Part 5 of Reading Course

Vector fields, integral curves, and integral surfaces

Thomas Church

Derivations and vector fields. Recall that $C^\infty(M)$ denotes the algebra[1] of infinitely-differentiable real-valued functions on a manifold M. A *derivation* Φ on any algebra R is a linear transformation $\Phi\colon R \to R$ satisfying the "product rule"

$$\Phi(fg) = \Phi(f)g + f\Phi(g) \qquad \text{for all } f, g \in R.$$

Theorem A. *If Φ is a derivation on $C^\infty(M)$, there exists a unique vector field V on M with $\Phi = \partial_V$.*

In Part 4 you hopefully proved Theorem A in the case $M = \mathbb{R}^2$. But in any case, you can consider the following two exercises:

Exercise 1. In the case $M = \mathbb{R}$, Theorem A states:

Let $\Phi\colon C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ be a linear transformation satisfying $\Phi(fg) = \Phi(f)g + f\Phi(g)$ for all functions $f, g \in C^\infty(\mathbb{R})$. Prove there exists a unique function u such that $\Phi(f) = u \cdot f'$ for all f.

Write out the proof, including details, in this case.

Exercise 2. Say that you know Theorem A holds for $M = \mathbb{R}^2$. Can you use this to prove Theorem A for the 2-sphere $M = S^2$, or other surfaces?

Integral curves. An *integral curve* for a vector field V on M is a curve $g(t) : \mathbb{R} \to M$ whose derivative agrees with V, meaning $g'(t) = V_{g(t)}$ for all t.

Exercise 3. **(Very Hard, Very Important)** If V is a vector field on $\mathbb{R}^2$ (infinitely-differentiable, as always) and the magnitude $|V_p|$ is globally bounded, prove that there is an integral curve passing through every point p.
(You may assume that V is never vertical, if you like.)
Can you prove that the integral curve is unique?

[1] "Algebra" means "vector space with a multiplication".

Integral surfaces. Let V and W be two vector fields in $\mathbb{R}^3$, which at every point are linearly independent. Say that an *integral surface* for V and W is a surface S in $\mathbb{R}^3$ such that V and W are both tangent to S at each point; in other words, at each point $p \in S$, the vectors V_p and W_p span the tangent plane of S. (We don't worry here about how to parametrize S.)

Exercise 4. Give an example of two vector fields V and W for which there exists *no integral surface* passing through the origin.

Exercise 5. Prove that if ∂_V and ∂_W *commute*, then there *is* an integral surface S passing through every point p. How can you say which points lie in the surface S?
[It's OK if the surface is just a small region; i.e. you only need to worry about this near p.]

Exercise 6. Prove that if there exist real-valued functions α and β such that

$$\partial_V(\partial_W f) - \partial_W(\partial_V(f)) = \alpha \partial_V(f) + \beta \partial_W(f),$$

then there is an integral surface passing through every point p.

Remark regarding the last exercise: it is straightforward to check that the operator

$$f \mapsto \partial_V(\partial_W f) - \partial_W(\partial_V f)$$

is a derivation (just plug in fg and check). So Theorem A says there is a unique vector field U such that

$$\partial_U(f) = \partial_V(\partial_W f) - \partial_W(\partial_V f).$$

The usual name for this vector field is $U = [V, W]$. So Exercise 6 says that if $[V, W]$ is a linear combination of V and W, then V and W have integral surfaces passing through every point.