Part 2 of Reading Course

Introduction to de Rham cohomology

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In Part 1 we had the following exercise: Let $\omega = \alpha\,dx + \beta\,dy$ be a covector field on $\mathbb{R}^2$. Describe precise conditions on α and β that guarantee that there *does* exist a function f with $df = \omega$.

Your answer[1] was that $\int \alpha\,dx - \int \beta\,dy$ should be a function of the form $g(x) + h(y)$. But this is a *global* condition, in that we need to look at the whole plane to detect it. (It would become quite a bit more complicated if our functions were not defined on the whole plane, for example.)

Let us now give a *local* condition, by differentiating instead of integrating.

Exercise 1. (Easy) Let $\omega = \alpha\,dx + \beta\,dy$ be a covector field on $\mathbb{R}^2$. Show that if $\omega = df$ for some infinitely differentiable function f, then

$$\frac{\partial\beta}{\partial x} = \frac{\partial\alpha}{\partial y}. \tag{$*$}$$

Exercise 2. Show that the condition $(*)$ is also *sufficient* on $\mathbb{R}^2$: if a covector field $\omega = \alpha\,dx + \beta\,dy$ satisfies $(*)$, then there exists a function f with $df = \omega$.

Combining these two exercises, we could define a function Φ from covector fields to functions:

$$\Phi(\alpha\,dx + \beta\,dy) \quad = \quad \frac{\partial\beta}{\partial x} - \frac{\partial\alpha}{\partial y}.$$

This has the property that ω can be written as $\omega = df$ if and only if $\Phi(\omega) = 0$. However, this function Φ depends too much on a particular choice of coordinates. The value of Φ should not really be thought of as a function, but rather a *2-covector field.*

Definition. Let V be the space of vectors (at a point in $\mathbb{R}^2$, say). A vector is an element of V, and a covector is an element of the dual space V^*. We now introduce a new definition: a *2-covector* is an element of $\bigwedge^2 V^*$.

If V is 2-dimensional, then V^* is 2-dimensional, and so we know from [linear algebra class] that $\bigwedge^2 V^*$ is $\binom{2}{2} = 1$-dimensional. If dx and dy give our basis for V^*, then $dx \wedge dy$ is the basis vector in $\bigwedge^2 V^*$.

A *2-covector field* consists of a 2-covector at every point. At every point $p \in \mathbb{R}^2$, the 2-covector must be some multiple of $dx \wedge dy$, so we can describe a 2-covector field on the plane as $\varphi = \gamma\ dx \wedge dy$ for some function γ.

Instead of Φ above (which you can now forget about forever), we define a function d from covector fields to 2-covector fields by:

$$d(\alpha\,dx + \beta\,dy) \quad = \quad \left(\frac{\partial\beta}{\partial x} - \frac{\partial\alpha}{\partial y}\right) dx \wedge dy$$

We can then restate Exercises 1 and 2:

Ex. 1: $d(df) = 0$ for any function f **Ex. 2:** if $d\omega = 0$, then $\omega = df$ for some function f.

Exercise 3. Let $\varphi = \gamma\,dx \wedge dy$ be a 2-covector field on $\mathbb{R}^2$.

Prove that there exists some 1-covector field $\omega = \alpha\,dx + \beta\,dy$ with $d\omega = \varphi$.

[You can read §7.6 of Eliashberg's notes if you have them, but not necessary.]

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[1] [For future readers: if you did not solve that exercise, stop skipping ahead and go back to Part 1! And take the time to understand why that answer is correct.]