1. Math 113 Homework 3 Solutions

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Exercises from the book.

Exercise 3B.2 Suppose V is a vector space and $S, T \in \mathcal{L}(V, V)$ are such that

$$\text{range } S \subset \text{null } T.$$

Prove that $(ST)^2 = 0$.

Proof. Suppose $v \in V$. Then

$$(ST)^2 v = ST(S(Tv))$$

Here we have $S(Tv) \in \text{range } S \subset \text{null } T$, thus $T(S(Tv)) = 0$. This implies $(ST)^2 v = ST(S(Tv)) = 0$, so we have $(ST)^2 = 0$. □

Exercise 3B.12 Suppose V is finite dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that $U \cap \operatorname{null} T = \{0\}$ and $\operatorname{range} T = \{Tu \mid u \in U\}$.

Proof. Proposition 2.34 says that if V is finite dimensional and W is a subspace of V then we can find a subspace U of V for which $V = W \oplus U$. Proposition 3.14 says that $\operatorname{null} T$ is a subspace of V. Setting $W = \operatorname{null} T$, we can apply Prop 2.34 to get a subspace U of V for which

$$V = \operatorname{null} T \oplus U$$

Now we want to prove any subspace U for which $V = \operatorname{null} T \oplus U$ satisfies the desired property. Since $V = \operatorname{null} T \oplus U$, we already have $\operatorname{null} T \cap U = \{0\}$. So we just need to show that $\operatorname{range} T = \{Tu \mid u \in U\}$. First we show that $\operatorname{range} T \subset \{Tu \mid u \in U\}$. So let $w \in \operatorname{range} T$. That means there is some $v \in V$ for which $T(v) = w$. Since $v \in V$ and we have that $V = \operatorname{null} T \oplus U$, we can find vectors $n \in \operatorname{null} T$ and $u \in U$ for which $v = n + u$. Thus,

$$\begin{aligned} T(v) &= T(n) + T(u) \\ &= 0 + T(u) \text{ since } n \in \operatorname{null} T \end{aligned}$$

We had that $w = T(v)$. So, $w = T(u)$ for some $u \in U$. That means that $w \in \{Tu \mid u \in U\}$. Thus $\operatorname{range} T \subset \{Tu \mid u \in U\}$.

Now we show that $\{Tu \mid u \in U\} \subset \operatorname{range} T$. But for any element $u \in U$, u is also in V as $U \subset V$. Thus Tu is in the image of T by definition. Therefore $\{Tu \mid u \in U\} \subset \operatorname{range} T$.

So we have shown that $\operatorname{range} T = \{Tu \mid u \in U\}$. Thus, there exists a subspace U of V s.t. $V = \operatorname{null} T \oplus U$ and $\operatorname{range} T = \{Tu \mid u \in U\}$.

□

Exercise 3.B.20 Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is injective if and only if there exists $S \in L(W, V)$ such that ST is the identity map on V.

Proof. First suppose T is injective. Define $S_1 : \operatorname{range} T \to V$ by

$$S_1(Tv) = v$$

because T is injective, each element of $\operatorname{range} T$ can be represented in the form Tv in only one way, so T is well defined.

First we will check S_1 is a linear map from T to V. For any $r \in \mathbb{F}$ and $Tx, Ty \in \operatorname{range} T$, we have

$$\begin{aligned} S_1(Tx+Ty) &= S_1(T(x+y)) \\ &= x+y \\ &= S_1(Tx)+S_1(Ty) \end{aligned}$$

and

$$\begin{aligned} S_1(rTx) &= S_1(T(rx)) \\ &= rx \\ &= rS_1(Tx) \end{aligned}$$

Therefore we know S_1 is a linear map from range T to V. Using Exercise 3.A.11 in our last homework. We know S_1 can be extended to $S \in \mathcal{L}(W, V)$, such that for any $u \in \operatorname{range} T$, we have $Su = S_1 u$. For any $v \in V$, since $Tv \in \operatorname{range} T$, $(ST)v = S(Tv) = S_1(Tv) = v$. Thus ST is the identity map on V, as we desired.

To prove the implication in the other direction, now suppose there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity map on V. If $u, v \in V$ are such that $Tu = Tv$, then

$$u = (ST)u = S(Tu) = S(Tv) = (ST)v = v.$$

Hence $u = v$. Thus T is injective, as desired. □

Exercise 3.B.21 Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is surjective if and only if there exists $S \in L(W, V)$ such that TS is the identity map on V.

Proof. First suppose T is surjective. Thus W, which equals range T is finite-dimensional (by Proposition 3.22). Let $w_1, ..., w_m$ be a basis of W. Since T is surjective, for each j there exists $v_j \in V$ such that $Tv_j = w_j$. By Proposition 3.5, there exists a unique linear map $S : W \to V$ such that

$$Sw_j = v_j$$

.

Hence for any $w \in W$, $w = a_1 w_1 + ... + a_m w_m$ for some $a_1, ..., a_m \in F$. We have

$$\begin{aligned} TS(w) &= T(S(a_1 w_1 + ... + a_m w_m)) \\ &= T(a_1 Sw_1 + ... + a_m Sw_m) \\ &= T(a_1 v_1 + ... + a_m v_m) \\ &= a_1 T(v_1) + ... + a_m T(v_m) \\ &= a_1 w_1 + ... + a_m w_m \\ &= w \end{aligned}$$

Hence we have TS is an identity map on W, as desired.

To prove the implication in the other direction, assume that there is some $S \in \mathcal{L}(W, V)$ such that TS is an identity map on W. Then for any $w \in W$, we have that $w = (TS)w = T(Sw) \in \operatorname{range} T$. So w is in the range of T, so T is surjective. □

Exercise 3.B.29 Suppose that $T \in \mathcal{L}(V, F)$. Suppose $u \in V$ is not in null T. Prove that

$$V = \operatorname{null} T \oplus \{au \mid a \in F\}$$

Proof. To show that $V = \operatorname{null} T \oplus \{au \mid a \in F\}$, we need to show that $V = \operatorname{null} T + \{au \mid a \in F\}$ and that $\operatorname{null} T \cap \{au \mid a \in F\} = \{0\}$.

First we show that $V = \operatorname{null} T + \{au \mid a \in F\}$. Let $v \in V$. We need to find $n \in \operatorname{null} T$ and $w \in \{au \mid a \in F\}$ for which $v = n + w$. Suppose $T(v) = a \in F$, and that $T(u) = b \in \mathbb{F}$. We know that $b \neq 0$ because u is not in $\operatorname{null} T$. Thus, b has an inverse in $\mathbb{F}$. Let $c = ab^{-1} \in \mathbb{F}$. Note that this means c times b gives back a. So,

$$\begin{aligned} T(cu) &= cT(u) \\ &= cb \\ &= a \end{aligned}$$

Thus, $T(cu) = T(v)$. Let $n = v - cu$. Then $T(n) = T(v) - T(cu) = 0$. Therefore, $n \in \operatorname{null} T$. Set $w = cu$. Then $w \in \{au \mid a \in F\}$, and $v = n + w$. So we can write v as the sum of elements of $\operatorname{null} T$ and $\{au \mid a \in F\}$. Therefore, $V = \operatorname{null} T + \{au \mid a \in F\}$.

Next we show that $\operatorname{null} T \cap \{au \mid a \in F\} = \{0\}$. Suppose $v \in \operatorname{null} T \cap \{au \mid a \in F\}$. Then $v \in \{au \mid a \in F\}$, so $v = au$ for some $a \in \mathbb{F}$.

Since $v \in \operatorname{null} T$, $T(v) = 0$. So $T(au) = 0$. But $T(au) = aT(u)$. Since $aT(u) = 0$, either $a = 0$ or $T(u) = 0$. But u is not in $\operatorname{null} T$, so $T(u) \neq 0$. This means a must equal 0. So $v = au$ implies that $v = 0$. Therefore, $\operatorname{null} T \cap \{au \mid a \in F\} = \{0\}$.

Since $V = \operatorname{null} T + \{au \mid a \in F\}$ and that $\operatorname{null} T \cap \{au \mid a \in F\} = \{0\}$, we have that $V = \operatorname{null} T \oplus \{au \mid a \in F\}$. □

Exercise 3C.3 Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exist a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ are 0 except the entries in row j, column j, equal 1 for $1 \leq j \leq \dim \operatorname{range} T$.

Proof. Let $u_1, ..., u_m$ be a basis of $\operatorname{null} T$. Extend $u_1, ..., u_m$ to a basis $u_1, ..., u_m, v_1, ..., v_n$ of V. Then $Tv_1, ..., Tv_n$ is a basis of $\operatorname{range} T$, as we have proved in the proof of Theorem 3.22. Therefore $n = \dim \operatorname{range} T$.

Because $Tv_1, ..., Tv_n$ is a basis of $\operatorname{range} T$, this list is linearly independent in W. Extend the linearly independent list $Tv_1, ..., Tv_n$ to a basis $Tv_1, ..., Tv_n, w_1, ..., w_p$ of W.

With respect to the basis $v_1, ..., v_n, u_1, ..., u_m$ of V (note that the v's now come before the u's) and the basis $Tv_1, ..., Tv_n, w_1, ..., w_p$ of W, the matrix of T has the desired form. Since $Tv_i = 1 \cdot Tv_i$ for any $i \in \{1, ..., n\}$, we have all the entries in the first n columns of the matrix are 0 except the entries in row i, column i, equal 1 for $1 \leq i \leq n = \dim \operatorname{range} T$. $Tu_j = 0$ for any $j \in \{1, ..., m\}$, therefore all the entries in the other m columns of the matrix are 0. □

Exercise 3C.4 Suppose $v_1, ..., v_m$ is a basis of V and W is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $w_1, ..., w_n$ of W such that all the entries in the first column of $\mathcal{M}(T)$ (with respect to the basis $v_1, ..., v_m$ and $w_1, ..., w_n$) are 0 except for possibly a 1 in the first row, first column.

Proof. Suppose $Tv_1 = 0$, then for any basis $w_1, ..., w_n$ of W, it will satisfy the requirement above since $Tv_1 = 0 = 0w_1 + ... + 0w_n$.

Suppose $Tv_1 \neq 0$, then let $w_1 = Tv_1$ and extend w_1 to a basis $w_1, ..., w_n$ of W. Then with respect to the basis $v_1, ..., v_m$ and $w_1, ..., w_n$, the first column of the matrix are all 0 expect for a 1 in the first row, first column. □

Exercise 3C.5 Suppose $w_1, ..., w_n$ is a basis of W and V is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $v_1, ..., v_m$ such that all the entries in the first row of $\mathcal{M}(T)$

(with respect to the basis $v_1, ..., v_m$ and $w_1, ..., w_n$) are 0 except for possibly a 1 in the first row, first column.

Proof. If range $T \subset \text{span}(w_2, ..., w_n)$, then the first row of $\mathcal{M}(T)$ will consist all 0's for every choice of basis of V (using, of course, $w_1, ..., w_n$ as basis of W).

Thus suppose range $T \not\subset \text{span}(w_2, ..., w_n)$. Let $\tilde{u}_1 \in V$ be such that $T\tilde{u}_1 \notin \text{span}(w_2, ..., w_n)$. Because $w_1, ..., w_n$ is a basis of W, we can write

$$T\tilde{u}_1 = \tilde{c}_1 w_1 + ... + \tilde{c}_n w_n$$

for some $\tilde{c}_1, ..., \tilde{c}_n \in F$. Since $T\tilde{u}_1 \notin \text{span}(w_2, ..., w_n)$, we know that $\tilde{c}_1 \neq 0$

Let $u_1 = (\tilde{c}_1)^{-1}\tilde{u}_1$, we have $Tu_1 = w_1 + c_2 w_2 + ... + c_n w_n$, here $c_l = (\tilde{c}_1)^{-1}\tilde{c}_l$ for $l \geq 2$.

Extend u_1 to a basis $u_1, u_2, ..., u_m$ of V. For each $k \in 2, ...m$, we can write

$$Tu_k = a_{1,k} w_1 + ... + a_{n,k} w_n.$$

Thus

$$T(u_k - a_{1,k} u_1) = (a_{2,k} - a_{1,k} c_2) w_2 + ... + (a_{n,k} - a_{1,k} c_n) w_n$$

Now we are going to prove $u_1, u_2 - a_{1,2}u_1, ..., u_m - a_{1,m}u_1$ is a basis of V.

First, suppose there is $b_1, ...b_m$ such that

$$b_1 u_1 + b_2(u_2 - a_{1,2}u_1) + b_m(u_m - a_{1,m}u_1) = 0$$

The equation above equals:

$$b_2 u_2 + b_3 u_3 + ... + b_m u_m + (b_1 - b_2 a_{1,2} - ... - b_m a_{1,m}) u_1 = 0$$

Therefore we have $b_2 = b_3 = ... = b_m = 0$ and $b_1 = 0$ since $u_1, u_2, ..., u_m$ is linearly independent. Meanwhile $u_1, u_2 - a_{1,2}u_1, ..., u_m - a_{1,m}u_1$ is of length m, which equals the dimension of V, Proposition 2.39 implies that $u_1, u_2 - a_{1,2}u_1, ..., u_m - a_{1,m}u_1$ is a basis of V. Let $v_1 = u_1, v_j = u_j - a_{1,j}u_1$ for $j \geq 2$. With respect to the basis $v_1, ..., v_m$ of V and $w_1, ..., w_n$ of W, we see that the first row of $\mathcal{M}(T)$ consists of all 0's except for a 1 in row 1, column 1. □

Exercise 3.D.7 Suppose V and W are finite-dimensional. Let $v \in V$. Let

$$E = \{T \in \mathcal{L}(V, W) : Tv = 0\}$$

(a) Show that E is a subspace of $\mathcal{L}(V, W)$.

(b) Suppose $v \neq 0$. What is $\dim E$

Proof. (a) First, the zero element in $\mathcal{L}(V, W)$ is the map $z : V \to W$ defined by $z(x) = 0 \in W$ for any $x \in V$, so we have $z(v) = 0$, therefore the zero element is contained in E.

Next, for any $f, g \in E$ and $\lambda \in \mathbb{F}$, $(f + g)(v) = f(v) + g(v) = 0$ since $f(v) = g(v) = 0$. $(\lambda f)(v) = \lambda f(v) = \lambda 0 = 0$, hence $f + g \in E$, $\lambda f \in E$. Thus we know E is a subspace of $\mathcal{L}(V, W)$

(b) Define $F : \mathcal{L}(W, V) \to W$ by

$$F(T) = Tv$$

First we check F is a linear map. For any $f, g \in E$ and $\lambda \in \mathbb{F}$, $F(f + g) = (f + g)(v) = f(v) + g(v) = F(f) + F(g)$, $F(\lambda f) = (\lambda f)(v) = \lambda f(v) = \lambda F(f)$, so we know that F is linear.

Note that null $F = E$ by definition. Note also that F is surjective (as follows from Proposition 3.5). Now

$$\begin{aligned} \dim E &= \dim F \\ &= \dim \mathcal{L}(W, V) - \dim \text{range}\, F \\ &= (\dim V)(\dim W) - \dim W \end{aligned}$$

when the second equality follows from the Fundamental Theorem of Linear Maps (Theorem 3.22), and the last equality follows from Proposition 3.61.

Thus
$$\dim E = (\dim V)(\dim W) - \dim W$$
□

Exercise 3.D.9 Suppose that V is finite dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST is invertible if and only if both S and T are invertible.

Proof. Let $S, T \in \mathcal{L}(V)$. Suppose ST is invertible. We need to show that S and T are both invertible.

Since ST is invertible, there is a maps $R : V \to V$ s.t. $R(ST) = I$. Composition of maps is associative, so the first equation means
$$(RS)T = I$$
Since $\mathcal{L}(V) = \mathcal{L}(V, V)$ by definition, and since V is finite dimensional, the previous exercise (Exercise 3.B.20) implies that since there is a linear function, RS for which $(RS)T = I$, we must have that T is injective. By Theorem 3.69, since V is finite dimensional, T is injective iff it is invertible. Therefore T is invertible.

Since T is invertible, we can write $S = STT^{-1}$. Multiplying both sides of this equation by R on the left, we get $RS = T^{-1}$. Multiplying by T on the left, we get that $T(RS) = TT^{-1}$. So, $(TR)S = I$. Again, Exercise 3.B.20 implies that since we have a linear function, TR for which $(TR)S = I$, then S is injective. Then Theorem 3.69 implies that since S is injective, it is invertible.

Thus, if ST is invertible, so are S and T.

Suppose S, T are both invertible. Then we show that $(ST)^{-1} = T^{-1}S^{-1}$.
$$\begin{aligned}(T^{-1}S^{-1})ST &= T^{-1}(S^{-1}S)T \\ &= T^{-1}T \\ &= I\end{aligned}$$
and
$$\begin{aligned}ST(T^{-1}S^{-1}) &= S(TT^{-1})S^{-1} \\ &= S^{-1}S \\ &= I\end{aligned}$$
Thus $(ST)^{-1} = T^{-1}S^{-1}$, so ST is invertible. □

Exercise 3.D.10 Suppose that V is finite dimensional and $S, T \in \mathcal{L}(V)$. Prove that $ST = I$ iff $TS = I$.

Proof. Note that since S and T are arbitrary linear functions, we only need to show that $ST = I$ implies $TS = I$ (the other direction follows by switching the labels of our linear transformations). So suppose $ST = I$. By Exercise 3.B.20, $ST = I$ implies T is injective. Since V is finite dimensional, Theorem 3.21 implies T is invertible.

Since $ST = I$, we can multiply this equation by T on the left to get
$$TST = T$$
Multiplying both sides of this equation by T^{-1} on the right, we get
$$(TST)T^{-1} = I$$
Since function composition is associative, $(TST)T^{-1} = TS(TT^{-1})$. So we really have
$$TS = I$$
as required. □

Exercise 3.D.16 Suppose that V is finite dimensional and $T \in \mathcal{L}(V)$. Prove that T is a scalar multiple of the identity iff $ST = TS$ for every $S \in \mathcal{L}(V)$.

Proof. Suppose $T = aI$ for some $a \in \mathbb{F}$. Let $S \in \mathcal{L}(V)$. Then $S(aI)(v) = S(av) = aS(v)$ and $aI(S(v)) = aS(v)$. Thus $ST = TS$ for each $S \in \mathcal{L}(V)$.

Now suppose $ST = TS$ for each $S \in \mathcal{L}(V)$. Since V is finite dimensional, let $v_1, \ldots, v_n$ be a basis for V. Define maps $S_{ij} : V \to V$ by

$$S_{ij}(a_1 v_1 + \cdots + a_n v_n) = a_i v_j$$

Clearly, this is a linear map. Now, for each v_i, let

$$T(v_i) = a_{i1}v_1 + a_{i2}v_2 + \cdots + a_{in}v_n$$

Then choosing numbers i and j between 1 and n,

$$\begin{aligned} S_{ij}T(v_i) &= a_{ii}v_j \text{ while} \\ TS_{ij}(v_i) &= T(v_j) \\ &= a_{j1}v_1 + \cdots + a_{jn}v_n \end{aligned}$$

So, since $S_{ij}T = TS_{ij}$,

$$a_{ii}v_j = a_{j1}v_1 + \cdots + a_{jn}v_n$$

Since $v_1, \ldots, v_n$ form a basis, these two sums are equal iff the coefficients are equal. On the left hand side, only the coefficient on v_j is non-zero. On the right hand side, that coefficient is a_{jj}. Thus, $a_{jk} = 0$ for all $k \neq j$ and $a_{jj} = a_{ii}$

Since i and j were chosen arbitrarily, we get that $a_{ij} = 0$ for all $i \neq j$ and $a_{ii} = a_{jj}$ for all i and j. Let $a = a_{11}$ (this is also equal to a_{ii} for any i, since all the a_{ii} are equal). Then we showed that

$$\begin{aligned} T(v_i) &= a_{ii}v_i \\ &= av_i \end{aligned}$$

for each i.

So, if $v = b_1v_1 + \cdots + b_nv_n \in V$, then $Tv = b_1T(v_1) + \cdots + b_nT(v_n)$. Thus,

$$\begin{aligned} Tv &= b_1(av_1) + \cdots + b_n(av_n) \\ &= a(b_1v_1 + \cdots + b_nv_n) \\ &= av \end{aligned}$$

Thus, $T = aI$, so we are done. □

Question 1. Assume that $T \in \mathcal{L}(V)$. Recall that T^2 denotes the composition $T \circ T$.

a) Give an example of a vector space V and a linear operator $T \in \mathcal{L}(V)$ such that $T^2 = T$. (Not $T = 1$ or 0.)
b) Prove that if $T^2 = T$ then $V = \operatorname{null} T \oplus \operatorname{null}(T - I)$.
c) Prove that if $V = \operatorname{null} T + \operatorname{null}(T - I)$ then $T^2 = T$.
d) Give an example of a vector space V and a linear operator $T \in \mathcal{L}(V)$ such that $T^2 = -I$.

Proof.

a) First we give an example of a vector space V and a linear operator $T \in \mathcal{L}(V)$ such that $T^2 = T$. Let $V = \mathbb{R}^2$ and let T be the linear map given by

$$T(x, y) = (x + y, 0)$$

Then $T^2(x, y) = T(x + y, 0) = (x + y, 0)$. So $T(x, y) = T^2(x, y)$ for all $(x, y) \in \mathbb{R}^2$.

b) Next we prove that if $T^2 = T$ then $V = \operatorname{null} T \oplus \operatorname{null}(T - I)$.

We have that $\operatorname{null}(T - I) = \{v \mid (T - I)v = 0\}$. But $(T - I)(v) = 0$ iff $Tv = Iv$, that is, iff $Tv = v$. Thus,

$$\operatorname{null}(T - I) = \{v \mid Tv = v\}$$

We want to show that $V = \operatorname{null} T \oplus \operatorname{null}(T - I)$. To do this, we first need to show that $V = \operatorname{null} T + \operatorname{null}(T - I)$. Let $v \in V$. We need to show that we can find $n \in \operatorname{null} T$ and $u \in \operatorname{null}(T - I)$ s.t. $v = n + u$. Since $T(Tv) = Tv$, we have that $Tv \in \operatorname{null}(T - I)$.

Consider the vector $Tv - v$. Then

$$T(Tv - v) = T^2 v - Tv = 0$$

since $T^2 v = Tv$. Thus, for any v, $Tv - v \in \operatorname{null} T$.

We can write $v = Tv - (Tv - v)$ where $Tv \in \operatorname{null}(T - I)$ and $-(Tv - v) \in \operatorname{null} T$. Thus $V = \operatorname{null} T + \operatorname{null}(T - I)$.

Next we need to show that $\operatorname{null} T \cap \operatorname{null}(T - I) = \{0\}$. Suppose $v \in \operatorname{null} T \cap \operatorname{null}(T - I)$. Then $v \in \operatorname{null}(T - I)$ implies that $Tv = v$. On the other hand, $v \in \operatorname{null} T$ implies that $Tv = 0$. Thus $v = 0$. Therefore $\operatorname{null} T \cap \operatorname{null}(T - I) = \{0\}$.

Since $\operatorname{null} T \cap \operatorname{null}(T - I) = \{0\}$ and $V = \operatorname{null} T + \operatorname{null}(T - I)$ we have shown that $V = \operatorname{null} T \oplus \operatorname{null}(T - I)$.

c) We want to show that if $V = \operatorname{null} T \oplus \operatorname{null}(T - I)$ then $T^2 = T$. Let $v \in V$. Since $V = \operatorname{null} T \oplus \operatorname{null}(T - I)$, we can find $n \in \operatorname{null} T$ and $u \in \operatorname{null}(T - I)$ s.t. $v = n + u$. Thus, we have

$$\begin{aligned} Tv &= T(n + u) \\ &= Tn + Tu \\ &= 0 + u \end{aligned}$$

So $Tv = u$, and since we showed above that $u \in \operatorname{null}(T - I)$ implies $Tu = u$,

$$\begin{aligned} T^2 v &= Tu \\ &= u \end{aligned}$$

So $T^2 v = u$, as well. Thus $V = \operatorname{null} T \oplus \operatorname{null}(T - I)$ implies $T^2 v = Tv$ for all $v \in V$.

d) Let $V = \mathbb{R}^2$ and let $T \in \mathcal{L}(V)$ be defined by $T(x, y) = (y, -x)$. Then $T^2(x, y) = T(y, -x) = (-x, -y)$. Thus $T^2(x, y) = -(x, y)$ for all $(x, y) \in \mathbb{R}^2$, so $T^2 = -I$. (Note that the linear operator T is just rotation by 90 degrees about the origin. Thus, squaring it, i.e. doing it twice, gives rotation by 180 degrees, which sends each vector to its opposite, negative, vector.)

$\square$

Question 2. Let V, W be finite dimensional, and consider $T \in \mathcal{L}(V, W)$ and $S \in \mathcal{L}(W, U)$.

a) Prove that $\dim(\operatorname{range} ST) \leq \dim(\operatorname{range} T)$.
b) Prove that $\dim(\operatorname{range} ST) = \dim(\operatorname{range} T)$ if and only if

$$\operatorname{range} T + \operatorname{null} S = \operatorname{range} T \oplus \operatorname{null} S$$

c) Prove that $\dim(\operatorname{null} ST) \leq \dim(\operatorname{null} S) + \dim(\operatorname{null} T)$.
d) Bonus: give a description (in terms of conditions on T, S, V, etc) of when $\dim(\operatorname{null} ST) = \dim(\operatorname{null} S) + \dim(\operatorname{null} T)$.

[Proof by TC.] Before tackling these questions themselves we state and prove some lemmas, since we will use them in multiple parts. This will simplify our proofs.

First, we can restrict the domain of S to obtain a map just from $\operatorname{range} T$ to U,

$$S_{\operatorname{ran} T} \colon \operatorname{range} T \to U$$

where $S_{\operatorname{ran} T}$ is defined to be the map from $\operatorname{range} T$ to U defined by $S_{\operatorname{ran} T}(w) = S(w)$ for each $w \in \operatorname{range} T$. Note that since S is linear, $S_{\operatorname{ran} T}$ must be linear as well.

Lemma 1. *$\operatorname{range} S_{\operatorname{ran} T} = \operatorname{range} ST$.*

Proof. Suppose that $v \in \operatorname{range} ST$. Then there is some $v' \in V$ s.t. $v = STv'$. Since $Tv' \in \operatorname{range} T$, we have that $S(Tv') = S_{\operatorname{ran} T}(Tv')$. Thus $v \in \operatorname{range} ST$ implies $v \in \operatorname{range} S_{\operatorname{ran} T}$.

Suppose $v \in \operatorname{range} S_{\operatorname{ran} T}$. Then there is a $w \in \operatorname{range} T$ s.t. $v = S(w)$. Since $w \in \operatorname{range} T$, there is a $v' \in V$ s.t. $Tv' = w$. Thus $v = STv'$. So $v \in \operatorname{range} S_{\operatorname{ran} T}$ implies $v \in \operatorname{range} ST$. Thus $\operatorname{range} ST = \operatorname{range} S_{\operatorname{ran} T}$. □

Lemma 2. *$\operatorname{null} S_{\operatorname{ran} T} = \operatorname{range} T \cap \operatorname{null} S$.*

Proof.

$$\begin{aligned}
\operatorname{null} S_{\operatorname{ran} T} &= \left\{w \in \operatorname{range} T \middle| S_{\operatorname{ran} T}(w) = 0\right\} \\
&= \left\{w \in \operatorname{range} T \middle| S(w) = 0\right\} \\
&= \left\{w \in W \middle| w \in \operatorname{range} T \text{ and } S(w) = 0\right\} \\
&= \left\{w \in W \middle| w \in \operatorname{range} T\right\} \cap \left\{w \in W \middle| S(w) = 0\right\} \\
&= \operatorname{range} T \cap \operatorname{null} S
\end{aligned}$$

□

Lemma 3. *$\dim \operatorname{range} T = \dim(\operatorname{range} T \cap \operatorname{null} S) + \dim \operatorname{range} ST$.*

Proof. Note that since W is finite dimensional, $\operatorname{range} T$ is finite dimensional. Therefore the Fundamental Theorem of linear maps (Theorem 3.22) applied to $S_{\operatorname{ran} T}$ states:

$$\dim \operatorname{range} T = \dim \operatorname{null} S_{\operatorname{ran} T} + \dim \operatorname{range} S_{\operatorname{ran} T}$$

Lemma 1 tells us that $\operatorname{range} S_{\operatorname{ran} T} = \operatorname{range} ST$, and Lemma 2 tells us that $\operatorname{null} S_{\operatorname{ran} T} = \operatorname{range} T \cap \operatorname{null} S$. Substituting these, we obtain the desired equation. □

Lemma 4. *$\dim \operatorname{null} ST = \dim \operatorname{null} T + \dim(\operatorname{range} T \cap \operatorname{null} S)$*

Proof. By the Fundamental Theorem of linear maps applied to T we have

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T.$$

Using Lemma 3 to subsitute for $\dim \operatorname{range} T$, this becomes

$$\dim V = \dim \operatorname{null} T + \dim(\operatorname{range} T \cap \operatorname{null} S) + \dim \operatorname{range} ST \tag{$*$}$$

However, applying Fundamental Theorem of linear maps to ST gives that

$$\dim V = \dim \operatorname{null} ST + \dim \operatorname{range} ST \tag{$**$}$$

Subtracting $(**)$ from $(*)$ yields

$$0 = \dim \operatorname{null} T + \dim(\operatorname{range} T \cap \operatorname{null} S) - \dim \operatorname{null} ST$$

which becomes

$$\dim \operatorname{null} ST = \dim \operatorname{null} T + \dim(\operatorname{range} T \cap \operatorname{null} S)$$

as required. □

We now begin the proofs of Question 2(a)–(d).

a) We need to show that $\dim(\operatorname{range} ST) \leq \dim(\operatorname{range} T)$. By Lemma 3, we have

$$\dim \operatorname{range} ST = \dim \operatorname{range} T - \dim(\operatorname{range} T \cap \operatorname{null} S).$$

Since the dimension of $\operatorname{range} T \cap \operatorname{null} S$ must be ≥ 0, this implies $\dim \operatorname{range} T \geq \dim \operatorname{range} ST$ as required. [1] □

b) By Lemma 3, we have $\dim \operatorname{range} ST = \dim \operatorname{range} T - \dim(\operatorname{range} T \cap \operatorname{null} S)$. Therefore $\dim \operatorname{range} ST = \dim \operatorname{range} T$ if and only if $\dim(\operatorname{range} T \cap \operatorname{null} S) = 0$. Since the only 0-dimensional vector space is $\{0\}$, this holds if and only if $\operatorname{range} T \cap \operatorname{null} S = \{0\}$. But Proposition 1.45' says that for two subspaces U, W

$$U \cap W = \{0\} \quad \Longleftrightarrow \quad U + W = U \oplus W.$$

Therefore applying Proposition 1.45' we have

$$\begin{aligned} \dim \operatorname{range} ST = \dim \operatorname{range} T &\Longleftrightarrow \operatorname{range} T \cap \operatorname{null} S = \{0\} \\ &\Longleftrightarrow \operatorname{range} T + \operatorname{null} S = \operatorname{range} T \oplus \operatorname{null} S, \end{aligned}$$

as required. [2] □

c) By Lemma 4 we know that $\dim \operatorname{null} ST = \dim \operatorname{null} T + \dim(\operatorname{range} T \cap \operatorname{null} S)$. Since $\operatorname{range} T \cap \operatorname{null} S$ is a subspace of $\operatorname{null} S$, Prop. 2.15 states that $\dim(\operatorname{range} T \cap \operatorname{null} S) \leq \dim \operatorname{null} S$. Therefore

$$\dim(\operatorname{null} ST) = \dim \operatorname{null} T + \dim(\operatorname{range} T \cap \operatorname{null} S) \leq \dim(\operatorname{null} T) + \dim(\operatorname{null} S)$$

as required. □

d) I claim that $\dim(\operatorname{null} ST) = \dim(\operatorname{null} T) + \dim(\operatorname{null} S)$ if and only if **Null S ⊂Image T**.

By Lemma 4, $\dim(\operatorname{null} ST) = \dim \operatorname{null} T + \dim(\operatorname{range} T \cap \operatorname{null} S)$. This will be equal to $\dim(\operatorname{null} T) + \dim(\operatorname{null} S)$ if and only if $\dim(\operatorname{range} T \cap \operatorname{null} S) = \dim(\operatorname{null} S)$. But $\operatorname{range} T \cap \operatorname{null} S$ is a subspace of $\operatorname{null} S$, so by Exercise 2.C.1 on HW2 we know that

$$\dim(\operatorname{range} T \cap \operatorname{null} S) = \dim \operatorname{null} S \quad \Longleftrightarrow \quad \operatorname{range} T \cap \operatorname{null} S = \operatorname{null} S.$$

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[1] Alternate proof of 2a by TC, not using lemmas: By Fundamental Theorem of linear maps applied to ST, $\dim(\operatorname{range} ST) = \dim V - \dim(\operatorname{null} ST)$. By Fundamental Theorem of linear maps applied to T, $\dim(\operatorname{range} T) = \dim V - \dim(\operatorname{null} T)$. We saw in class that $\operatorname{null} T \subset \operatorname{null} ST$, so by Prop. 2.15, $\dim(\operatorname{null} T) \leq \dim(\operatorname{null} ST)$. Therefore $-\dim(\operatorname{null} ST) \leq -\dim(\operatorname{null} T)$ [negating reverses inequalities]. Adding $\dim V$ to both sides gives the desired inequality:

$$\dim(\operatorname{range} ST) = \dim V - \dim(\operatorname{null} ST) \leq \dim V - \dim(\operatorname{null} T) = \dim(\operatorname{range} T).$$

[2] Alternate proof of 2b by TC (sketch): By the Fundamental Theorem of linear maps in previous footnote,

$$\dim(\operatorname{range} ST) = \dim(\operatorname{range} T) \iff \operatorname{null} ST = \operatorname{null} T.$$

This means $ST(v) = 0 \iff T(v) = 0$. Therefore $T(v) \neq 0 \implies ST(v) \neq 0$ (contrapositive of forwards implication). This means that if $w \in \operatorname{range} T$ is nonzero, $S(w) \neq 0$; in other words, $\operatorname{range} T \cap \ker S = \{0\}$. By Prop. 1.45, this is the condition to have a direct sum:

$$\operatorname{range} T + \ker S = \operatorname{range} T \oplus \ker S$$

But this last condition holds if and only if $\operatorname{null} S \subset \operatorname{range} T$, as claimed. [This is a general fact about sets: for any sets X and Y it's true that $X \cap Y = Y \iff Y \subset X$. The proof is quite straightforward. -TC] $\square$