Math 113: Linear Algebra and Matrix Theory

Thomas Church (tfchurch@stanford.edu)

http://math.stanford.edu/~church/teaching/113-F15

Homework 1

Due Wednesday, September 30 in class.

Do all the following exercises:

1A.2	1A.3	1A.10
1B.1	1B.2	
1C.4	1C.20	1C.24

You may assume the following facts from calculus without proof:

- Sums of continuous functions are continuous.
- A scalar multiple of a continuous function is continuous.
- Sums of differentiable functions are differentiable.
- A scalar multiple of a differentiable function is differentiable.

Question 1. Let S be a set, and let U be a vector space over $\mathbf{F}$. Recall that U^S is the set of functions $f\colon S \to U$. Given functions $f, g \in U^S$ and $a \in \mathbf{F}$, we define $f + g \in U^S$ and $a \cdot f \in U^S$ by

$$(f+g)(x) = f(x) + g(x)$$

$$(a \cdot f)(x) = a \cdot (f(x))$$

Prove that U^S is a vector space over $\mathbf{F}$.

Question 2. Let $U_1 = \{(a, 0, 0) \mid a \in \mathbf{F}\}$ and $U_2 = \{(b, b, 0) \mid b \in \mathbf{F}\}$. These are both subsets of $\mathbf{F}^3$.

a) Prove that U_1 and U_2 are subspaces of $\mathbf{F}^3$.

b) Prove that $U_1 + U_2 = \{(x, y, 0) \mid x, y \in \mathbf{F}\}$.

Question 3. Let V be a vector space, and let U_1 and U_2 be subspaces of V.

a) Their intersection $U_1 \cap U_2$ consists of all vectors that belong to *both* subspaces:

$$U_1 \cap U_2 = \{v \in V \mid v \in U_1 \textbf{ and } v \in U_2\}.$$

Prove that $U_1 \cap U_2$ is always a subspace of V.

b) Their union $U_1 \cup U_2$ consists of all vectors that belong to *either* subspace:

$$U_1 \cup U_2 = \{v \in V \mid v \in U_1 \textbf{ or } v \in U_2\}.$$

Prove that $U_1 \cup U_2$ is a subspace of V *if and only if* one subspace is contained in the other.[1]

[1] i.e. either $U_1 \subset U_2$ or $U_2 \subset U_1$. Notice that this means that the union of two subspaces is *usually not* a subspace.

Question 4. Let $U_1 = \{(a, -a, 0) \,|\, a \in \mathbf{F}\}$, let $U_2 = \{(0, b, -b) \,|\, b \in \mathbf{F}\}$, and let $U_3 = \{(c, 0, -c) \,|\, c \in \mathbf{F}\}$. These are all subspaces of $\mathbf{F}^3$ (you may assume this without proof).

a) Describe the subspace $U_1 + U_2 + U_3$ by filling in the blank by an equation involving x, y, and z:

$$U_1 + U_2 + U_3 = \{(x, y, z) \in \mathbf{F}^3 \,|\, \underline{\hspace{4cm}}\,\}$$

b) Let $W = U_1 + U_2 + U_3$. Is W the direct sum of U_1, U_2, and U_3? Prove or disprove.

Question 5. Let U be the following subset of $\mathbf{F}^\infty$:

$$U = \big\{(v_1, v_2, v_3, \ldots) \in \mathbf{F}^\infty \,\big|\, v_{i+3} = v_i \text{ for all } i\big\}$$

Prove that U is a subspace of $\mathbf{F}^\infty$.

Challenge Problem 6. Say that a sequence $v = (v_1, v_2, v_3, \ldots) \in \mathbf{F}^\infty$ is *periodic* if there exists some positive number $k \in \mathbb{N}$ such that $v_{i+k} = v_i$ for all i. Let W be the set of all periodic sequences:

$$W = \big\{v \in \mathbf{F}^\infty \,\big|\, v \text{ is periodic}\big\}$$

Is W a subspace of $\mathbf{F}^\infty$? Prove or disprove.