Math 113: Linear Algebra and Matrix Theory

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Homework 4

Due Wednesday, October 21 in class.

Do all the following exercises, but write up only the **unstarred exercises** and the **questions** below. (Starred exercises are valuable and worth working out, but they will not be collected or graded.)

3E.13*						
3F.7*	3F.8*	**3F.15**				
5A.12*	5A.15*	5A.18*	**5A.20**	**5A.22**	**5A.30**	5A.32*
5B.1*	**5B.2**					

Question 1. Suppose U is a subspace of V such that $\dim V/U = 1$.
Prove that there exists a linear functional $f \in V'$ such that

$$\operatorname{null} f = U. \qquad \text{(typo corrected here)}$$

(Note that V is not assumed to be finite-dimensional, so we cannot do this by choosing a basis.)

Question 2. Let $C^\infty(\mathbb{R})$ denote the vector space (over $\mathbb{R}$) of infinitely-differentiable[1] real-valued functions $f\colon \mathbb{R} \to \mathbb{R}$.

a) Let U denote the subspace of $C^\infty(\mathbb{R})$ consisting of functions which vanish at 42 and at π (you do not have to prove that U is a subspace):

$$U = \left\{ f \in C^\infty(\mathbb{R}) \,\middle|\, f(42) = 0,\ f(\pi) = 0 \right\}$$

Prove that the quotient vector space $C^\infty(\mathbb{R})/U$ is finite-dimensional. What is its dimension? (Note that $C^\infty(\mathbb{R})$ itself is very *infinite*-dimensional!)

b) Let W denote the subspace of $C^\infty(\mathbb{R})$ consisting of functions which "vanish to second order at 0":

$$W = \left\{ f \in C^\infty(\mathbb{R}) \,\middle|\, f(0) = 0,\ f'(0) = 0,\ f''(0) = 0 \right\}$$

Prove that the quotient vector space $C^\infty(\mathbb{R})/W$ is finite-dimensional, and find a basis for $C^\infty(\mathbb{R})/W$.

[1] A function $f\colon \mathbb{R} \to \mathbb{R}$ is "infinitely differentiable" if f is continuous, its derivative f' exists and is continuous, *its* derivative f'' exists and is continuous, and so on. Almost all the standard functions you know are infinitely differentiable: for example, all polynomials, exponentials, sin, cos, etc. You do not need to prove that $C^\infty(\mathbb{R})$ or $C^\infty(\mathbb{R}, \mathbb{C})$ is a vector space; in fact, you shouldn't have to worry about the details of $C^\infty(\mathbb{R})$ at all.

Question 3. Let $C^\infty(\mathbb{R},\mathbb{C})$ be the vector space (over $\mathbb{C}$) of complex-valued functions $f\colon \mathbb{R}\to\mathbb{C}$ that are infinitely differentiable. Let V be the subspace consisting of functions $f\in C^\infty(\mathbb{R},\mathbb{C})$ satisfying the differential equation $f''=-f$:

$$V=\left\{\, f\in C^\infty(\mathbb{R},\mathbb{C}) \,\middle|\, f''=-f \,\right\}$$

(You do not have to prove that V is a subspace of $C^\infty(\mathbb{R},\mathbb{C})$.)

If you take a course on differential equations, you'll learn how to prove that the space of solutions V is at most 2-dimensional, from the form of the differential equation $f''=-f$. However, since this is a linear algebra course, just trust me on this, and assume without proof that $\dim V\le 2$.

a) Prove that the functions $\sin x$ and $\cos x$ both lie in V, and moreover that $(\sin x,\cos x)$ forms a basis for V. [2]

b) Let D be the operator on $C^\infty(\mathbb{R},\mathbb{C})$ defined by $D(f)=f'$.
Prove that V is an invariant subspace for D.

c) Now consider $D\in\mathcal{L}(V)$ as an operator on V (still defined by $D(f)=f'$).
Find a basis for V consisting of eigenvectors for D. What are their eigenvalues?

[2] Remember your derivatives: $(\sin x)'=\cos x$, and $(\cos x)'=-\sin x$.