 0 & p^{-1}\end{smallmatrix}\right)$ under i_α, where p is a uniformizer in F. Then

$$c(w,\chi) = \prod_{\alpha>0,\, w(\alpha)<0} \frac{(1-\chi(a_\alpha)^{n(\alpha)})^{-1}}{(1-q^{-1}\chi(a_\alpha)^{n(\alpha)})^{-1}},$$

where q is the cardinality of the residue field. The extension of the Gindikin-Karpelevich formula as in Casselman [C] to the metaplectic group is justified by Kazhdan and Patterson [KP], Section I.2.

We are particularly interested in the principal series representations $\mathrm{Ind}(\delta^{1/2}\chi_s)$ when $s = s_\theta$,

$$s_\theta = \left(\frac{n}{2}, \frac{n-1}{2}, \cdots, \frac{1}{2}\right).$$

We will denote $\chi_\theta = \chi_{s_\theta}$. Since χ_θ is regular, that is, not fixed by any element of the Weyl group except the identity,

$$\dim \mathrm{Hom}_{\widetilde{SO}_{2n+1}(F)}\big(\mathrm{Ind}(\delta^{1/2\ w_1}\chi_\theta), \mathrm{Ind}(\delta^{1/2\ w_2}\chi_\theta)\big) = 1 \tag{4}$$

for any two elements w_1 and w_2 of W. See Proposition I.2.1 and I.2.2 of Kazhdan and Patterson, and the references there to Bernstein and Zelevinsky. We will denote by l the length function on W.

Lemma 1. *Let σ be a simple reflection in W, and let w be any Weyl group element such that $l(w\sigma) = l(w) - 1$. Then the composition*

$$\mathrm{Ind}(\delta^{1/2\ \sigma}\chi_\theta) \xrightarrow{M_\sigma} \mathrm{Ind}(\delta^{1/2}\ \chi_\theta) \xrightarrow{M_w} \mathrm{Ind}(\delta^{1/2\ w}\chi_\theta) \tag{5}$$

is the zero map.

Proof. Since $l(w\sigma) = l(w) - 1$ we have $M_w = M_{w\sigma} \circ M_\sigma$ so we reduce to the case where $w = \sigma$. It follows from (4) that if $w = \sigma$ the composite (5) is a multiple of the identity map.

If the residue characteristic of F is odd, we note that (3) implies that M_σ kills the K-fixed vector in $\mathrm{Ind}(\delta^{1/2\ \sigma}\chi_\theta)$. Indeed, σ is attached to a simple root α. If α is the unique nonmetaplectic simple root, we have

$$\sigma(s_\theta) = \left(\frac{n}{2}, \frac{n-1}{2}, \cdots, 1, -\frac{1}{2}\right),$$
$$a_\alpha = D(1, \cdots, 1, \varpi^2), \qquad n(\alpha) = 1,$$

while if α is metaplectic, then for some i

$$\sigma(s_\theta) = \left(\frac{n}{2}, \frac{n-1}{2}, \cdots, \frac{i}{2}, \frac{i+1}{2}, \cdots, \frac{1}{2}\right),$$
$$a_\alpha = D(1, 1, \cdots, \varpi, \varpi^{-1}, \cdots, 1), \qquad n(\alpha) = 2,$$

where ϖ is a uniformizer. In either case, $1 - q^{-1\ \sigma}\chi_\theta(a_\alpha) = 0$.

If F is dyadic there is no K-fixed vector, but the lemma reduces easily to a computation in either SO_3 or $\widetilde{SL}_2$, where the statement is known. ■

Let J be the Jacquet functor from the category of smooth representations of finite length of $\widetilde{SO}_{2n+1}(F)$ to smooth representations of finite length of $\widetilde{T}(F)$. Let w_0 denote the long element of the Weyl group.

Theorem 2. *The image V_Θ of $M_{w_0} : \mathrm{Ind}(\delta^{1/2}\chi_\theta) \to \mathrm{Ind}(\delta^{1/2}\chi_\theta^{-1})$ is irreducible and self-contragredient. It is the unique irreducible quotient of $\mathrm{Ind}(\delta^{1/2}\chi_\theta)$, and the unique irreducible subrepresentation of $\mathrm{Ind}(\delta^{1/2}\,{}^{w_0}\chi_\theta)$. Its Jacquet module $J(\Theta)$ is $\rho(\delta^{1/2}\chi_\theta^{-1})$.*

We will denote by $\Theta = \Theta_{SO_{2n+1}}$ the representation on the space V_Θ. The proof of this parallels the proof of the "Periodicity Theorem," Kazhdan and Patterson [KP] Theorem I.2.9.

Proof. Since s_θ is regular, $J(\mathrm{Ind}(\delta^{1/2}\chi_\theta))$ is semisimple and in fact is the direct sum of the $\rho(\delta^{1/2}\,{}^{w}\chi_\theta)$ as w runs through W. By exactness of the Jacquet functor, $J(\Theta)$ is the direct sum of some of these factors. We show that only $\rho(\delta^{1/2}\,{}^{w_0}\chi_\theta)$ occurs in $J(\Theta)$. (Note that ${}^{w_0}\chi_\theta = \chi_{\theta^{-1}}$.) Note that Θ is nonzero since M_{w_0} is nonzero.

Suppose that $\rho(\delta^{1/2}\,{}^{w}\chi_\theta)$ occurs in $J(\Theta)$. We prove that $w = w_0$. If not, there is a simple reflection $\sigma \in W$ such that $l(w\sigma) = l(w) + 1$. Since

$$\mathrm{Hom}_{\widetilde{SO}_{2n+1}(F)}(\Theta, \mathrm{Ind}(\delta^{1/2}\,{}^{w}\chi_\theta)) \cong \mathrm{Hom}_{\widetilde{T}(F)}(J(\Theta), \delta^{1/2}\,{}^{w}\chi_\theta)$$

is nonzero we let $\alpha : \Theta \to \mathrm{Ind}(\delta^{1/2}\,{}^{w}\chi_\theta)$ be a nonzero element. Since $M_{w_0} : \mathrm{Ind}(\delta^{1/2}\chi_\theta) \to \Theta$ is surjective and α is nonzero, (4) implies that $\alpha \circ M_{w_0}$ is a nonzero multiple of $M_w : \mathrm{Ind}(\delta^{1/2}\chi_\theta) \to \mathrm{Ind}(\delta^{1/2}\,{}^{w}\chi_\theta)$. Composing this with $M_\sigma : \mathrm{Ind}(\delta^{1/2}\,{}^{\sigma}\chi_\theta) \to \mathrm{Ind}(\delta^{1/2}\chi_\theta)$ gives a contradiction because $M_w \circ M_\sigma$ is nonzero yet $M_{w_0} \circ M_\sigma = 0$ by Lemma 1.

We have proven that $J(\Theta) \cong \rho(\delta^{1/2}\,{}^{w_0}\chi_\theta)$. It follows that Θ is irreducible, as we will now argue. Any composition factor of $\mathrm{Ind}(\delta^{1/2}\chi_\theta)$ has no cuspidal constituents in its Jacquet module with respect to any nonminimal parabolic subgroup, so its Jacquet module with respect to the Borel subgroup is nonzero. By exactness of the Jacquet functor, it follows that if a subquotient of $\mathrm{Ind}(\delta^{1/2}\chi_\theta)$ has an irreducible Jacquet module, it must itself be irreducible.

Finally we must show that Θ is the unique irreducible quotient of $\mathrm{Ind}(\delta^{1/2}\chi_\theta)$. Let π be another irreducible quotient. As a composition factor of $\mathrm{Ind}(\delta^{1/2}\chi_\theta)$, its Jacquet module $J(\pi)$ is nonzero and must be a quotient of

$$J(\mathrm{Ind}(\delta^{1/2}\chi_\theta)) = \bigoplus_w \rho(\delta^{1/2}\,{}^{w}\chi_\theta).$$

Hence there is a nonzero homomorphism $J(\pi) \to \rho(\delta^{1/2}\,{}^{w}\chi_\theta)$ for some w. But

$$\mathrm{Hom}_{\widetilde{SO}_{2n+1}(F)}(\pi, \mathrm{Ind}(\delta^{1/2}\,{}^{w}\chi_\theta)) \cong \mathrm{Hom}_{\widetilde{T}(F)}(J(\pi), \delta^{1/2}\,{}^{w}\chi_\theta),$$

so there is a nonzero intertwining map $\pi \to \mathrm{Ind}(\delta^{1/2}\,{}^{w}\chi_\theta)$. The composition

$$\mathrm{Ind}(\delta^{1/2}\chi_\theta) \to \pi \to \mathrm{Ind}(\delta^{1/2}\,{}^{w}\chi_\theta)$$

is nonzero and by (4), it is a constant multiple of M_w. Therefore the composition

$$\mathrm{Ind}(\delta^{1/2}\chi_\theta) \longrightarrow \pi \longrightarrow \mathrm{Ind}(\delta^{1/2}\,{}^{w}\chi_\theta) \xrightarrow{M_{w_0 w^{-1}}} \mathrm{Ind}(\delta^{1/2}\,{}^{w_0}\chi_\theta)$$

is M_{w_0} and its image is therefore Θ. We see that Θ is a quotient of π, and since π is irreducible they are equal.

Since ${}^{w_0}\chi_\theta = \chi_\theta^{-1}$, the contragredient of $\mathrm{Ind}(\delta^{1/2}\chi_\theta)$ is $\mathrm{Ind}(\delta^{1/2}\,{}^{w_0}\chi_\theta)$. The former has a unique irreducible quotient, so by duality the latter has a unique irreducible subrepresentation, which is obviously Θ. In particular Θ is self-contragredient. ∎

Suppose that $P = MU_P$ is a maximal parabolic subgroup of SO_{2n+1}, with Levi factor $M = GL_m \times SO_{2(n-m)+1}$ and unipotent radical U_P. The preimage $\widetilde{M}(F)$ of $M(F)$ in $\widetilde{SO}_{2n+1}(F)$ is the direct product (with amalgamated μ_4) $\widetilde{GL}_m(F) \times \widetilde{SO}_{2(n-m)+1}(F)$, where $\widetilde{GL}_m$ denotes the metaplectic double cover in the sense of Kazhdan and Patterson [KP]. For $(g_i, h_i) \in GL_m \times SO_{2(n-m)+1}$ $(i = 1, 2)$ using the block compatibility property of the cocycle (see Banks, Levy and Sepanski [BLS]) we have

$$\sigma\big((g_1, h_1), (g_2, h_2)\big) = \sigma_{GL_m}(g_1, g_2)^2 \left(\det(g_1), \det(g_2)\right)_4 \sigma_{SL_{2(n-m)+1}}(h_1, h_2).$$

Since the square of the cocycle made with the 4-th order Hilbert symbols is the cocycle made with the quadratic Hilbert symbol this is a twist of the double cover. The theory of Kazhdan and Patterson [KP] including the construction of the exceptional representations goes through without essential change.

Let us define a character of the torus in GL_m by

$$\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_m \end{pmatrix} \to \prod_{i=1}^{m} |t_i|^{(n-i)/2}. \tag{6}$$

This is an exceptional character in the sense of [KP], so we may form the exceptional representation of $\widetilde{GL}_m(F)$, which we will denote by Θ_{GL_m}.

Let J_P be the Jacquet module with respect to the parabolic subgroup P. It is a functor from the category of smooth representations of finite length of $\widetilde{SO}_{2n+1}(F)$ to the category of smooth representations of finite length of $\widetilde{M}(F)$.

Theorem 3. *The Jacquet module* $J_P(\Theta_{SO_{2n+1}}) \cong \Theta_{GL_m} \otimes \Theta_{SO_{2(n-m)+1}}$.

(If $2(n-m)+1 \leq 3$ the metaplectic cover splits and $\Theta_{SO_{2(n-m)+1}}$ should be interpreted as the trivial representation.)

Proof. Applying Theorem 2 to $\Theta_{2(n-m)+1}$ and the similar Periodicity Theorem I.2.9 (e) of [KP], the Jacquet module with respect to the Borel subgroup $B \cap M$ of M is an irreducible representation of $\widetilde{T}(F)$, and we may compute that it equals $\rho(\delta_B^{1/2\, w_0}\chi_\theta)$. We may factor the Jacquet module from $\widetilde{SO}_{2n+1}(F)$ relative to the Borel subgroup through P, so using Theorem 2 we see that the two representations $J_P(\Theta)$ and $\Theta_{GL_m} \otimes \Theta_{SO_{2(n-m)+1}}$ have the same Jacquet module relative to $B \cap M$, and that $\widetilde{T}(F)$-module is irreducible. As in the proof of Theorem 2 this implies that both are irreducible.

Suppose that π denotes one of these representations. By Frobenius reciprocity

$$\mathrm{Hom}_{\widetilde{M}(F)}(\pi, \mathrm{Ind}(\rho(\delta_B^{1/2\, w_0}\chi_\theta))) \cong \mathrm{Hom}_{\widetilde{T}(F)}(J(\pi), \rho(\delta_B^{1/2\, w_0}\chi_\theta))$$

is nonzero. Here Ind is parabolic induction on M. Using Theorem 2 and [KP] I.2.9 (a), $\mathrm{Ind}(\rho(\delta_B^{1/2\, w_0}\chi_\theta))$ has a unique irreducible subrepresentation. Therefore π is uniquely characterized by its irreducibility and the nonvanishing of this Hom space. ■

Let δ_P be the modular quasicharacter of P. In the following Proposition, "Ind" denotes parabolic induction.

Proposition 4. *The representation $\Theta_{SO_{2n+1}}$ can be realized as a subrepresentation of* $\mathrm{Ind}(\Theta_{GL_m} \otimes \Theta_{SO_{2(n-m)+1}})$ *and, for suitable s_0, as a quotient representation of* $\mathrm{Ind}(\Theta_{GL_m} \otimes \Theta_{SO_{2(n-m)+1}} \otimes \delta_P^{s_0})$.

Proof. By Theorem 3, $J_P(\Theta_{SO_{2n+1}})$ has the form $\Theta_{GL_m} \otimes \Theta_{SO_{2(n-m)+1}}$. By Frobenius reciprocity we get an embedding into $\mathrm{Ind}(\Theta_{GL_m} \otimes \Theta_{SO_{2(n-m)+1}})$. Dualizing and noting that $\Theta_{SO_{2n-1}}$ and $\Theta_{SO_{2(n-m)+1}}$ are each self-contragredient, while the contragredient of Θ_{GL_m} is just a twist by a power of $|\det|$, we realize Θ as a quotient. ■

Let ψ be a nontrivial additive character of F.

Proposition 5. *Suppose that the length $2r_1r_3 + r_2^2$ of (r_1, r_2, r_3) is nonzero. There does not exist a nonzero linear functional Λ on the space of Θ_{SO_5} such that*

$$\Lambda\left(\Theta_{SO_5}\begin{pmatrix} 1 & x_1 & x_2 & x_3 & * \\ & 1 & 0 & 0 & -x_3 \\ & & 1 & 0 & -x_2 \\ & & & 1 & -x_1 \\ & & & & 1 \end{pmatrix} v\right) = \psi(x_1r_1 + x_2r_2 + x_3r_3)\,\Lambda(v). \tag{7}$$

Here and elsewhere a $*$ in any matrix means an unspecified value. In many cases such as this, the value of the $*$ entry is determined by the requirement that the matrix be orthogonal.

Proof. Recall that the derived group $SO_5(F)'$ is the kernel of the spinor norm map. Its index is therefore the number of square classes. There is a surjective homomorphism $Sp_4(F) \to SO_5(F)'$. The pullback of the cover $\widetilde{SO}_5(F)$ is a double cover of $Sp_4(F)$ and since there is a unique such double cover, we get a homomorphism $\widetilde{Sp}_4(F) \to \widetilde{SO}_5(F)'$.

For every nonzero $a \in F$, there is a Weil representation ω_a of $\widetilde{Sp}_4(F)$ acting on the Bruhat-Schwartz space $S(F^2)$. The isomorphism class of ω_a depends on the square class of a. We take the usual polarization so that the unipotent radical of the Siegel parabolic acts by

$$\omega\begin{pmatrix} I_4 & X \\ & I_4 \end{pmatrix}\phi(v) = \psi\left(\tfrac{a}{2}\,{}^t v X v\right)\phi(v), \qquad X = {}^tX \tag{8}$$

for $\phi \in S(F^2)$.

The Weil representation splits into odd and even functions. Let ω_a^+ be the even Weil representation. If a/b is not a square then ω_a^+ and ω_b^+ are not isomorphic. Indeed, it follows from (8) that their restrictions to the unipotent subgroup are nonisomorphic.

Since the kernel $-I_4$ of the homomorphism $\widetilde{Sp}_4(F) \to \widetilde{SO}_5(F)'$ acts trivially we may regard ω_a^+ as a representation of $\widetilde{SO}_5(F)'$. On the other hand we may pull back the representation Θ_5 to $\widetilde{Sp}_4(F)$. We will show that this restriction is the sum of the ω_a^+ as a runs through the square classes of $F^\times$. Since the ω_a^+ are nonisomorphic, and since the number of irreducible representations ω_a^+ is equal to the index $[F^\times : (F^\times)^2] = [\widetilde{SO}_5(F) : \widetilde{SO}_5(F)']$, it is sufficient to show that each ω_a^+ may be embedded in Θ_5 over $\widetilde{SO}_5(F)'$.

Let P be the parabolic subgroup of SO_5 with Levi factor $GL_1 \times SO_3$. The metaplectic cover splits over $P(F)' = P(F) \cap SO_5(F)'$, which is a subgroup of index $[F^\times : (F^\times)^2]$ in $P(F)$. Using this splitting, we may induce $\delta_P^{1/6}$ from $P(F)'$ to $\widetilde{SO}_5(F)$. Noting that the restriction of $\delta_P^{1/6}$ to B is $\delta_B^{1/2\, w_0}\chi_\theta$, we therefore get an embedding of $\mathrm{Ind}_{\widetilde{P(F)'}}^{\widetilde{SO}_5(F)}(\delta_P^{1/6})$ into $\mathrm{Ind}(\delta_B^{1/2\, w_0}\chi_\theta)$. By Theorem 2 the latter has a unique irreducible subrepresentation and it follows that $\mathrm{Ind}_{\widetilde{P(F)'}}^{\widetilde{SO}_5(F)}(\delta_P^{1/6})$ does also. If $\phi \in S(F^2)$ let $L(\phi) = \phi(0)$. This linear functional gives an embedding into $\mathrm{Ind}_{\widetilde{P(F)'}}^{\widetilde{SO}_5(F)}(\delta_P^{1/6})$ over $\widetilde{SO}_5(F)'$ and it follows that ω_a^+ may be embedded in Θ_5 over $SO_5(F)'$.

We now show that the restriction of a functional Λ satisfying (7) must be zero on each ω_a^+. The matrix in (7) corresponds to

$$X = \begin{pmatrix} 1 & & 2x_1 & x_2 \\ & 1 & x_2 & -x_3 \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

so

$$(\omega_a^+(X)\phi)(v) = \psi(\tfrac{a}{2}\, {}^t v X v)\phi(v) = \psi(\tfrac{a}{2}\,\mathrm{tr}(X v\, {}^t v))\phi(v).$$

Writing $R = \left(\begin{smallmatrix} r_1 & r_2 \\ r_2 & -2r_3 \end{smallmatrix}\right)$ we must show that there exists no nonzero linear functional λ on $S(F^2)$ with

$$\lambda(\omega^+(X)\phi) = \psi(\tfrac{1}{2}\mathrm{tr}(XR))\lambda(\phi).$$

Suppose $\lambda(\phi) \neq 0$. Since R has rank two, $a\, v\, {}^t v = R$ has no solutions and it is easy to see that we may find $X_1, \cdots, X_r$ and $\phi_1, \cdots, \phi_r$ such that

$$\phi(v) = \sum_i \left(\tfrac{1}{2}\psi(\mathrm{tr}(X_i R)) - \psi(\tfrac{a}{2}\mathrm{tr}(X_i v\, {}^t v)\right)\phi_i(v)$$

for all v. Applying λ produces a contradiction. ∎

Let U_{2n+1} be the subgroup of SO_{2n+1} consisting of matrices

$$u = \begin{pmatrix} 1 & X & * \\ & I_{2n-1} & * \\ & & 1 \end{pmatrix}, \qquad X = (x_1, \cdots, x_{2n-1}). \tag{9}$$

Let ψ_n be the character $\psi_n(X) = \psi(x_n)$ of this group.

Theorem 6. *Let $n \geq 2$. The twisted Jacquet module*

$$J_{U_{2n+1}, \psi_n}(\Theta_{SO_{2n+1}}) = 0.$$

Proof. The idea of the proof is to show that this Jacquet module, if nonzero, is a supercuspidal representation of $\widetilde{SO}_{2n-2}(F)$. This will lead to a contradiction. We prove the Theorem in a series of Lemmas.

If $n = 2$ then Theorem 6 is Proposition 5. We assume $n \geq 3$ in Lemma 7, Lemma 8 and Lemma 9.

Let R_k be the subgroup of matrices in SO_{2n+1} of the form

$$r = \begin{pmatrix} z & y & * \\ & I_{2n+1-2k} & * \\ & & z^* \end{pmatrix},$$

where z is unipotent and upper triangular in GL_k. Define a character of R_k by

$$\psi_\epsilon(r) = \psi(\epsilon_1 z_{1,2} + \cdots + \epsilon_{k-1} z_{k-1,k} + y_{k,n-k+1}),$$

where $\epsilon = (\epsilon_1, \cdots, \epsilon_{k-1}) \in F^{k-1}$.

Lemma 7. *For $k \geq 2$ the Jacquet module $J_{R_k,\psi_\epsilon}(\Theta_{SO_{2n+1}}) = 0$.*

Proof. We prove this by induction on n. Assume it is true for strictly smaller orthogonal groups. If any $\epsilon_i = 0$, this Jacquet module factors through the ordinary Jacquet module to $GL_{i-1} \times SO_{2n+3-2i}$. A similar integral appears in the middle $SO_{2n+3-2i}$ block and, using Theorem 3 and the induction hypothesis, this is zero. Therefore we may assume that the ϵ_i are all nonzero. Since the torus of GL_k acts transitively on the nondegenerate characters we may assume the $\epsilon_i = 1$ in which case denote the character as ψ_1. A linear form on this Jacquet module amounts to an element of $\mathrm{Hom}_{R_k}(\Theta_{SO_{2n+1}}, \psi_1)$. Let P be the parabolic subgroup of SO_{2n+1} with Levi subgroup $GL_1 \times SO_{2n-1}$. Since the theta representation is a quotient of $\mathrm{Ind}_P^{SO_{2n+1}}(\delta_P^s \otimes \Theta_{SO_{2n-1}})$ for some s (Proposition 4), it is sufficient to show that for all s

$$\mathrm{Hom}_{R_k}(\mathrm{Ind}_P^{SO_{2n+1}}(\delta_P^s \otimes \Theta_{SO_{2n-1}}), \psi_1) = 0.$$

The double cosets of $P(SO_{2n-1})\backslash SO_{2n+1}/R_k \cdot SO_{2(n-k)}$ are as follows. (Note: We are entitled to include the $SO_{2(n-k)}$ because this group normalizes R_k and stabilizes the character.) We have double coset representatives

$$w_1 \cdots w_r, \qquad 0 \leq r \leq k,$$

$$w_1 \cdots w_{k-1}$$

$$w_1 \cdots w_n \, x_{\alpha_n}(m),$$

$$w_1 \cdots w_{n-1} w_n w_{n-1} \cdots w_r, \qquad 1 < r \leq k.$$

$$w_1 \cdots w_{n-1} w_n w_{n-1} \cdots w_1.$$

Here $w_1, \cdots, w_n$ are the simple reflections in W. The first and fourth cases may be dispensed with because they conjugate a root in R_k where the character is nontrivial into the unipotent radical of the parabolic subgroup P. The remaining cases conjugate a part of the character on R_k into a character of the same type for SO_{2n-1}, and by induction, we are done. $\square$

Lemma 8. *The Jacquet module* $J_{U_{2n+1},\psi_n}(\Theta_{2n+1})$ *is supercuspidal as an* $\widetilde{SO}_{2n-2}(F)$ *module.*

Proof. For any m and k such that $2(m+k)=2n-2$ let

$$U_{m,k} = \begin{pmatrix} I_k & y & * \\ & I_{2m} & * \\ & & I_k \end{pmatrix} \subset SO_{2n-2}.$$

The supercuspidality of $J_{U_{2n+1},\psi_n}(\Theta_{SO_{2n+1}})$ will follow if we prove that

$$J_{U_{m,k}U_{2n+1},\psi_n}(\Theta_{2n+1}) = 0.$$

In SO_{2n+1},

$$U_{m,k}U_{2n+1} = \begin{pmatrix} 1 & r_1 & r_2 & r_3 & * \\ & I_k & \overline{y} & x & * \\ & & I_{2m+1} & * & * \\ & & & I_r & * \\ & & & & 1 \end{pmatrix}, \qquad \overline{y} = (y_1, 0, y_2).$$

Here r_1, r_2, r_3 are row vectors, r_1 and r_3 of length k, r_2 of length $2m+1$, and the character is in the middle position of r_2. Following Ginzburg, Rallis and Soudry [GRS2], Section 2.2, this Jacquet module is equal to the twisted Jacquet module with respect to the group

$$\begin{pmatrix} 1 & 0 & r_2 & r_3 & * \\ & I_k & z & * & * \\ & & I_{2m+1} & * & * \\ & & & I_k & 0 \\ & & & & 1 \end{pmatrix},$$

with the character now appearing on the r_2 as before. Conjugating we may move the character to the middle entry of the last row of z. This is a representation of the mirabolic subgroup of GL_{k+1} stabilizing a line, which we can analyze using the Bernstein-Zelevinsky theory of derivatives ([BZ]). One may analyze this in stages. At each stage, there are two contributions, one involving a trivial Jacquet module, and one involving a nontrivial one. Using Theorem 3, the trivial module will induce a similar functional on a smaller orthogonal group which is zero by induction. Finally we are left with the Jacquet module of Lemma 7, which is zero. $\square$

Lemma 9. *If* π *is supercuspidal on* $\widetilde{SO}_{2n-2}(F)$ *then*

$$\mathrm{Hom}_{\widetilde{SO}_{2n-2}(F)}\big(\pi, J_{U_{2n+1},\psi_n}(\mathrm{Ind}_{P_{GL_n}}^{SO_{2n+1}}(\Theta_{GL_n}\otimes\delta_P^s))\big) = 0$$

for any s.

Proof. The proof of this parallels the global argument for the Rankin-Selberg convolution for $SO_{2n-2} \times GL_n$ in Soudry [S], Section 0.3 (pp. 4–9). Using the cuspidality of π, if the representation of GL_n had a Whittaker model there would be a single double coset of $SO_{2n-2}U_{2n+1}\backslash SO_{2n+1}/P$ which could contribute. In this case, since we are assuming $n \geq 3$, the representation of GL_n has no Whittaker model, and so there is no contribution. $\square$

To complete the proof of Theorem 6, we observe that by Lemma 9, using the exactness of the Jacquet functor and the fact that Θ can be embedded in $\mathrm{Ind}_{P_{GL_n}}^{SO_{2n+1}}(\Theta_{GL_n} \otimes \delta_P^s)$ for suitable s (Proposition 4),

$$\mathrm{Hom}_{\widetilde{SO}_{2n-2}(F)}(\pi, J_{U_{2n+1},\psi_n}(\Theta_{2n+1})) = 0$$

for any supercuspidal π on $\widetilde{SO}_{2n-2}(F)$. But taking $\pi = J_{U_{2n+1},\psi_2}(\Theta_{2n+1})$ and using Lemma 8 we obtain a contradiction. $\blacksquare$

2. Residues of Eisenstein series. Let F be a global field of characteristic not equal to 2, such that F contains μ_4. Let A be the adele ring of F. Let $\widetilde{SO}_{2n+1}(A)$ be the global metaplectic cover of $SO_{2n+1}(A)$, defined as in Section 1 to be the restriction of the metaplectic four-fold cover of $SL_{2n+1}(A)$ to SO_{2n+1}. Let $K = \prod_v K_v$ be the standard maximal compact subgroup of SO_{2n+1}, and let $\widetilde{K}$ be its preimage in $\widetilde{SO}_{2n+1}(A)$.

The metaplectic cover splits over $SO_{2n+1}(F)$ and we will identify $SO_{2n+1}(F)$ with its image in $\widetilde{SO}_{2n+1}(A)$.

As in Section 1, if $s = (s_1, \cdots, s_n) \in \mathbb{C}^n$, we may parametrize a representation $\mathrm{Ind}(\delta^{1/2}\chi_s)$. If v is a place of F we denote by $\mathrm{Ind}_v(\delta^{1/2}\chi_{s,v})$ the representation of $\widetilde{SO}_{2n+1}(F_v)$ as in Section 1. Then

$$\mathrm{Ind}(\delta^{1/2}\chi_s) = \bigotimes_v \mathrm{Ind}_v(\delta^{1/2}\chi_{s,v})$$

(restricted tensor product). We have intertwining integrals $M_w = M_w(s) : \mathrm{Ind}(\delta^{1/2}\chi_s) \to \mathrm{Ind}(\delta^{1/2}\,{}^w\chi_s)$ defined by

$$M_w(s)f(g) = \int_{(U(A)\cap wU(A)w^{-1})\backslash U(A)} f(w^{-1}ug)\,du.$$

An important difference between the local theory and the global theory is that the global intertwining integral M_{w_0} has a pole at $s = s_\theta$. Let $f \in \mathrm{Ind}(\delta^{1/2}\chi_\theta)$. There is a unique global section $s \to f_s$ mapping $s \in \mathbb{C}^n$ to $f_s \in \mathrm{Ind}(\delta^{1/2}\chi_s)$ such that $f_{s_\theta} = f$ with the property that the restriction of f_s to $\widetilde{K}$ is independent of s.

Proposition 10. *If $w = w_0$, then $M_w f_s$ has poles along the lines*

$$s_n = \tfrac{1}{2}, \qquad 2s_i - 2s_{i+1} = 1,$$

and the multiresidue

$$\mathrm{res}_{s_1=\frac{n}{2}} \cdots \mathrm{res}_{s_{n-2}=\frac{3}{2}}\ \mathrm{res}_{s_{n-1}=1}\ \mathrm{res}_{s_n=\frac{1}{2}} M_w(s)f_s(g)$$

is nonzero. If $w \neq w_0$ this multiresidue vanishes.

Proof. We assume that $f_s = \otimes_v f_{s,v}$ is a pure tensor with $f_{s,v} \in \mathrm{Ind}_v(\delta^{1/2}\chi_{s,v})$.

Let S be a finite set of places such that for $v \notin S$, v is nonarchimedean, of odd residue characteristic and $f_{s,v} = f^\circ_{s,v}$ is the standard K_v-fixed vector. This set is independent of s. Using (3),

$$M_w f_s(g) = \left\{ \prod_{v \notin S} c(w, \chi_{s,v}) \right\} \prod_{v \notin S} f^\circ_{w(s),v}(g_v) \prod_{v \in S} M_w f_{s,v}(g_v).$$

If $w = w_0$ the term in braces is

$$\frac{\prod_{i<j} \{\zeta_S(2s_i - 2s_j)\zeta_S(2s_i + 2s_j)\} \prod_i \zeta_S(2s_i)}{\prod_{i<j} \{\zeta_S(2s_i - 2s_j + 1)\zeta_S(2s_i + 2s_j + 1)\} \prod_i \zeta_S(2s_i + 1)}$$

where ζ_S is the partial Dedekind zeta function of F, and due to the pole of ζ_S at $s = 1$, the advertised pole occurs. On the other hand if $w \neq w_0$, one of the zeta functions corresponding to a simple root will be missing, and consequently there is no pole. ∎

Although $M_{w_0} : \mathrm{Ind}(\delta^{1/2}\chi_s) \to \mathrm{Ind}(\delta^{1/2\, w_0}\chi_s)$ has a pole when $s = s_\theta$, there is still an intertwining operator $I_\theta : \mathrm{Ind}(\delta^{1/2}\chi_\theta) \to \mathrm{Ind}(\delta^{1/2\, w_0}\chi_\theta)$ given by the residue in Proposition 10. Its image is the restricted tensor product of representations Θ_v which agree with the representations of Theorem 2 when v is nonarchimedean. Even when v is archimedean Θ_v is irreducible, by Theorem 1 on p. 75 of Miličić [Mi], which is applicable because the archimedean places are complex, so the cover splits, and because the character ${}^{w_0}\chi_s$ is in the negative Weyl chamber. We denote this representation $\otimes_v \Theta_v$, which is the image of I_θ, as $\Theta = \Theta_{SO_{2n+1}}$.

If $f \in \mathrm{Ind}(\delta^{1/2}\chi_\theta)$, define define the Eisenstein series

$$E(g, f_s) = \sum_{\gamma \in B(F) \backslash SO_{2n+1}(F)} f_s(\gamma g),$$

where f_s is as above. As usual the constant term

$$\int_{U(F)\backslash U(A)} E(ug, f_s)\, du = \sum_{w \in W} M_w f_s(g). \tag{10}$$

Thus if

$$\theta_f(g) = \mathrm{res}_{s_1 = \frac{n}{2}} \cdots \mathrm{res}_{s_{n-2} = \frac{3}{2}} \mathrm{res}_{s_{n-1} = 1} \mathrm{res}_{s_n = \frac{1}{2}} E(g, s, f_s), \tag{11}$$

then by Proposition 10

$$\int_{U(F)\backslash U(A)} \theta_f(ug)\, du = I_\theta(f)(g). \tag{12}$$

The automorphic form θ_f is therefore nonzero if $I_\theta(f) \neq 0$ and square integrable by the criterion of Jacquet [J].

Theorem 11. *The representation on the subspace of $L^2(SO_{2n+1}(F)\backslash\widetilde{SO}_{2n+1}(A))$ spanned by the θ_f is irreducible and isomorphic to Θ.*

Proof. Suppose that $f \in \mathrm{Ind}(\delta^{1/2}\chi_\theta)$ is in the kernel of I_θ. Then (12) shows that θ_f has zero constant term along U. As a residue of a minimal parabolic Eisenstein series it cannot have a cuspidal constituent along any nonminimal parabolic subgroup so it is a cusp form. However a residue of an Eisenstein series cannot be cuspidal so it is zero. Consequently there is a well defined equivariant map $\Theta \to L^2(SO_{2n+1}(F)\backslash\widetilde{SO}_{2n+1}(A))$ mapping $I_\theta(f)$ to θ_f. It is surjective by construction and since Θ is irreducible, it is an isomorphism. ∎

Let $1 \le m \le n$ and let $P = MU_P$ be the standard parabolic with Levi factor $GL_m \times SO_{2(n-m)+1}$. As in [KP] we may form the global automorphic exceptional representation of $\widetilde{GL}_m(A)$ attached to the character (6), and we call this Θ_{GL_m}.

Conjecture 12. *If $\theta \in V_\Theta$ the constant term*

$$m \to \int_{U_P(F)\backslash U_P(A)} \theta(um)\,du, \qquad m \in \widetilde{M}(A),$$

is in the space $\Theta_{GL_m} \otimes \Theta_{SO_{2(n-m)+1}}$.

In the function field case, this follows from its local version Theorem 3. However we have not proved Theorem 3 in the archimedean case, so we will only prove the case we need. The general case probably can be proved similarly with more work.

Proposition 13. *If $m = 1$ then Conjecture 12 is true.*

Proof. Let P be the parabolic subgroup of SO_{2n+1} with Levi subgroup $M = GL_1 \times SO_{2n-1}$, and let U be its unipotent radical. Taking the residues in (11) in reverse order, we may represent θ_f as a residue at $s = (3n-1)/(4n-2)$ of an Eisenstein series $E_P(g, s, F_s)$ in one variable, where $F_s \in \mathrm{Ind}_{\widetilde{P}(A)}^{\widetilde{SO}_{2n+1}(A)}(\delta_P^s \otimes \Theta_{SO_{2n-1}})$. A set of representatives for $P\backslash SO_{2n+1}/P$ can be chosen as:

$$1, \qquad w_1 = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & I_{2n-3} & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix}, \qquad w_0 = \begin{pmatrix} & & & & & & 1 \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & -1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & \\ 1 & & & & & & \end{pmatrix}.$$

It follows as in Kudla and Rallis [KR], Section 1.2 that for $\mathrm{re}(s)$ large and $(a, h) \in \widetilde{GL}_1(A) \otimes \widetilde{SO}_{2n-1}(A)$ we have

$$\int_{U(F)\backslash U(A)} E_P(u(a,h), s, F_s)\,du = \\ |a|^{(2n-1)s} F_s(h) + |a|^{n/2} E_{P'}(h, M_{w_1} F'_{s'}, s') + |a|^{(2n-1)(1-s)} M_{w_0} F_s(h),$$

where P' is the parabolic subgroup of SO_{2n-1} with Levi factor $GL_1 \times SO_{2n-3}$, $s' = ((2n-1)s-1)/(2n-3)$,

$$F'_{s'}(h) = \int_{(A/F)^{2n-3}} F_s\left(\begin{pmatrix} 1 & & & & & & \\ & 1 & u_2 & \cdots & u_{2n-3} & * & \\ & & 1 & & & -u_{2n-3} & \\ & & & \ddots & & \vdots & \\ & & & & 1 & -u_2 & \\ & & & & & 1 & \\ & & & & & & 1 \end{pmatrix} h\right) du_i,$$

and the intertwining operators M_w are defined by the usual integrals. Taking the residue, the first two terms have no pole and the last contributes an element of $\Theta_{GL_1} \otimes \Theta_{SO_{2n-1}}$. $\blacksquare$

3. Fourier Coefficients of $\Theta_{SO_{2n+1}}$. Let us first briefly review the connection between unipotent classes and Fourier coefficients. For the group SO_{2n+1} the unipotent classes are parametrized by partitions of $2n+1$ with the property that each even number occurs with even multiplicity. See [Ca] or [CM]). Let $\mathcal{O} = (r_1, r_2, \ldots, r_\ell)$ be a unipotent class. To this class we associate a one-parameter torus given by

$$\mathrm{diag}(t^{r_1-1}, t^{r_1-3}, \ldots, t^{1-r_1}, t^{r_2-1}, t^{r_2-3}, \ldots, t^{1-r_2}, \ldots) \tag{13}$$

Let $h_{\mathcal{O}}(t)$ denote the one-parameter torus in SO_{2n+1} which is conjugate to (13) such the the powers of t are in decreasing order. For example if $\mathcal{O} = (322) = (32^2)$ then (13) is given by $\mathrm{diag}(t^2, 1, t^{-2}, t, t^{-1}, t, t^{-1})$ (which is not a torus in SO_7) and

$$h_{\mathcal{O}}(t) = \mathrm{diag}(t^2, t, t, 1, t^{-1}, t^{-1}, t^{-2}).$$

The elements $h_{\mathcal{O}}(t)$ act on the upper unipotent subgroup $V_n = U$ as follows. If α is a positive root, x_α will denote the standard one-parameter unipotent subgroup corresponding to α. Then $h_{\mathcal{O}}(t)x_\alpha(r)h_{\mathcal{O}}(t)^{-1} = x_\alpha(t^i r)$ and $i \geq 0$. Similarly $h_{\mathcal{O}}(t)$ acts on the one dimensional unipotent subgroup corresponding to negative roots with negative powers of t. Let $G_0(\mathcal{O})$ denote the centralizer in SO_{2n+1} of the one parameter subgroup $h_{\mathcal{O}}(t)$. Let V_n^2 be the unipotent subgroup of V_n given by

$$V_n^2 = \{x_\alpha(r) \in V_n : h_{\mathcal{O}}(t)x_\alpha(r)h_{\mathcal{O}}(t)^{-1} = x_\alpha(t^i r) \quad \text{and} \quad i \geq 2\}\ .$$

The group $L_n^2 = V_n^2/[V_n^2, V_n^2]$ is abelian and $G_0(\mathcal{O})$ acts on it by conjugation. In fact L_n^2 can be identified with a sum of the form $(\bigoplus_{m_1,m_2} \mathrm{Mat}_{m_1\times m_2}) \oplus \mathrm{Mat}^0_{k_1\times k_2}$ where $\mathrm{Mat}_{m_1\times m_2}$ is the group of $m_1 \times m_2$ matrices and

$$\mathrm{Mat}^0_{k_1\times k_2} = \{m \in \mathrm{Mat}_{k_1\times k_2} : {}^t mJ + Jm = 0\}$$

and the sum is determined by the unipotent class $\mathcal{O}$. We are interested in constructing additive characters for the group $V_n^2(F)\backslash V_n^2(A)$. To do so, we shall construct a set of characters for $L_n^2(F)\backslash L_n^2(A)$ and extend trivially to $V_n^2(F)\backslash V_n^2(A)$. Let ψ be a nontrivial additive character of $F\backslash A$. Since L_n^2 is abelian then we can identify the set of all characters of $L_n^2(F)\backslash L_n^2(A)$ with the group $L_n^2(F)$. We consider the set of orbits in $L_n^2(F)$ under the action of $G_0(\mathcal{O})(F)$. We are interested in the set of all representatives in $L_n^2(F)$, under the action of $G_0(\mathcal{O})(F)$, which correspond to the generic points. By this we mean the following. The group $G_0(\mathcal{O})(K)$, where K is an algebraically closed field, acts on $L_n^2(K)$ and under this action there is an open orbit. Let u_0 be a representative of this open orbit and denote by $G_0^{u_0}(\mathcal{O})(K)$ the stabilizer of u_0 in $G_0(\mathcal{O})(K)$. It is well-known (see [Ca]) that $G_0^{u_0}(\mathcal{O})$ is reductive. We say that $\xi \in L_r^2(F)$ is a *generic point* if $G_0^{\xi}(\mathcal{O})(F)$ is the group of rational points of a reductive group of type $G_0^{u_0}(\mathcal{O})$.

Let ψ_ξ be the character of $V_n^2(F)\backslash V_n^2(A)$ described above. Given an automorphic function $\varphi(g)$ on $SO_{2n+1}(A)$ or its cover we define the *Fourier coefficient associated with* ξ to be the integral

$$\int\limits_{V_n^2(F)\backslash V_n^2(A)} \varphi(vg)\, \psi_\xi(v)\, dv \ .$$

In this way, we associate with each unipotent class $\mathcal{O}$ a set of Fourier coefficients.

Example. In SO_5 consider $\mathcal{O} = (31^2)$. Then $h_{\mathcal{O}}(t) = \mathrm{diag}(t^2, 1, 1, 1, t^{-2})$. In this case $(n = 2)$,

$$V_2^2 = \left\{ \begin{pmatrix} 1 & x & y & z & * \\ & 1 & 0 & 0 & * \\ & & 1 & 0 & * \\ & & & 1 & * \\ & & & & 1 \end{pmatrix} \right\} .$$

In this case V_2^2 is abelian and $G_0(\mathcal{O}) = GL_1 \times SO_3$. We may take $u_0 = \begin{pmatrix} 1 & 0 & 1 & 0 & * \\ & 1 & 0 & 0 & 0 \\ & & 1 & 0 & * \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}$

and $G_0^{u_0}(\mathcal{O}) = SO_2$. Thus if φ is an automorphic form of SO_5 or its metaplectic cover, the set of Fourier coefficients associated with $\mathcal{O} = (31^2)$ is given by

$$\int\limits_{(F\backslash A)^3} \varphi \left[\begin{pmatrix} 1 & x & y & z & * \\ & 1 & 0 & 0 & * \\ & & 1 & 0 & * \\ & & & 1 & * \\ & & & & 1 \end{pmatrix} g \right] \psi(x + \beta z)\, dx\, dy\, dz$$

where $\beta \in (F^\times)^2\backslash F^\times$. $\blacksquare$

Let us recall the partial order defined on unipotent classes. Given $\mathcal{O}_1 = (r_1, r_2, \cdots, r_\ell)$ and $\mathcal{O}_2 = (m_1, m_2, \cdots, m_\ell)$ we say that $\mathcal{O}_1 \geq \mathcal{O}_2$ if $r_1 + \cdots + r_i \geq m_1 + \cdots + m_i$ for all $1 \leq i \leq \ell$. If $\mathcal{O}_1$ is not greater than $\mathcal{O}_2$ and $\mathcal{O}_2$ is not greater than $\mathcal{O}_1$ we say that $\mathcal{O}_1$

and $\mathcal{O}_2$ are not comparable. For example, $\mathcal{O}_1 = (31^6)$ and $\mathcal{O}_2 = (2^4 1)$ in SO_9 are not comparable.

The main goal of this section is to determine which unipotent classes support nonzero Fourier coefficients for Θ. Let $\mathcal{O}(\Theta_{SO_{2n+1}}) = (2^{2m}1)$ if $n = 2m$ and $\mathcal{O}(\Theta_{SO_{2n+1}}) = (2^{2m}1^3)$ if $n = 2m+1$. Our main result is

Theorem 14. *(i) Let $\mathcal{O}$ be a unipotent class. If any function in the space of $\Theta_{SO_{2n+1}}$ has a nonzero Fourier coefficient corresponding to $\mathcal{O}$, then $\mathcal{O} \le \mathcal{O}(\Theta_{SO_{2n+1}})$.*
(ii) The representation $\Theta_{SO_{2n+1}}$ has a nontrivial Fourier coefficient corresponding to the unipotent class $\mathcal{O}(\Theta_{SO_{2n+1}})$.

We will prove the Theorem in several steps. Let $\mathcal{O} = (31^{2n-2})$. This unipotent class corresponds to the following set of Fourier coefficients. We parametrize the unipotent radical of the parabolic subgroup of SO_{2n+1} whose Levi part is $GL_1 \times SO_{2n-1}$ by

$$(r_1, \ldots, r_{2n-1})^\wedge =$$

$$I_{2n+1} + \sum_{i=1}^{2n-1} r_i e'_{1,i+1} - \left(r_1 r_{2n-1} + r_2 r_{2n-2} + \cdots + r_{n-1} r_{n+1} + \tfrac{1}{2} r_n^2\right) e_{1,2n+1}$$

where $e'_{1,j} = e_{1,j} - e_{2n-j+2,2n+1}$ and $e_{i,j}$ is the $(2n+1)\times(2n+1)$ matrix whose (i,j)-th entry is one and zero otherwise. For $\mathcal{O} = (31^{2n-2})$ we have $h_{\mathcal{O}}(t) = \mathrm{diag}(t^2, 1, \ldots, 1, t^{-2})$ and hence the set of Fourier coefficients of an automorphic form φ corresponding to (31^{2n-2}) is given by

$$\int\limits_{(F\backslash A)^{2n-1}} \varphi((r_1, \ldots, r_{2n-1})^\wedge g)\, \psi(r \cdot v)\, dr_i. \tag{14}$$

where $r = (r_1, \ldots, r_{2n-1})$ and v is a vector in F^{2n-1} with *nonzero* length. Clearly (31^{2n-2}) is not comparable with $\mathcal{O}(\Theta_{SO_{2n+1}})$ if $n \ge 4$.

Proposition 15. *Let θ be in the space of $\Theta_{SO_{2n+1}}$. Then integral (14) with $\varphi = \theta$ is zero for all choices of data and all $v \in F^{2n-1}$ which have nonzero length.*

Proof. It is enough to show that there is a place such that the local theta representation $\Theta_{SO_{2n+1}}$ does not support a functional of type (14). Let w be a place of F such that F_w is a nonarchimedean local field of odd residue characteristic where $\Theta_{SO_{2n+1}}$ is unramified. We may also choose w such that the length of the vector r in (14) is a square in $F_w^\times$.

If (14) is nonzero, then the functional

$$\theta \mapsto \int\limits_{(F\backslash A)^{2n-1}} \theta((r_1, \ldots, r_{2n-1})^\wedge)\, \psi(r \cdot v)\, dr_i$$

induces a nonzero functional on Θ_w which factors through the Jacquet module for the character $\psi(r \cdot v)$ on the group $U_{2n+1}(F_w)$ defined in (9). On our assumption that the length of r is a local square, this character can be conjugated into ψ_n in the notation of Theorem 6. This Jacquet module is therefore zero, contradiction. ■

Any class not $\le \mathcal{O}(\Theta_{SO_{2n+1}})$ is $\ge (31^{2n-2})$. Therefore Theorem 14 (i) will be established when we prove the following result.

Proposition 16. *Let $\varphi \in \pi$ be an automorphic function on $SO_{2n+1}(A)$ or $\widetilde{SO}_{2n+1}(A)$ such that the Fourier coefficients (14) corresponding to the unipotent class (31^{2n-2}) are zero for all choices of data. Let $\mathcal{O}$ be any unipotent class which is greater than (31^{2n-2}). Then any Fourier coefficient of $\varphi \in \pi$ corresponding to the unipotent class $\mathcal{O}$ vanishes for all choices of data.*

Proof. Write $\mathcal{O} = (n_1^{r_1}, n_2^{r_2}, \cdots, n_m^{r_m})$ where all the n_i are distinct. Since $\sum_{i=1}^{m} r_i n_i = 2n+1$ then at least one of the n_i is odd. Assume first that there exists an i such that $n_i > 1$. This means that if $h_{\mathcal{O}}(t) = \text{diag}(\dots, t^2, \dots, t^2, t, \dots t, 1, \dots, 1, \dots)$ then there are k times t^2 and $\ell + k$ times 1 where $k > 0$ and $\ell \geq 0$ and k and ℓ are determined by $\mathcal{O}$. Also $\ell + k$ is an odd number. The Fourier coefficient of $\varphi \in \pi$ corresponding to $\mathcal{O}$ is

$$(15) \qquad \int\limits_{V_n^2(F)\backslash V_n^2(A)} \varphi\left[\begin{pmatrix} \ddots & & & & & & \\ & I_k & 0 & x & * & * & \\ & & I_p & 0 & * & * & \\ & & & I_{\ell+k} & 0 & * & \\ & & & & I_p & 0 & \\ & & & & & I_k & \\ & & & & & & \ddots \end{pmatrix} g\right] \psi_\xi(v)\, dv.$$

Here $x \in \text{Mat}_{k\times(\ell+k)}$, p is the number of t's which occur in $h_{\mathcal{O}}(t)$ and ξ is any representative corresponding to the unipotent class $\mathcal{O}$. What is important to us is the restriction of ψ_ξ to x. To compute this we study the action of $GL_k(F) \times SO_{\ell+k}(F)$ on $\text{Mat}_{k\times(\ell+k)}(F)$ and determine those matrices such that the stabilizer in $GL_k(F) \times SO_{\ell+k}(F)$ is given by $SO_k^\Delta(F) \times SO_\ell(F)$. Here SO_k and $SO_{\ell+k}$ are orthogonal groups, not necessarily the split form, and SO_k^Δ is the diagonal embedding of SO_k in $GL_k \times SO_{\ell+k}$. Hence restricting ψ_ξ to x we obtain the character $\psi(x_1 \cdot v_1 + \cdots + x_k \cdot v_k)$ where x_i are the rows of x and v_i are vectors in $F^{\ell+k}$. Since ξ is generic and $\ell + k$ is odd we may choose at least one of the v_i's to be of nonzero length. So assume v_1 is of nonzero length. If we conjugate the first row of x in (15) to the first row of the matrix we obtain as an inner integral

$$(16) \qquad \int\limits_{(F\backslash A)^{2n-1-q}} \varphi((0, \dots, 0, x_{1,1}, \dots x_{1,\ell+k}, y_1, \dots, y_q)^\wedge)\psi(x_1 \cdot v_1)\, dx_{1,i}\, dy_j$$

where $q = \frac{1}{2}(2n-1-\ell-k)$ (recall that $\ell+k$ is odd) and $x_1 = (x_{1,1}, \dots, x_{1,\ell+k})$. Expanding further this equals

$$\sum_{\delta_i \in F} \int\limits_{(F\backslash A)^{2n-1}} \varphi((z_1, \dots, z_q, x_{1,1}, \dots, x_{1,\ell+k}, y_1, \dots, y_\gamma)^\wedge)\, \psi(\sum \delta_i z_i + x_1 \cdot v_1) dx_{1,i}\, dy_j\, dz_r\,.$$

Since the vector $(\delta_1, \dots, \delta_q, v_1, 0, \dots, 0) \in F^{2n-1}$ is of nonzero length, it follows from the assumption that (14) vanishes that this integral is zero. From this it follows that (15) is zero for all choices of data.

Next we consider the case where $\mathcal{O} = ((2n_1)^{2r_1} \cdots (2n_m)^{2r_m} 1^{2\ell+1})$ i.e. when all numbers are even or 1. Since we assume that $\mathcal{O} > (31^{2n-2})$ then there is at least one i such that $n_i \geq 2$. We shall prove our result in two steps.

First we shall prove that a certain Fourier coefficient of $\varphi \in \pi$ is zero for all choices of data. The unipotent radical of the maximal parabolic subgroup of SO_{2n+1} whose Levi part is $GL_2 \times SO_{2n-3}$ is given by all matrices of the form

$$\begin{pmatrix} I_2 & Y & Z \\ & I_{2n-3} & Y^* \\ & & I_2 \end{pmatrix}$$

where $Y \in \mathrm{Mat}_{2\times(2n-3)}$ and $Z = \left\{ \begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix} \right\}$. We shall denote this matrix by

$$\begin{pmatrix} y_{1,1} & \cdots & y_{1,2n-3} \\ y_{2,1} & \cdots & y_{2,2n-3} \end{pmatrix}^{\wedge} x(z).$$

We shall prove that the integral

(17)
$$\int \varphi \left[\begin{pmatrix} 0 & \cdots & 0 & y_{1,r} & \cdots & y_{1,p-1} & y_{1,p} & \cdots & y_{1,2n-3} \\ 0 & \cdots & 0 & 0 & \cdots & 0 & y_{2,p} & \cdots & y_{2,2n-3} \end{pmatrix}^{\wedge} x(z) \right] \psi(y_{1,r} + y_{2,p})\, dy_{ij}$$

is zero for all choices of data. Here $r \leq n-2$, $p = 2n-3-r+1$ and we integrate all $y_{i,j}$ over $F\backslash A$. To show this, we expand along the variables $y_{2,r}$ to $y_{2,p-1}$. Thus it is enough to prove that the integral

$$\int \varphi \left[\begin{pmatrix} 0 & \cdots & 0 & y_{1,r} & \cdots & y_{1,p} & \cdots & y_{1,2n-3} \\ 0 & \cdots & 0 & y_{2,r} & \cdots & y_{2,p} & \cdots & y_{2,2n-3} \end{pmatrix}^{\wedge} x(z) \right]$$
$$\psi(y_{1,r} + \epsilon_r y_{2,r} + \cdots + \epsilon_{p-1} y_{2,p-1} + y_{2,p})\, dy_{ij}$$

is zero for all choices of data and all $\epsilon_i \in F$. Conjugating from the left by the matrix $I_{2n+1} + \epsilon_r(e_{1,2} - e_{n,n+1})$ and changing variables, it is enough to show that

$$\int \varphi \left[\begin{pmatrix} 0 & \cdots & 0 & y_{1,r} & \cdots & y_{1,p} & \cdots & y_{1,2n-3} \\ 0 & \cdots & 0 & y_{2,r} & \cdots & y_{2,p} & \cdots & y_{2,2n-3} \end{pmatrix}^{\wedge} x(z) \right]$$
$$\psi(y_{1,r} + \epsilon_{r+1} y_{2,r+1} + \cdots + \epsilon_{p-1} y_{2,p-1} + y_{2,p})\, dy_{ij}$$

is zero for all choices of data. If the length of the vector $(\epsilon_{r+1}, \ldots, \epsilon_{p-1})$ is nonzero then arguing as in (16) we obtain the vanishing result. If the length is zero then using a suitable matrix in $SO_{2n-3}(F)$ is enough to prove that the integral

$$\int \varphi \left[\begin{pmatrix} 0 & \cdots & 0 & y_{1,r} & \cdots & y_{1,2n-3} \\ 0 & \cdots & 0 & y_{2,r} & \cdots & y_{2,2n-3} \end{pmatrix}^{\wedge} x(z) \right] \psi(y_{1,r} + \epsilon y_{2,r+1} + y_{2,p})\, dy_{ij}$$

is zero for all choices of data and all $\epsilon \in F$. Conjugating by $I_{2n+1} + (e_{1,2} - e_{n,n+1})$ from left to right and changing variables, we once again obtain an integral of the form (16) as inner integration. Arguing as before we deduce the vanishing of (17) for all choices of data.

To prove any Fourier coefficient which corresponds to $\mathcal{O}$ vanishes for all choices of data we will show that integral (17) is an inner integration for a suitable choice of r. Recall that $\mathcal{O} = ((2n_1)^{r_1} \cdots (2n_m)^{r_m} 1^{2\ell+1})$. Hence $h_{\mathcal{O}}(t) = \mathrm{diag}(\ldots, t^3, \ldots, t^3, t, \ldots, t, 1, \ldots, 1, \ldots)$ where we have $\ell+1$ ones, t occurs $2(r_1+\cdots+r_m)$ times and t^3 occurs $2(r_1+\cdots+r_{m+1}+p)$ times, where $p=0$ if $n_m=1$ and $p=r_m$ if $n_m>1$. Notice that since there is n_i which is greater than 1 then $2(r_1+\cdots+r_{m+1}+p)$ is nonzero. The integral we consider is given by

$$\int_{V_n^2(F)\backslash V_n^2(A)} \varphi \left[\begin{pmatrix} \ddots &&&&&& \\ & I_k & x & * & * & * & \\ && I_s & 0 & R & * & \\ &&& I_{2\ell+1} & 0 & * & \\ &&&& I_s & * & \\ &&&&& I_k & \\ &&&&&& \ddots \end{pmatrix} g \right] \psi_\xi(v)\, dv \tag{18}$$

where $x \in \mathrm{Mat}_{k\times s}$, $R \in \mathrm{Mat}^0_{s\times s}$ and $k = 2(r_1+\cdots+r_{m+1}+p)$ and $s = 2(r_1+\cdots+r_m)$. Also ξ is any representative corresponding to the unipotent class $\mathcal{O}$ as described in the beginning of this Section. We are concerned with the restriction of ψ_ξ to X and R. For this we need to analyze the action of $GL_k \times GL_s$ on x and that of GL_s on R. The action of GL_s on R is via the exterior square representation. Since s is even, the stabilizer in GL_s of a generic point is Sp_s. Thus, if $r=(r_{ij}) \in R$ we may choose the restriction of ψ_ξ to R as $\psi(r_{1,1}+r_{2,2}+\cdots+r_{s/2,s/2})$. The stabilizer of this choice is clearly Sp_s. Next we let $GL_k \times Sp_s$ act on $\mathrm{Mat}_{k\times s}(F)$ which can be identified with all characters of $x \in \mathrm{Mat}_{k\times s}$ $(F\backslash A)$. Again we need to choose a character such that the stabilizer is $Sp_k^\Delta \times Sp_{s-k}$ (recall that s and k are even). If $x=(x_{i,j}) \in \mathrm{Mat}_{k\times s}$ then we can clearly choose ξ so that ψ_ξ restricted to x will be of the form $\psi(x_{1,1}+\cdots)$ and $x_{1,1}$ is the only variable in the first row of x such that ψ_ξ is nontrivial. If we conjugate the rows in V_n^2 which contain the first row of x and the last row of R to the first and second rows of V_n^2, we obtain as inner integration to (18) the integral

$$\int \varphi \left[\begin{pmatrix} 0 & \cdots & 0 & x_{1,1} & \cdots & * & * & \cdots & * & \cdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & r_{s,1} & \cdots & r_{s,s} & \cdots \end{pmatrix}^\wedge x(z) \right] \psi(x_{1,1}+r_{s,s})\, d(\cdots)$$

where we integrate all variables over $F\backslash A$. Here $*$ indicates arbitrary entries in the SO_{2n+1} matrix. Clearly we obtain (17) as inner integration to the above integral. Hence the above integral and also the integral (18) vanishes for all choices of data. This completes the proof of the Proposition. ∎

We now prove (ii) of Theorem 14.

Recall that $\mathcal{O}(\Theta_{SO_{2n+1}}) = (2^{2m}1)$ if $n=2m$ and equals $(2^{2m}1^3)$ if $n=2m+1$. Let us start with the case $n=2m$, when the group is SO_{4m+1}. We argue by induction on m. We have $h_{\mathcal{O}}(t) = \mathrm{diag}(t, \ldots, t, 1, t^{-1}, \ldots, t^{-1})$ where t occurs $2m$ times along the diagonal. Thus the Fourier coefficient corresponding to $(2^{2m}1)$ is given by

$$\int\limits_{X_m(F)\backslash X_m(A)} \theta_{SO_{4m+1}} \left(\begin{pmatrix} I_{2m} & 0 & x \\ & 1 & 0 \\ & & I_{2m} \end{pmatrix} g \right) \psi'(x)\, dx \tag{19}$$

for $\theta_{4m+1} \in V_{\Theta_{4m+1}}$, where $X_m = \mathrm{Mat}^0_{2m\times 2m}$ and for $x = (x_{ij}) \in \mathrm{Mat}^0_{2m\times 2m}(A)$ we define

$$\psi'(x) = \psi(x_{1,2m-1} + x_{3,1} + x_{4,2} + \cdots + x_{m+1,m-1}).$$

For $m = 1$ we have

$$\int\limits_{F\backslash A} \theta_{SO_5} \left(\begin{pmatrix} 1 & 0 & 0 & y & 0 \\ & 1 & 0 & 0 & -y \\ & & 1 & 0 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix} g \right) \psi(y)\, dy.$$

This is clearly nonzero, for if not we would get by Fourier expansion that

$$\theta_{SO_5}(g) = \int\limits_{F\backslash A} \theta_{SO_5} \left(\begin{pmatrix} 1 & 0 & 0 & y & 0 \\ & 1 & 0 & 0 & -y \\ & & 1 & 0 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix} g \right) dy$$

which means that $\theta_{SO_5}(g)$ is constant along the copy of $\widetilde{SL}_2(A)$ generated by a long root. This is impossible.

We now consider the general case. Let $X^1_m = \{x \in \mathrm{Mat}^0_{2m\times 2m} : x_{1,2m-1} = 0\}$. Consider the integral

$$\int\limits_{X^1_m(F)\backslash X^1_m(A)} \theta_{SO_{4m+1}} \left[\begin{pmatrix} I_{2m} & 0 & x_1 \\ & 1 & 0 \\ & & I_{2m} \end{pmatrix} g \right] \psi'(x_1)\, dx_1\,. \tag{20}$$

Here we view ψ' as a character of X^1_m by restriction. We claim that (20) is nonzero for some choices of data. Indeed, suppose (20) is zero for all choices of data. Let U_{2m} be the unipotent radical of the parabolic subgroup of SO_{4m+1} whose Levi part is $GL_1 \times SO_{4m-1}$. Similarly, define U_{2m-2} as the unipotent radical of the parabolic subgroup of SO_{4m-1} whose Levi part is $GL_1 \times SO_{4m-3}$. We embed U_{2m-2} in SO_{4m+1} via the embedding of SO_{4m-1} in SO_{4m+1}. Consider the integral

$$\int\limits_{U_{2m}(F)\backslash U_{2m}(A)} \int\limits_{U_{2m-2}(F)\backslash U_{2m-2}(A)} \int\limits_{X^1_m(F)\backslash X^1_m(A)} \theta_{SO_{4m+1}} \left(u_1 u_2 \begin{pmatrix} I_{2m} & 0 & x_1 \\ & 1 & 0 \\ & & I_{2m} \end{pmatrix} \right)$$
$$\psi'(x_1)\, dx_1\, du_1\, du_2.$$

This integral is zero for all choices of data since (20) is an inner integration. On the other hand if we apply Proposition 13 twice the above integral equals

$$\int\limits_{X_{m-1}(F)\backslash X_{m-1}(A)} \theta_{SO_{4m-3}}\left(\begin{pmatrix} I_{2m-2} & 0 & x \\ & 1 & 0 \\ & & I_{2m-2} \end{pmatrix}\right)\psi'(x)\,dx$$

which by our induction assumption is nonzero for some choices of data. This implies that (20) is nonzero for some choices of data, as claimed. Notice that this is not enough to assert that (19) is nonzero for some choices of data.

Given a matrix $x_1 \in X_m^1$ we write $x_1 = \begin{pmatrix} y & 0 \\ x_2 & y^* \end{pmatrix}$ where $y = (y_1, y_2)$ with $y_1, y_2 \in \mathrm{Mat}_{2\times(m-1)}$, and $x_2 \in X_{m-1}$ is embedded in X_m in the obvious way. We let

$$Y_1 = \left\{ \begin{pmatrix} y_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y_1^* \end{pmatrix} \in X_m^1 : y_1 \in \mathrm{Mat}_{2\times(m-1)} \right\},$$

$$Y_2 = \left\{ \begin{pmatrix} 0 & y_2 & 0 \\ 0 & 0 & y_2^* \\ 0 & 0 & 0 \end{pmatrix} \in X_m^1 : y_2 \in \mathrm{Mat}_{2\times(m-1)} \right\},$$

and $Y = Y_1 \times Y_2$. Then can write the integral (20) as

$$\int \theta_{SO_{4m+1}}\left[\begin{pmatrix} I_2 & 0 & 0 & y & 0 \\ & I_{2m-2} & 0 & x_2 & * \\ & & 1 & 0 & 0 \\ & & & I_{2m-2} & 0 \\ & & & & I_2 \end{pmatrix} g\right] dy\,dx_2 \tag{21}$$

Here y is integrated over $Y(F)\backslash Y(A)$ and x_2 over $X_{m-1}(F)\backslash X_{m-1}(A)$. Denote the above matrix by $r(y_1, y_2, x_2)$. Let $a \in GL_{2m}$ be embedded in SO_{4m+1} as $a \to \begin{pmatrix} a & & \\ & 1 & \\ & & a^* \end{pmatrix}$. We denote the embedding by $a^\vee$. Let

$$L = \left\{ \begin{pmatrix} I_2 & & \\ \ell & I_{m-1} & \\ 0 & 0 & I_{m-1} \end{pmatrix} : \ell \in \mathrm{Mat}_{(m-1)\times 2} \right\}$$

We embed L in SO_{4m+1} via the embedding of GL_{2m} as described above. Expanding (21) along $L(F)\backslash L(A)$, changing variables, and collapsing summation with integration, (21) equals

$$\int \theta_{SO_{4m+1}}\left[\begin{pmatrix} I_2 & & \\ \ell & I_{m-1} & \\ 0 & 0 & I_{m-1} \end{pmatrix}^\vee r(y_1, y_2, x_2)\, g\right]\psi'(x_2)\,d\ell\,dx_2\,dy_2\,dy_1 \ .$$

Here $y_1^\wedge$ is the embedding of y_1 in SO_{4m+1} via the embedding of Y_1 as described above. Also ℓ is integrated over $L(F)\backslash L(A)$, y_2 over $Y_2(F)\backslash Y_2(A)$, x_2 over $X_{m-1}(F)\backslash X_{m-1}(A)$ and y_1 over $Y_1(A)$. Since (20) and hence (21) is nonzero for some choices of data it follows that the above integral is nonzero for some choices of data. Hence the function

$$\widetilde{f}(g) = \int \theta_{SO_{4m+1}} \left[\begin{pmatrix} I_2 & & \\ \ell & I_{m-1} & \\ 0 & 0 & I_2 \end{pmatrix}^\vee r(0, y_2, x_2)\, g \right] \psi'(x_2)\, d\ell\, dx_2\, dy_2$$

is not identically zero.

Let g be a matrix in the $SL_2(A)$ copy which is generated by $x_{\pm\alpha}(z)$ where α is the highest root in SO_{4m+1}. We claim that $\widetilde{f}(g)$ is actually invariant under $SL_2(F)$. This can be verified by matrix multiplication. It is enough to check it for $x_\alpha(z)$ where $z \in F$ and for the Weyl element $\begin{pmatrix} & & I_2 \\ & I_{4m-3} & \\ I_2 & & \end{pmatrix}$. Let us remark that this is the reason for the above Fourier coefficient manipulation replacing the Y_1 integration by an L integration. Then $\widetilde{f}(g)$ is an automorphic function on $\widetilde{SL}_2(A)$. Since it is nonzero, $\widetilde{f}(g)$ has a nontrivial Fourier coefficient. Hence the integral

$$\int\limits_{F\backslash A} \widetilde{f} \left[\begin{pmatrix} 1 & z \\ & 1 \end{pmatrix} \right] \psi(az)\, dz =$$

$$\int \theta_{SO_{4m+1}} \left[\begin{pmatrix} I_2 & & \\ \ell & I_{m-1} & \\ 0 & 0 & I_2 \end{pmatrix}^\vee r(0, y_2, x_2)\, x_\alpha(z) \right] \psi'(x_2)\, \psi(az)\, d\ell\, dy_2\, dx_2\, dz$$

is nonzero for some choices of $a \in F^\times$. Here z is integrated over $F\backslash A$ and all the other variables as before. Conjugating by $\mathrm{diag}(a, 1, \ldots, 1, a^{-1}) \in SO_{4m+1}(F)$, which acts linearly on $x_\alpha(z)$, and changing variables we deduce that the above integral is nonzero for some choices of data and for all $a \in F^\times$. Now we repeat a similar Fourier expansion to show that (19) is nonzero for some choices of data. Indeed, expanding the above integral along $y \in Y_1(F)\backslash Y_1(A)$ and collapsing summation with integration we deduce that the above integral equals

$$\int \theta_{SO_{4m+1}} \left[r(y_1, y_2, x_2) x_\alpha(z) \begin{pmatrix} I_2 & & \\ \ell & I_{m-1} & \\ 0 & 0 & I_2 \end{pmatrix}^\vee \right] \psi'(x_2)\, \psi(z)\, dy_2\, dx_2\, dz\, d\ell.$$

Here ℓ is integrated over $L(A)$ and all the other variables as before. Thus the integral is nonzero. But (19) is an inner integration to this integral. Hence we deduce that (19) is nonzero for some choices of data. This completes the proof of Theorem 14 (ii) for SO_{4m+1}.

Next we consider SO_{4m+3}. We will prove Theorem 14 (ii) in this case using the case of $\Theta_{SO_{4m+1}}$ for which it is already proved. Since $\mathcal{O}(\Theta_{SO_{4m+3}}) = (2^{2m}1^3)$,

$$h_{\mathcal{O}}(t) = \mathrm{diag}(t, \ldots, t, 1, 1, 1, t^{-1}, \ldots, t^{-1}).$$

Hence the integral we consider is

$$\int\limits_{X_m(F)\backslash X_m(A)} \theta_{SO_{4m+3}}\left[\begin{pmatrix} I_{2m} & 0 & x \\ & I_3 & 0 \\ & & I_{2m} \end{pmatrix}\right]\psi'(x)\,dx\ .$$

Conjugating by a suitable Weyl element this is nonzero for some choices of data if and only if the integral

$$\int\limits_{X_m(F)\backslash X_m(A)} \theta_{SO_{4m+3}}\left[\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ & I_{2m} & 0 & x & 0 \\ & & 1 & 0 & 0 \\ & & & I_{2m} & 0 \\ & & & & 1 \end{pmatrix}\right]\psi'(x)\,dx$$

is nonzero for some choices of data. Assume that this integral is zero for all choices of data. Then the integral

$$\int\limits_{X_m(F)\backslash X_m(A)}\ \int\limits_{U_{2m+2}(F)\backslash U_{2m+2}(A)} \theta_{SO_{4m+3}}\left[u\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ & I_{2m} & 0 & x & 0 \\ & & 1 & 0 & 0 \\ & & & I_{2m} & 0 \\ & & & & 1 \end{pmatrix}\right]\psi'(x)\,du\,dx$$

is zero for all choices of data. Using Proposition 13 this implies that the integral

$$\int\limits_{X_m(F)\backslash X_m(A)} \theta_{SO_{4m+1}}\left[\begin{pmatrix} I_{2m} & 0 & x \\ & 1 & 0 \\ & & I_{2m} \end{pmatrix}\right]\psi'(x)\,dx$$

is zero for all choices of data. This integral is (19) and hence we derive a contradiction. ∎

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