Math 113 Homework 2 Solutions

Solutions by Guanyang Wang, with edits by Tom Church.
Exercises from the book.

Exercise 2.A.11 Suppose v_1, ..., v_m is linearly independent in V and $w \in V$. Show that $v_1, ..., v_m, w$ is linearly independent if and only if

$$w \notin \text{span}(v_1, ..., v_m)$$

Proof. First suppose $v_1, ..., v_m, w$ is linearly independent. Then if $w \in \text{span}(v_1, ..., v_m)$, we can write w as the linear combination of $v_1, ..., v_m$, that is $w = a_1v_1 + ... + a_mv_m$. Adding both sides of the equation by $-w$, we have

$$a_1v_1 + ... + a_mv_m + (-w) = 0$$

Therefore we can write 0 as $a_1v_1+...+a_mv_m+(-w)$, so there exists $a_1, a_2, ..., a_m, -1$, not all 0, such that $a_1v_1 + ... + a_mv_m + (-w) = 0$. by the definition of linear dependence, we have $v_1, ...v_m, w$ is linearly dependent, which contradicts our initial assumption. Thus we have $w \notin \text{span}(v_1, ..., v_m)$.

Conversely, suppose $w \notin \text{span}(v_1, ..., v_m)$. If $v_1, ..., v_m, w$ is linearly dependent, then by the linear dependence lemma(Lemma 2.21), we have $v_j \in span(v_1, ..., v_{j-1})$ for some j or $w \in \text{span}(v_1, ..., v_m)$. But since $v_1, ..., v_m$ is linearly independent, there is no $j \in \{1, ..., m\}$ such that $v_j \in\text{span}(v_1, ..., v_{j-1})$. Meanwhile we have $w \notin \text{span}(v_1, ..., v_m)$ by our assumption. Therefore $v_1, ..., v_m, w$ is linearly independent. □

Exercise 2.B.5 Prove or disprove: there exists a basis p_0, p_1, p_2, p_3 of $P_3(\mathbb{F})$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2.

Proof. We will show that

$$\begin{aligned} p_0 &= 1 \\ p_1 &= x \\ p_2 &= x^3 + x^2 \\ p_3 &= x^3 \end{aligned}$$

is a basis for $P_3(\mathbb{F})$. Note that none of these polynomials has degree 2.

Proposition 2.42 in the book states that if V is a finite dimensional vector space, and we have a spanning list of vectors of length $\dim V$, then that list is a basis. It is shown in the book that $P_3(\mathbb{F})$ has dimension 4. Since this list has 4 vectors, we only need to show that it spans $P_3(\mathbb{F})$.

Suppose $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \in P_3(\mathbb{F})$. We need to find $b_0, \ldots, b_3$ s.t. $p(x) = b_0p_0 + \cdots + b_3p_3$. Note that $p_2 - p_3 = x^2$. So let $b_0 = a_0, b_1 = a_1$, $b_2 = a_2$ and $b_3 = a_3 - a_2$. Then,

$$\begin{aligned} b_0p_0 + b_1p_1 + b_2p_2 + b_3p_3 &= a_0 + a_1x + a_2(x^2 + x^3) + (a_3 - a_2)x^3 \\ &= a_0 + a_1x + a_2x^2 + a_2x^3 + a_3x^3 - a_2x^3 \\ &= a_0 + a_1x + a_2x^2 + a_3x^3 \\ &= p(x) \end{aligned}$$

So we can can write $p(x)$ as a linear combination of p_0, p_1, p_2 and p_3. Thus p_0, p_1, p_2 and p_3 span $P_3(\mathbb{F})$. Thus, they form a basis for $P_3(\mathbb{F})$. Therefore, there exists a basis of $P_3(\mathbb{F})$ with no polynomial of degree 2. □

Exercise 2.B.7 Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis of V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then v_1, v_2 is a basis of U.

Proof. The statement above is false. Take $V = \mathbb{R}^4$, let $v_1 = (1,0,0,0)$, $v_2 = (0,1,0,0)$, $v_3 = (0,0,1,0)$, $v_4 = (0,0,0,1)$, it is the standard basis of $\mathbb{R}^4$ (see example 2.28 (a)). Let

$$U = \{(a,b,c,c) : a,b,c \in R\}$$

We have $v_1 \in U, v_2 \in U, v_3 \notin U, v_4 \notin U$. Now we will prove v_1, v_2 does not span U. For any $w \in \text{span}(v_1, v_2)$, $w = a_1v_1 + a_2v_2 = a_1(1,0,0,0) + a_2(0,1,0,0) = (a_1, a_2, 0, 0)$. Let $u = (0,0,1,1)$, we have $u \in U$ but $u \notin \text{span}(v_1, v_2)$.

By definition of basis, we have v_1, v_2 is not a basis of U. □

Exercise 2.C.1 Suppose that V is finite dimensional and U is a subspace of V such that $\dim U = \dim V$. Prove that $U = V$.

Proof. Suppose $\dim U = \dim V = n$. Then we can find a basis $u_1, \ldots, u_n$ for U.

Since $u_1, \ldots, u_n$ is a basis of U, it is a linearly independent set. Proposition 2.39 says that if V is finite dimensional, then every linearly independent list of vectors in V of length $\dim V$ is a basis for V. The list $u_1, \ldots, u_n$ is a list of n linearly independent vectors in V (because it forms a basis for U, and because $U \subset V$.) Since $\dim V = n$, $u_1, \ldots, u_n$ is a basis of V.

This means that $u_1, \ldots, u_n$ spans V. Thus, we can express any $v \in V$ as a linear combination of $u_1, \ldots, u_n$. But each u_i is an element of U. Since U is a vector space, any linear combination of elements of U is also in U. Thus any $v \in V$ is also an element of U. Therefore $V \subset U$.

We have $U \subset V$ since U is a subspace of V, and we have just shown that $V \subset U$. Therefore, $U = V$. □

Exercise 2.C.7 (a) Let $U = \{p \in P_4(\mathbb{F}) : p(2) = p(5) = p(6)\}$. Find a basis of U.

(b) Extend the basis in part (a) to a basis of $P_4(\mathbb{F})$.

(c) Find a subspace W of $P_4(\mathbb{F})$ such that $P_4(\mathbb{F}) = U \oplus W$.

Proof. (a) A basis of U is

$$1, (x-2)(x-5)(x-6), (x-2)^2(x-5)(x-6)$$

Each polynomial in the list above is in U. To verify that the list above is indeed a basis of U, first note that the list above is linearly independent. Suppose $a, b, c \in \mathbb{R}$ and

$$a + b(x-2)(x-5)(x-6) + c(x-2)^2(x-5)(x-6) = 0$$

for every $x \in R$. Without explicitly expanding the left side of the equation above, we can see that the left side has a cx^4 term. Because the right side has no x^4 term, this implies that $c = 0$. Because $c = 0$, we see that the left side has a bx^3 term, which implies that $b = 0$. Because $b = c = 0$, the equation becomes $a = 0$.

Therefore the equation above implies $a = b = c = 0$. Hence the list $1, (x-2)(x-5)(x-6), (x-2)^2(x-5)(x-6)$ is linearly independent in U. Now we are going to prove $\dim U = 3$, then Proposition 2.39 implies that $1, (x-2)(x-5)(x-6), (x-2)^2(x-5)(x-6)$ is a basis of U. Since we already know $1, (x-2)(x-5)(x-6), (x-2)^2(x-5)(x-6)$ is linearly independent in U, we have $\dim U \geq 3$, thus we just need to prove $\dim U \leq 3$.

Define $V = \{p \in P_4(\mathbb{F}) : p(2) = p(5)\}$. We know that V is a proper subspace of $P_4(\mathbb{F})$, since e.g. $f(x) = x$ is a polynomial in $P_4(\mathbb{F})$ that is not in V (since $f(2) = 2$ while $f(5) = 5$). We already know $\dim(P_4(\mathbb{F})) = 5$ from Example 2.37. Using the result in Exercise 2.C.1, we know that $\dim V < \dim P_4(\mathbb{F})$ since V is a proper subspace of $P_4(\mathbb{F})$, so $\dim V \leq 4$. Similarly, we know U is a proper subspace of V, because e.g. $q(x) = (x-2)(x-5)$ is a polynomial that is in V but not in U (since $q(5) = 0$ while $q(6) = 4$). Applying Exercise 2.C.1 again, we conclude that $\dim U < \dim V$, so $\dim U \leq 3$.

We conclude that $\dim U = 3$. By Prop. 2.39, we can conclude that $1, (x-2)(x-5)(x-6), (x-2)^2(x-5)(x-6)$ is a basis of U.

(b)The list

$$1, (x-2)(x-5)(x-6), (x-2)^2(x-5)(x-6), x, x^2$$

is a basis of $P_4(\mathbb{F})$.

First we prove that $1, (x-2)(x-5)(x-6), (x-2)^2(x-5)(x-6), x, x^2$ is linear independent.

Suppose $a, b, c, d, e \in \mathbb{R}$ and

$$a + b(x-2)(x-5)(x-6) + c(x-2)^2(x-5)(x-6) + dx + ex^2 = 0$$

Without explicitly expanding the left side of the equation above, we can see that the left side has a cx^4 term. Because the right side has no x^4 term, this implies that $c = 0$. Because $c = 0$, we see that the left side has a bx^3 term, which implies that $b = 0$. Because $b = c = 0$, the left side has a ex^2 term which implies that $b = 0$. Because $b = c = e = 0$, the left side has a dx term which implies that $d = 0$. Because $b = c = d = e = 0$, the equation above becomes $a = 0$.

Therefore the equation above implies $a = b = c = d = e = 0$. Hence the list $1, (x-2)(x-5)(x-6), (x-2)^2(x-5)(x-6), x, x^2$ is linearly independent in $P_4(\mathbb{F})$.

Notice that this linearly independent list has length 5, meanwhile $\dim P_4(\mathbb{F}) = 5$ (see Example 2.37). Using Proposition 2.39 , we can conclude that $1, (x-2)(x-5)(x-6), (x-2)^2(x-5)(x-6), x, x^2$ is a basis of $P_4(\mathbb{F})$.

(c) Denote the subspace $span(x, x^2)$ by W. Since

$$1, (x-2)(x-5)(x-6), (x-2)^2(x-5)(x-6), x, x^2$$

forms a basis of $P_4(\mathbb{F})$, we know that x, x^2 is linearly independent, thus x, x^2 is a basis of W(see Definition 2.27) and we have $\dim W = 2$ (see Definition 2.36). From (a) we know that $1, (x-2)(x-5)(x-6), (x-2)^2(x-5)(x-6)$ is a basis of U, and $\dim U = 3$. Now we want to prove $P_4(\mathbb{F}) = U \oplus W$.

First we prove that U and W is a direct sum. Suppose $f \in U \cap W$, then we can write f as

$$f = a_1 + a_2(x-2)(x-5)(x-6) + a_3(x-2)^2(x-5)(x-6)(\text{ since } f \in U)$$

and

$$f = a_4 x + a_5 x^2(\text{ since } f \in W)$$

Combining the two equalities together we have

$$a_1 + a_2(x-2)(x-5)(x-6) + a_3(x-2)^2(x-5)(x-6) = a_4x + a_5x^2$$

Adding both sides of the equality by $-(a_4x + a_5x^2)$ and using the property of additive inverse, we have

$$\begin{aligned}&a_1 + a_2(x-2)(x-5)(x-6) + a_3(x-2)^2(x-5)(x-6) + (-(a_4x + a_5x^2))\\ =&a_4x + a_5x^2 + (-(a_4x + a_5x^2))\\ =&0\end{aligned}$$

So we have

$$a_1 + a_2(x-2)(x-5)(x-6) + a_3(x-2)^2(x-5)(x-6) + (-a_4)x + (-a_5)x^2 = 0$$

Since $1, (x-2)(x-5)(x-6), (x-2)^2(x-5)(x-6), x, x^2$ is linearly independent, using Definition 2.17 we have:

$$a_1 = a_2 = a_3 = -a_4 = -a_5 = 0$$

Which is equivalent to

$$a_1 = a_2 = a_3 = a_4 = a_5 = 0$$

So we have $f = 0x^2 + 0x = 0$, thus U and W is a direct sum (see Proposition 1.45).

Then we prove $U \oplus W = P_4(\mathbb{F})$. Using Theorem 2.43 , we have

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$$

Consider the right side of the equation. Since $\dim U = 3$, $\dim W = 2$, $\dim(U \cap W) = 0$. We have $\dim(U + W) = 5$. The vector space $U + W$ is a subspace of $P_4(\mathbb{F})$, so using the result in Exercise 2.C.1, we have $U + W = P_4(\mathbb{F})$. We have proved that U and W is a direct sum, therefore we have $U \oplus W = P_4(\mathbb{F})$. □

Exercise 2.C.11 Suppose that U and W are subspaces of $\mathbb{R}^8$ such that $\dim U = 3$, $\dim W = 5$, and $U + W = \mathbb{R}^8$. Prove that $\mathbb{R}^8 = U \oplus W$.

Proof. We know from Theorem 2.43 that

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$$

First consider the left hand side of the equation. Here we have $U + W = \mathbb{R}^8$, so $\dim(U + W) = \dim(\mathbb{R}^8) = 8$.

Now consider the right hand side of the equation. Since $\dim U = 3$, $\dim W = 5$, the right hand of the equation equals to $8 - \dim(U \cap W)$.

Therefore we have $8 - \dim(U \cap W) = 8$, so $\dim(U \cap W) = 0$, which implies $U \cap W = \{0\}$, so we have $\mathbb{R}^8 = U \oplus W$ (see Proposition 1.45 from our textbook). □

Exercise 2.C.12 Suppose U and W are both five-dimensional subspaces of $\mathbb{R}^9$. Prove that $U \cap W \neq \{0\}$.

Proof. Suppose that $U \cap W = \{0\}$. By Theorem 2.43,

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$$

First consider the right hand side of this equation. Since $U \cap W = \{0\}$, the dimension of $U \cap W$ is zero. Since $\dim U = \dim V = 5$, the right hand side of this equation is 10.

Now consider the left hand side of the equation. The vector space $U+W$ is a subspace of $\mathbb{R}^9$. By Proposition 2.38 in the book, the dimension of a subspace of $\mathbb{R}^9$ is at most the dimension of $\mathbb{R}^9$. Since $\dim(\mathbb{R}^9)=9$, we have $\dim(U+W)\leq 9$. But this is impossible since the right hand side of the equality is 10.

Therefore, $U\cap W\neq\{0\}$. $\square$

Exercise 3.A.11 Suppose V is finite-dimensional. Prove that every linear map on a subspace of V can be extended to a linear map on V. In other words, show that if U is a subspace of V and $S\in\mathcal{L}(U,W)$, then there exists $T\in\mathcal{L}(V,W)$ such that $Tu=Su$ for all $u\in U$.

Proof. Suppose U is a subspace of V and $S\in\mathcal{L}(U,W)$. Choose a basis $u_1,...,u_m$ of U. Then $u_1,...,u_m$ is a linearly independent list of vectors in V, and so can be extended to a basis $u_1,...,u_m,v_1,...,v_n$ of V (by Proposition 2.33). Using Proposition 3.5, we know that there exists a unique linear map $T\in\mathcal{L}(V,W)$ such that

$$Tu_i=Su_i \text{ for all } i\in\{1,2,...,m\}$$
$$Tv_j=0 \text{ for all } j\in\{1,2,...n\}$$

Now we are going to prove $Tu=Su$ for all $u\in U$.

For any $u\in U$, u can be written as $a_1u_1+...+a_mu_m$, since $S\in\mathcal{L}(U,W)$, $Su=a_1Su_1+a_2Su_2+...+a_mSu_m$ (see Definition 3.2).

Since $T\in\mathcal{L}(V,W)$, we have

$$\begin{aligned}Tu&=T(a_1u_1+...+a_mu_m)\\&=a_1Tu_1+a_2Tu_2+...+a_mTu_m\\&=a_1Su_1+a_2Su_2+...+a_mSu_m\\&=Su\end{aligned}$$

Therefore we have $Tu=Su$ for all $u\in U$, so we have proved that every linear map on a subspace of V can be extended to a linear map on V. $\square$

Exercise 3.A.14 Suppose V is finite-dimensional with $\dim V\geq 2$. Prove that there exist $S,T\in\mathcal{L}(V,V)$ such that $ST\neq TS$.

Proof. Let $v_1,...,v_n$ be a basis of V. We can use Proposition 3.5 to define $S,T\in\mathcal{L}(V,V)$ such that

$$Sv_k=\begin{cases}v_2 & \text{if } k=1\\ 0 & \text{if } k\neq 1\end{cases}$$

and

$$Tv_k=\begin{cases}v_1 & \text{if } k=2\\ 0 & \text{if } k\neq 2\end{cases}$$

Then

$$(ST)(v_1)=S(Tv_1)=S0=0$$

but

$$(TS)(v_1)=T(Sv_1)=Tv_2=v_1\neq 0$$

Thus $ST\neq TS$.

$\square$

Question 1. If V is a vector space over the field $\mathbb{F}$, consider its dual vector space V^*. Assume that $\dim V = n$, and that $v_1, \ldots, v_n$ is a basis for V. Find a basis V^*. What is $\dim V^*$?

Proof. Let $v_1, \ldots, v_n$ be a basis for V. Let $v \in V$. Then we can write v as a linear combination of the basis vectors. That is, $v = a_1 v_1 + \cdots + a_n v_n$ for some $a_1, \ldots, a_n \in \mathbb{F}$. Let $f_i : V \to \mathbb{F}$ be the map such that $f_i(v) = a_i$. We will show that $f_1, \ldots, f_n$ is a basis for V^*.

First, we should show that $f_i \in V^*$ for all i. Suppose $v, w \in V$. Write $v = a_1 v_1 + \cdots + a_n v_n$ and $w = b_1 v_1 + \cdots + b_n v_n$. Thus $f_i(v) = a_i$ and $f_i(w) = b_i$. We have $v + w = (a_1 + b_1)v_1 + \ldots (a_n + b_n)v_n$. So, $f_i(v + w) = a_i + b_i$. This is the same as $f_i(v) + f_i(w)$. Thus $f_i(v + w) = f_i(v) + f_i(w)$. Next, if $c \in \mathbb{F}$, then $cv = ca_1 v_1 + \cdots + ca_n v_n$. So, $f_i(cv) = ca_i$, which is the same as $cf_i(v)$. Thus $f_i(cv) = cf_i(v)$. Therefore, $f_i \in V^*$ for all i.

Next, we need to show that $f_1, \ldots, f_n$ span V^*. Let $f \in V^*$. We will show that if $c_i = f(v_i)$, (where $v_1, \ldots, v_n$ is our basis for V), then $f = c_1 f_1 + \cdots + c_n f_n$.

Let $v = a_1 v_1 + \cdots + a_n v_n \in V$. Then

$$\begin{aligned} f(v) &= a_1 f(v_1) + \ldots a_n f(v_n) \text{ because } f \in V^* \\ &= a_1 c_1 + \cdots + a_n c_n \text{ since we defined } c_i = f(v_i) \\ &= c_1 f_1(v) + \cdots + c_n f_n(v) \text{ because } f_i(v) = a_i, \text{ by definition} \end{aligned}$$

Thus $f(v) = (c_1 f_1 + \ldots c_n f_n)(v)$ for all $v \in V$, so $f = c_1 f_1 + \ldots c_n f_n$. This shows that any element f of V^* can be written as a linear combination of $f_1, \ldots, f_n$.

Lastly, we need to show that $f_1, \ldots, f_n$ are linearly independent. Suppose that we can find constants $c_1, \ldots, c_n$ s.t. $c_1 f_1 + \cdots + c_n f_n = 0$. Then for any element $v \in V$, $c_1 f_1(v) + \cdots + c_n f_n(v) = 0$. In particular, for any i,

$$\begin{aligned} c_1 f_1(v_i) + \cdots + c_n f_n(v_i) = 0 &\text{ but } f_j(v_i) = 0 \text{ for all } j \neq i \text{ so} \\ c_i f_i(v_i) = 0 &\text{ and since } f_i(v_i) = 1. \\ c_i = 0 & \end{aligned}$$

Therefore $c_i = 0$ for all i. Thus the f_i are linearly independent.

This means $f_1, \ldots, f_n$ form a basis for V^*. Since there are n elements of this basis, the dimension of V^* is n. □

Question 2. Let V be a vector space with basis v_1, v_2, and let W be a vector space with basis w_1, w_2, w_3. Find a basis for $\mathcal{L}(V, W)$. What is $\dim \mathcal{L}(V, W)$?

Proof. Let $v \in V$. Then we can write $v = a_1 v_1 + a_2 v_2$. Define $f_{ij} : V \to W$ for $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$ to be

$$f_{ij}(v) = a_i w_j$$

For example, $f_{12}(v) = a_1 w_2$, and so on.

We claim that $f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}$ form a basis for $\mathcal{L}(V, W)$. First we need to show that f_{ij} is linear for any i, j. Suppose $v, w \in V$. Write $v = a_1 v_1 + a_2 v_2$ and $w = b_1 v_1 + b_2 v_2$. Thus $f_{ij}(v) = a_i w_j$ and $f_{ij}(w) = b_i w_j$. We have $v + w = (a_1 + b_1)v_1 + (a_2 + b_2)v_2$. So, $f_{ij}(v + w) = (a_i + b_i)w_j$. This is the same as $f_{ij}(v) + f_{ij}(w)$. Thus $f_{ij}(v + w) = f_{ij}(v) + f_{ij}(w)$. If $c \in \mathbb{F}$, then $cv = ca_1 v_1 + ca_2 v_2$. So, $f_{ij}(cv) = ca_i w_j$, which is the same as $cf_{ij}(v)$. Thus $f_{ij}(cw) = cf_{ij}(w)$. Therefore, $f_{ij} \in \mathcal{L}(V, W)$ for all i, j.

Next, we need to show that $f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}$ span $\mathcal{L}(V, W)$. Let $f \in \mathcal{L}(V, W)$. For each i, $f(v_i)$ is an element of W. That means we can express it as a linear combination of w_1,w_2 and w_3. For each $i \in \{1, 2\}$ we define elements $c_{i1}, c_{i2}, c_{i3} \in \mathbb{F}$ s.t. $f(v_i) = c_{i1}w_1 + c_{i2}w_2 + c_{i3}w_3$ (where v_1, v_2 is our basis for V.) We claim that

$$f = c_{11}f_{11} + c_{12}f_{12} + c_{13}f_{13} + c_{21}f_{21} + c_{22}f_{22} + c_{23}f_{23}$$

Let $v = a_1v_1 + a_2v_2 \in V$. Then

$$f(v) = a_1f(v_1) + a_2f(v_2)$$

because f is linear

$$=a_1(c_{11}w_1 + c_{12}w_2 + c_{13}w_3) + a_2(c_{21}w_1 + c_{22}w_2 + c_{23}w_3)$$

by the definition of c_{ij} for each i, j

$$=c_{11}f_{11}(v) + c_{12}f_{12}(v) + c_{13}f_{13}(v) + c_{21}f_{21}(v) + c_{22}f_{22}(v) + c_{23}f_{23}(v)$$

because $f_{ij}(v) = a_iw_j$, by definition

Thus any element of $\mathcal{L}(V, W)$ can be written as a linear combination of $f_{11}, f_{12}, f_{13}, f_{21}, f_{22}$ and f_{23}.

Lastly, we need to show that $f_{11}, f_{12}, f_{13}, f_{21}, f_{22}$ and f_{23} are linearly independent. Suppose that we can find constants $c_{11}, c_{12}, c_{13}, c_{21}, c_{22}$ and c_{23} s.t.

$$c_{11}f_{11} + c_{12}f_{12} + c_{13}f_{13} + c_{21}f_{21} + c_{22}f_{22} + c_{23}f_{23} = 0$$

Then for any element $v \in V$,

$$c_{11}f_{11}(v) + c_{12}f_{12}(v) + c_{13}f_{13}(v) + c_{21}f_{21}(v) + c_{22}f_{22}(v) + c_{23}f_{23}(v) = 0$$

In particular, for any i,

$$c_{11}f_{11}(v_i) + c_{12}f_{12}(v_i) + c_{13}f_{13}(v_i) + c_{21}f_{21}(v_i) + c_{22}f_{22}(v_i) + c_{23}f_{23}(v_i) = 0$$

but $f_{kj}(v_i) = 0$ for all $k \neq i$ so

$$c_{i1}f_{i1}(v_i) + c_{i2}f_{i2}(v_i) + c_{i3}f_{i3}(v_i) = 0$$

and since $f_{ij}(v_i) = w_j$,

$$c_{i1}w_1 + c_{i2}w_2 + c_{i3}w_3 = 0$$

but w_1, w_2 form a basis for W, so this is only possible if

$$c_{i1} = c_{i2} = c_{i3} = 0$$

Therefore $c_{ij} = 0$ for all i, j. Thus the f_{ij} are linearly independent.

This means $f_{11}, f_{12}, f_{13}, f_{21}, f_{22}$ and f_{23} form a basis for $\mathcal{L}(V, W)$. Since there are 6 elements of this basis, the dimension of $\mathcal{L}(V, W)$ is 6. □

Question 3. Let U be a subset of $\mathbb{R}^\infty$ consisting of all sequences that satisfy

$$v_i + v_{i+2} = v_{i+1} \text{ for all } i$$

(1) Prove that U is a subspace of $\mathbb{R}^\infty$.

(2) Let $x, y \in U$ be the elements

$$x = (0, 1, 1, 0, -1, -1, 0, 1, 1, \dots)$$
$$y = (1, 0, -1, -1, 0, 1, 1, 0, -1, \dots)$$

Prove that the list x, y is a linearly independent set.

(3) Prove that x, y is a basis for U.

(4) Let W be the subspace of $\mathbb{R}^\infty$ consisting of all sequences with $v_1 = 0$ and $v_2 = 0$. Prove that $\mathbb{R}^\infty = U \oplus W$.

Proof. (1) First we prove that U is a subspace of $\mathbb{R}^\infty$. To do this, we show that it has the following properties.

Zero: The sequence $(0, 0, \dots)$ satisfies $v_i + v_{i+2} = v_{i+1}$ because $v_i = v_{i+1} = v_{i+2} = 0$. Therefore $0 \in U$.

Closed Under Vector Addition: Suppose $v = (v_1, v_2, \dots), w = (w_1, w_2, \dots) \in U$. Then $v_i + v_{i+2} = v_{i+1}$ and $w_i + w_{i+2} = w_{i+1}$. Thus $(v_i + w_i) + (v_{i+2} + w_{i+2}) = (v_{i+1} + w_{i+1})$. Since the i^{th} term of $v + w$ is $v_i + w_i$ for each i, this means that $v + w \in U$. Therefore U is closed under vector addition.

Closed Under Scalar Multiplication: Suppose $v = (v_1, v_2, \dots) \in U$ and $a \in \mathbb{R}$. Since $v_i + v_{i+2} = v_{i+1}$, we have that $av_i + av_{i+2} = av_{i+1}$. Since the i^{th} term of av is av_i for each i, this means that $av \in U$. Therefore U is closed under scalar multiplication.

Since U satisfies these properties, it is a subspace of $\mathbb{R}^\infty$.

(2) Let $x, y \in U$ be the elements

$$x = (0, 1, 1, 0, -1, -1, 0, 1, 1, \dots)$$
$$y = (1, 0, -1, -1, 0, 1, 1, 0, -1, \dots)$$

We will show that (x, y) is a linearly independent set.
Suppose not. Then we can find $a, b \in \mathbb{R}$ s.t. $ax + by = 0$. Note that

$$ax = (0, a, a, 0, -a, -a, 0, a, a, \dots)$$
$$by = (b, 0, -b, -b, 0, b, b, 0, -b, \dots) \text{ so,}$$
$$ax + by = (b, a, \dots)$$

If $ax + by = 0$ then $b = 0$ and $a = 0$ since two sequences are equal iff their terms are all equal. This means that x and y are linearly independent.

(3) Next we show that (x, y) is a basis for U. Since we have already shown that (x, y) is a linearly independent set, we just need to show that it spans U.

Let $u \in U$. Write $u = (u_1, u_2, \dots)$. Then we claim that $u = u_1 y + u_2 x$. Note that

$$u_1 y + u_2 x = (u_1, u_2, u_2 - u_1, -u_1, -u_2, -u_2 + u_1, u_1, u_2, u_2 - u_1, \dots)$$

We will show that all the terms of u and $u_1 y + u_2 x$ match up by induction. We will use the fact that since $u_i + u_{i+2} = u_{i+1}$, then $u_{i+2} = u_{i+1} - u_i$. First of all, this means that $u_3 = u_2 - u_1$. Thus, u and $u_1 y + u_2 x$ match up on the first three terms.

Now suppose the first $3n$ terms of u and $u_1 y + u_2 x$ are the same. We need to show that this implies the first $3(n+1)$ terms are the same. There are two cases: n is either odd or even. First suppose n is odd. Then

$$u = (u_1, \dots, -u_2 + u_1, u_1, u_2, u_2 - u_1, u_{3n+1}, u_{3n+2}, u_{3n+3}, \dots)$$

where the $u_2 - u_1$ is its $3n^{th}$ term. Since $u_{3n+1} = u_{3n} - u_{3n-1}$, we have that $u_{3n+1} = -u_1$. Next, since $u_{3n+2} = u_{3n+1} - u_{3n}$, we have that $u_{3n+2} = -u_2$. Lastly, since $u_{3n+3} = u_{3n+2} - u_{3n+1}$, we have that $u_{3n+3} = -u_2 + u_1$. Thus u and $u_1 y + u_2 x$ match up for $3n + 3 = 3(n + 1)$ terms.

Now suppose that n is even. Then,

$$u = (u_1, \ldots, u_2 - u_1, -u_1, -u_2, -u_2 + u_1, u_{3n+1}, u_{3n+2}, u_{3n+3}, \ldots)$$

where the $u_2 - u_1$ is its $3n^{th}$ term. Since $u_{3n+1} = u_{3n} - u_{3n-1}$, we have that $u_{3n+1} = u_1$. Next, since $u_{3n+2} = u_{3n+1} - u_{3n}$, we have that $u_{3n+2} = u_2$. Lastly, since $u_{3n+3} = u_{3n+2} - u_{3n+1}$, we have that $u_{3n+3} = u_2 - u_1$. Thus u and $u_1 y + u_2 x$ match up for $3n + 3 = 3(n + 1)$ terms.

Therefore, by induction, $u_1 y + u_2 x = u$.

This means that x and y span U. Since we have shown that they are linearly independent, they form a basis for U.

(4) Let W be the subspace of $\mathbb{R}^\infty$ consisting of all sequences with $v_1 = 0$ and $v_2 = 0$. We need to show that $\mathbb{R}^\infty = U \oplus W$. By Proposition 1.9 from the book, $\mathbb{R}^\infty = U \oplus W$ iff $\mathbb{R}^\infty = U + W$ and $U \cap W = \{0\}$.

To show that $\mathbb{R}^\infty = U + W$ we need to show that any sequence can be written as the sum of an element of U and an element of W. Let $x = (x_1, x_2, \ldots) \in \mathbb{R}^\infty$. Let $u = (x_1, x_2, x_2 - x_1, -x_1, -x_2, x_1 - x_2, x_1, x_2, \ldots)$ be the element of U that starts with x_1 and x_2. Let $w = x - u$. Since u and x have the same first and second term, $w = (0, 0, w_3, w_4, \ldots)$. So, $w \in W$. Since $x = u + w$, we can write any element of $\mathbb{R}^\infty$ as the sum of an element of U plus an element of W. Thus, $\mathbb{R}^\infty = U + W$.

To show that $U \cap W = \{0\}$, suppose $v \in U \cap W$. We will show that $v = 0$ by induction. Write $v = (v_1, v_2, \ldots)$. Since $v \in W$, $v_1 = v_2 = 0$. Suppose $v_{n-1} = v_n = 0$. Then we need to show that $v_{n+1} = 0$. Since $v_{n+1} = v_n - v_{n-1}$, we have that $v_{n+1} = 0$. So by induction, $v = 0$. Therefore, $U \cap W = \{0\}$.

Since we proved $\mathbb{R}^\infty = U + W$ and $U \cap W = \{0\}$, we have shown that $\mathbb{R}^\infty = U \oplus W$.

$\square$