Math 113 — Fall 2015 — Prof. Church
Final Exam 8:30–11:30am 12/11/2015

Name: ____________________________________

Signature: ________________________________

This exam is closed-book and closed-notes. In your proofs you may use any theorem from class or from the sections that we covered of the book and lecture notes (not including exercises or homework questions). You do not need to cite theorems by number; just give the statement of the theorem you wish to cite. When giving counterexamples, you may describe linear maps or operators either by a formula or by a matrix.

There are 4 questions worth 100 points total on this exam, plus a 5-point bonus question. You should finish *all* the other questions before attempting the bonus question.

Question 1 (20 points)**.** Let V be a vector space, and let U and W be subspaces of V.

Assume that $\dim V = 4$ and that $\dim U = 3$ and $\dim W = 3$.
Prove that $\dim(U \cap W) \geq 2$.

Total	1	2a	2b	2c	3a	3b	4a	4b	4c	4d	Bonus
100 points	20 points	30 points			20 points		30 points				

Question 2 (30 points)**.** Given a linear map T, we define its *rank* to be the dimension of its range:

$$\operatorname{rank} T = \dim(\operatorname{range} T)$$

Let U, V, and W be finite-dimensional vector spaces over $\mathbb{R}$.
Let $T\colon U \to V$ be a linear map from U to V, and
let $S\colon V \to W$ be a linear map from V to W.

$$U \xrightarrow{T} V \xrightarrow{S} W$$

2(a): Prove that $\operatorname{rank}(ST) \leq \operatorname{rank}(T)$. [10 points]

Let V and W be finite-dimensional vector spaces over $\mathbb{R}$.
Let $Q, R, S \in \mathcal{L}(V, W)$ be three linear maps from V to W,
and assume that they each have rank 1:

$$\operatorname{rank}(Q) = 1 \qquad\qquad \operatorname{rank}(R) = 1 \qquad\qquad \operatorname{rank}(S) = 1$$

We'll be considering the linear map $Q + R + S \in \mathcal{L}(V, W)$, so let's give it a name:

$$\text{let } \ T = Q + R + S \qquad \in \mathcal{L}(V, W).$$

2(b): Prove that $\operatorname{rank}(Q + R + S) \leq 3$. [10 points]

Recall our assumptions:
V and W are finite-dimensional;
Q, R, $S \in \mathcal{L}(V, W)$
$\operatorname{rank}(Q) = 1$, $\operatorname{rank}(R) = 1$, $\operatorname{rank}(S) = 1$;
$T = Q + R + S$.

2(c): Is the following assertion $(*)$ true? [10 points]

$$\operatorname{rank}(Q + R + S) < 3 \quad \Longleftrightarrow \quad Q,\ R,\ S \text{ are linearly dependent in } \mathcal{L}(V, W) \qquad (*)$$

Prove or give a counterexample.

Question 3 (20 points)**.** Let us say that a real $n \times n$ matrix is a *radical matrix* if for every column of the matrix, the entries in that column add up to 1.

For example, $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -4 & -6 & -8 \end{bmatrix}$ is a radical matrix.

3(a): Let A and B be real $n \times n$ matrices. [10 points]

Prove that if A and B are both radical matrices, then AB is a radical matrix.

[Hint: consider the corresponding operators on $\mathbb{R}^n$,
and the linear map $f\colon \mathbb{R}^n \to \mathbb{R}$ defined by $f(x_1, \ldots, x_n) = x_1 + \cdots + x_n$.]

3(b): Let A be a real $n \times n$ matrix, [10 points]

and let $T \colon \mathbb{R}^n \to \mathbb{R}^n$ be the corresponding operator with matrix A.

Prove that if A is a radical matrix, then 1 is an eigenvalue of T.

Question 4 (30 points)**.** If V is an inner product space over $\mathbb{R}$ or $\mathbb{C}$, we define an operator $S \in \mathcal{L}(V)$ to be *skew-self-adjoint* if it is equal to the *negative* of its adjoint:

$$S^* = -S$$

4(a): Prove that *every* operator $R \in \mathcal{L}(V)$ can be written as a sum $R = T + S$ where $T \in \mathcal{L}(V)$ is self-adjoint and $S \in \mathcal{L}(V)$ is skew-self-adjoint. [7 points]

For the remaining parts, assume that V is an inner product space over $\mathbb{R}$, and $S \in \mathcal{L}(V)$ is skew-self-adjoint.

4(b): Prove that if S is injective, then S has no eigenvectors. [8 points]

Recall our assumptions:

V is an finite-dimensional inner product space over $\mathbb{R}$,

$S \in \mathcal{L}(V)$ is skew-self-adjoint.

4(c): Prove that the operator $S^2 \in \mathcal{L}(V)$ is diagonalizable. [8 points]

Let $\mathrm{SSA}(V) \subset \mathcal{L}(V)$ be the subspace of skew-self-adjoint operators.
(You do not need to prove that this is a subspace.)

4(d): Let V be a 3-dimensional inner product space over $\mathbb{R}$ [7 points]
with orthonormal basis v_1, v_2, v_3.

Find an explicit basis for $\mathrm{SSA}(V)$. What is the dimension of $\mathrm{SSA}(V)$?

Bonus Question (5 points)**.** Let me describe a very important empirical phenomenon. Suppose you have an $n \times n$ matrix A which

- is a *radical matrix* in the sense of Question 2 (every column adds up to 1);
- is a *symmetric* matrix;
- and each entry of A is ≥ 0.

If you randomly pick an initial vector $u \in \mathbb{R}^n$ whose entries add up to 1 and repeatedly multiply it by A, the sequence of vectors u, Au, $A^2u, \ldots,$ $A^ku, \ldots$ will converge to some vector $v \in \mathbb{R}^n$ (meaning that the distance $\|v - A^ku\|$ goes to 0 as k goes to infinity).

Moreover, with extremely high probability, no matter *which* initial vector u you start with, the sequence A^ku will end up converging to the *same* vector v.

Explain this phenomenon using linear algebra, to whatever degree you are able.

(If you want to explain it from another perspective as well, I would be delighted to read it; but you should also explain it from the perspective of linear algebra.)