Math 113: Linear Algebra and Matrix Theory

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Homework 3

Due Wednesday, October 14 in class.

Do all the following exercises, but write up only the **unstarred exercises** and the **questions** below. (Starred exercises are valuable and worth working out, but they will not be collected or graded.)

3B.2*	3B.12*	**3B.20**	3B.21*	**3B.29**
3C.3	3C.4*	3C.5*		
3D.7	3D.9*	3D.10*	**3D.16**	

Question 1. Let $T \in \mathcal{L}(V)$ be an operator on V. Recall that T^2 denotes the composition $T \circ T$.

a) Give an example of a vector space V and an operator $T \in \mathcal{L}(V)$ such that $T^2 = T$ (an example other than $T = I$ or $T = 0$, those are too easy).

b) Prove that if $T^2 = T$, then $V = \operatorname{null} T \oplus \operatorname{null}(T - I)$.

c) Prove that if $V = \operatorname{null} T + \operatorname{null}(T - I)$, then $T^2 = T$.

d) Give an example of a vector space V and an operator $T \in \mathcal{L}(V)$ such that $T^2 = -I$.

Question 2. Let V and W be finite dimensional, and consider $T \in \mathcal{L}(V, W)$ and $S \in \mathcal{L}(W, U)$.

a) Prove that $\dim(\operatorname{range} ST) \leq \dim(\operatorname{range} T)$.

b) Prove that $\dim(\operatorname{range} ST) = \dim(\operatorname{range} T)$ if and only if

$$\operatorname{range} T + \operatorname{null} S = \operatorname{range} T \oplus \operatorname{null} S.$$

c) Prove that $\dim(\operatorname{null} ST) \leq \dim(\operatorname{null} S) + \dim(\operatorname{null} T)$.

d) Challenge problem: Can you give some description (in terms of conditions on T, S, V, etc.) of when we get equality in the previous part, i.e. $\dim(\operatorname{null} ST) = \dim(\operatorname{null} S) + \dim(\operatorname{null} T)$?