MATH 113 HOMEWORK 5 SOLUTIONS (STARRED PROBLEMS)

Solutions by Guanyang Wang, with edits by Tom Church.

Exercise 5.C.1 Suppose $T \in \mathcal{L}(V)$ is diagonalizable. Prove that $V = \operatorname{null} T \oplus \operatorname{range} T$.

Proof. Let $v_1, ..., v_n$ be a basis of V with respect to which T has a diagonal matrix. So for every $j \in 1, 2, ...n$, we have some $\lambda_j \in \mathbb{F}$ such that $Tv_j = \lambda_j v_j$. By renumbering, we can choose $m \in \{1, 2, ..., n\}$ such that

$$\lambda_j = 0 \text{ for } j = 1, 2...m$$

and

$$\lambda_j \neq 0 \text{ for } j = m+1, ...n$$

So we have $V = \operatorname{span}\{v_1, ..., v_m\} \oplus \operatorname{span}\{v_{m+1}, ...v_n\}$. Now we claim that $\operatorname{null} T = \operatorname{span}\{v_1, ..., v_m\}$ and $\operatorname{range} T = \operatorname{span}\{v_{m+1}, ...v_n\}$

First we prove $\operatorname{null} T = \operatorname{span}\{v_1, ..., v_m\}$. Notice that $\operatorname{span}\{v_1, ..., v_m\}$ is the eigenspace of T corresponding to 0. Hence every element $v \in \operatorname{span}\{v_1, ..., v_m\}$ satisfies $Tv = 0$, so $v \in \operatorname{null} T$. Meanwhile any $u \in \operatorname{null} T$, we have $Tu = 0$, therefore u is an eigenvector of T corresponding to 0. So $u \in \operatorname{span}\{v_1, ..., v_m\}$. Hence we have $\operatorname{null} T = \operatorname{span}\{v_1, ..., v_m\}$.

Then we prove $\operatorname{range} T = \operatorname{span}\{v_{m+1}, ...v_n\}$. For any $j \in m+1, ...n$, we have $T(\lambda_j^{-1} v_j) = v_j$, therefore $\operatorname{range} T \supset \operatorname{span}\{v_{m+1}, ...v_n\}$. Meanwhile for every $y \in \operatorname{range} T$, we have $y = Tx$ for some $x \in V$. Since $v_1, ..., v_n$ is a basis of V, we have $x = a_1 v_1 + ... + a_n v_n$ for some $a_1, \cdots, a_n \in \mathbb{F}$. Therefore

$$\begin{aligned} y = Tx &= T(a_1 v_1 + ... + a_n v_n) \\ &= \lambda_{m+1} a_{m+1} v_{m+1} + \cdots + \lambda_n a_n v_n \in \operatorname{span}\{v_{m+1}, ..., v_n\} \end{aligned}$$

Hence we have $\operatorname{range} T = \operatorname{span}\{v_{m+1}, ...v_n\}$. Therefore we have proved our claim, so we can conclude that $V = \operatorname{null} T \oplus \operatorname{range} T$. □

Exercise 5.C.2 Prove the converse of the statement in the exercise above or give a counterexample to the converse.

Proof. The converse of the statement in the exercise above is false. As an example, define $T \in \mathcal{L}(\mathbb{F}^2)$ by

$$T(w, z) = (w + z, z)$$

The eigenvector-eigenvalue equation $T(w, z) = \lambda(w, z)$ is equivalent to the system of equations

$$w + z = \lambda w \text{ and } z = \lambda z$$

After solving the equations, we have 1 is the only eigenvalue of T and that

$$E(1, T) = \{(w, 0) : w \in \mathbb{F}\}.$$

Since 1 is the only eigenvalue of T, 5.41 shows that T is not diagonalizable.

Because 0 is not an eigenvalue of T, we know that T is invertible. Thus $\operatorname{null} T = \{0\}$ and $\operatorname{range} T = \mathbb{F}^2$. Hence $\mathbb{F}^2 = \operatorname{null} T \oplus \operatorname{range} T$, providing a counterexample to the converse of the previous exercise.

□

Exercise 5.C.8. Suppose $T \in \mathbb{F}^5$ and $\dim E(8,T) = 4$. Prove that $T - 2I$ and $T - 6I$ is invertible.

Proof. From 5.38, we know that

$$\dim E(8,T) + \dim E(2,T) + \dim E(6,T) \leq \dim \mathbb{F}^5.$$

Since $\dim E(8,T) = 4$ and $\dim \mathbb{F}^5 = 5$, the inequality above can be written as

$$\dim E(2,T) + \dim E(6,T) \leq 1$$

Thus we have $\dim E(2,T) = 0$ or $\dim E(6,T) = 0$. In other words, 2 is not an eigenvalue of T or 6 is not an eigenvalue of T. Hence $T - 2I$ or $T - 6I$ is invertible. □

Exercise 5.C.14 Find $T \in \mathcal{L}(\mathbb{C}^3)$ such that 6 and 7 are eigenvalues of T and such that T does not have a diagonal matrix with respect to any basis of $\mathbb{C}^3$.

Proof. Define $T \in \mathcal{L}(\mathbb{C}^3)$ by

$$T(z_1, z_2, z_3) = (6z_1 + z_2, 6z_2, 7z_3)$$

The eigenvector-eigenvalue equation $T(z_1, z_2, z_3) = \lambda(z_1, z_2, z_3)$ is equivalent to the system of equations

$$\begin{aligned} 6z_1 + z_2 &= \lambda z_1 \\ 6z_2 &= \lambda z_2 \\ 7z_3 &= \lambda z_3 \end{aligned}$$

After solving the equations, we have 6 and 7 are the only eigenvalues of T and from our definition we have z_2 is the eigenvector of T corresponding to 6 and z_3 is the eigenvector of T corresponding to 7. We also have

$$E(6,T) = \operatorname{span}((1,0,0)) \text{ and } E(7,T) = \operatorname{span}((0,0,1))$$

Thus

$$\dim E(6,T) = \dim E(7,T) = 1.$$

Now 5.41 shows that T is not diagonalizable since $\dim E(6,T) + \dim E(7,T) = 2 < 3 = \dim(\mathbb{C}^3)$. □

Exercise 8.C.8 Suppose $T \in \mathcal{L}(V)$. Prove that T is invertible if and only if the constant term in the minimal polynomial of T is nonzero.

Proof. For any polynomial $f(x)$, the constant term is the value $f(0)$ at 0 (since if $f(x) = a_n x^n + \cdots + a_1 x + a_0$, then $f(0) = a_n 0^n + \cdots + a_1 0 + a_0 = a_0$). Therefore the constant term in the minimal polynomial $p(x)$ is nonzero if and only if $p(0)$ is nonzero; in other words, if and only if 0 is not a root of $p(x)$. Since the roots of the minimal polynomial are the eigenvalues of T, we conclude that:

$$\begin{aligned} \text{constant term of } p(x) \text{ is nonzero} &\iff 0 \text{ is not a root of } p(x) \\ &\iff 0 \text{ is not an eigenvalue of } T \\ &\iff (T - 0I) = \{0\} \\ &\iff T \text{ is injective} \\ &\iff T \text{ is invertible} \end{aligned}$$

□

Question 1.

- Give an example of an operator T on $V = \mathbb{C}^3$ whose minimal polynomial is $(x+2)^2$.
- Give an example of an operator S on $W = \mathbb{C}^4$ whose minimal polynomial is $(x^2+1)(x-3)^2$.
- What are the eigenvalues of the operators T and S in parts a) and b)?

Proof.

- Let $T\colon \mathbb{C}^3 \to \mathbb{C}^3$ be defined by

$$T(x,y,z) = (-2x, -2y, -2(y+z))$$

We claim that the minimal polynomial of T is $f(x) = (x+2)^2$. First we show that $f(T) = (T+2I)^2 = 0$. We have that

$$(T+2I)(x,y,z) = (0,0,-2y)$$

Thus, $(T+2I)^2(x,y,z) = (T+2I)(0,0,-2y)$. Since

$$(T+2I)(0,0,-2y) = (0,0,0)$$

we get that $(T-I)^2(x,y,z) = (0,0,0)$ for all $(x,y,z) \in \mathbb{C}^3$. That is, $f(T) = 0$.

We have found one polynomial $f(x) \in U_T$, which has degree 2 (recall from class that U_T is the set of all polynomials with $F(T) = 0$). It is obvious that T is not a multiple of the identity, so no degree-1 polynomial $x - \lambda$ is contained in U_T. Therefore $f(x)$ has the smallest possible degree in U_T, and so the minimal polynomial of T is $m_T(x) = (x+1)^2$.

- Give an example of an operator S on $W = \mathbb{C}^5$ whose minimal polynomial is $(x^2+1)(x-3)^2$.

Let $S\colon \mathbb{C}^5 \to \mathbb{C}^5$ be defined by

$$S(x,y,z,w,t) = (ix, -iy, 3z, 3w, 3t+w)$$

We begin by noting that i $-i$ and 3 are eigenvalues of S:

$$S(1,0,0,0,0) = (i,0,0,0,0) = i \cdot (1,0,0,0,0)$$
$$S(0,1,0,0,0) = (0,-i,0,0,0) = -i \cdot (0,1,0,0,0)$$
$$S(0,0,1,0,0) = (0,0,3,0,0) = 3 \cdot (0,0,1,0,0)$$

So we compute

$$(S-3I)(x,y,z,w,t) = ((i-3)x, (-i-3)y, 0, 0, w)$$

and therefore.

$$(S-3I)^2(x,y,z,w,t) = ((i-3)^2x, (-i-3)^2y, 0, 0, 0)$$

Meanwhile,

$$(S^2+I)(x,y,z,w,t) = (0,0,10z,10w,10t+6w)$$

Thus applying (S^2+I) to the result of $(S-3I)^2(x,y,z,w,t)$ we get

$$(S^2+I)(S-3I)^2(x,y,z,w,t) = (0,0,0,0,0).$$

This shows that $(S^2+I)(S-3I)^2 = 0$, so if $f(x) = (x^2+1)(x-3)^2$ then $f(x) \in U_S$. However, since $-i$, i and 3 are eigenvalues of S, we know that $-i$,i and 3 are roots of the minimal polynomial. Therefore the only smaller possibility for the minimal polynomial is $(x-i)(x+i)(x-3) = (x^2+1)(x-3)$, since this is the only polynomial of degree < 4 with both $-i$, i and 3 as

roots. So we just need to show that $(S^2+I)(S-3I) \neq 0$, and this we can do by direct computation. Indeed, applying S^2+I to the result of $S-3I$ that we found above, we get

$$(S^2+I)(S-3I)(x,y,z,w,t) = (0,0,0,0,10w)$$

which is non-zero. Thus the minimal polynomial of S is indeed

$$m_S(x) = (x^2+1)(x-3)^2.$$

- Since all the eigenvalues are precisely all the zeros of the minimal polynomial (8.49), we just need to compute the zeros of the minimal polynomial. In the first case, the minimal polynomial $(x+2)^2$, so -2 is the only eigenvalue of operator T. In the second case, the minimal polynomial of S is $(x^2+1)(x-3)^3$, the roots of this polynomial are $-i, i$ and 3. Therefore $-i, i$ and 3 are all the eigenvalues of S.

□

Question 2. Let $V = \mathbb{R}^4$, and let $T \in \mathcal{L}(V)$ be the operator with matrix

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Find the minimal polynomial of T.

Proof. We claim that $f(x) = (x-2)(x-3)^2$ is the minimal polynomial. First we show that $(T-2I)(T-3I)^2 = 0$

$$\begin{aligned}(T-2I)(T-3I)^2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\end{aligned}$$

Therefore we have $(T-2I)(T-3I)^2 = 0$, so $f(x) = (x-2)(x-3)^2 \in U_T$.

However, since 2, and 3 are eigenvalues of T, we know that 2 and 3 are roots of the minimal polynomial. Therefore the only smaller possibility for the minimal polynomial is $(x-2)(x-3)$, since this is the only polynomial of degree < 3 with both 2 and 3 as roots. So we just need to show that $(T-2I)(T-3I) \neq 0$, and this

we can do by direct computation.

$$(T-2I)(T-3I) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which is not 0. Thus the minimal polynomial of T is indeed

$$m_T(x) = (x-2)(x-3)^2.$$

□