Math 112-40, Mr. Church, Homework 5

Due at the beginning of class on Friday, October 30.

Please staple your homework.

This homework assignment has two pages.

1. Note that

$$1 + 3 + 5 = 9 = 3^2$$

$$1 + 3 + 5 + 7 = 16 = 4^2$$

$$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 = 100 = 10^2$$

Prove using mathematical induction that

$$P(n): \qquad 1 + 3 + 5 + 7 + 9 + \cdots + (2n-3) + (2n-1) = n^2$$

is true for all $n \geq 1$.

Make sure to highlight what your base case is, and how you prove the inductive step.

2. In 2 to 4 sentences, explain the mistake in the following proof. Note: this doesn't mean explain why the conclusion is wrong (I hope that is obvious). It means explain where the logical mistake in the argument happens.

"Theorem": All horses are the same color.

"Proof": We prove the theorem by mathematical induction. Our claim $P(n)$ will be:

$P(n)$: In any collection of n horses, all the horses are the same color.

Our base case is $P(1)$, which asserts that in any collection of 1 horse, that horse is the same color as itself. This is clearly true.[1]

For our inductive step, we need to prove that $P(n)$ implies $P(n+1)$. I will give an example to show how the argument goes. Let's show that $P(6) \implies P(7)$. So we assume that any collection of 6 horses all has the same color, and we try to prove it for 7. So given a collection of 7 horses, consider the collection of all-horses-but-the-last. This is a collection of 6 horses, so by $P(6)$ all horses (except possibly the last) have the same color as each other. Similarly, we can consider the collection of all-horses-but-the-first. This is a collection of 6 horses, so by $P(6)$ all horses (except possibly the first) have the same color as each other. But the five horses in the middle fall into both groups, so the first has the same color as the middle horses, and the middle horses have the same color as the last horse, so they all have the same color.[2]

[1] Hint: this is not the mistake.

[2] Hint: this is not the mistake either.

The same argument works in general to show that $P(n)$ implies $P(n+1)$. We assume $P(n)$: In any collection of n horses, all the horses are the same color. We want to prove that this is true for any collection of $n+1$ horses. Consider the collection of all-horses-but-the-last; this is a collection of n horses, so they all have the same color. Similarly consider the collection of all-horses-but-the-last; this is a collection of n horses, so they all have the same color. Thus the first horse has the same color as the middle horses (all $n-1$ of them), and the middle horses have the same color as the last horse, so they all have the same color.

Conclusion: since we have proved that $P(1)$ is true, and that $P(n) \implies P(n+1)$ for all n, we can conclude by mathematical induction that $P(n)$ is true for all n. Thus all horses are the same color.

3. Exercise 3.2.

4. Exercise 3.7.

5. Exercise 3.8.

6. Exercise 3.9.

7. Exercise 3.10.

8. Exercise 3.12.