Math 113: Linear Algebra and Matrix Theory

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Homework 6

Due Wednesday, November 4 in class.

Do all the following exercises.

6A.16 over $\mathbb{R}$

6B.5 **6B.7** **6B.8**

6A.16: you can assume that V is a vector space over $\mathbb{R}$.

6B.5: uses the Gram-Schmidt algorithm, which we may not cover until Monday. If you want to start early, it's on p183.

6B.7 and 6B.8: you may find these much easier after solving 6B.5.

Question 1. Let V be a finite-dimensional vector space over $\mathbb{C}$, and let $T \in \mathcal{L}(V)$. Let U and W be nonzero subspaces such that $V = U \oplus W$.

Assume that U and W are invariant under T, so we can *restrict* the operator $T: V \to V$ to an operator $T|_U : U \to U$, and similarly we can restrict T to an operator $T|_W : W \to W$.

a) Prove without using minimal polynomials that if $\lambda \in \mathbb{C}$ is an eigenvalue of T, then either λ is an eigenvalue of $T|_U$ or λ is an eigenvalue of $T|_W$ (or both).

   [Hint: start with a nonzero eigenvector $v \in V$ such that $T(v) = \lambda v$, and somehow construct either an eigenvector $u \in U$ such that $T(u) = \lambda u$, or an eigenvector $w \in W$ such that $T(w) = \lambda w$.]

Let $f(x)$ be the minimal polynomial of $T|_U$, and let $g(x)$ be the minimal polynomial of $T|_W$.

b) Prove that $f(T)g(T) = 0$ in $\mathcal{L}(V)$.

c) Prove that if $f(x)$ and $g(x)$ have no shared roots (meaning no $\lambda \in \mathbb{C}$ is a root of both $f(x)$ and $g(x)$), then $f(x)g(x)$ is the minimal polynomial of T.

d) Prove that if $f(x)$ and $g(x)$ have a shared root $\lambda \in \mathbb{C}$, then $f(x)g(x)$ is **not** the minimal polynomial of T.

Question 2. Let V be an inner product space over $\mathbb{R}$, and suppose that $T \in \mathcal{L}(V)$ satisfies $\|Tv\| = \|v\|$ for all $v \in V$. Prove that T has at most two eigenvalues.

(Question 3 provides an example showing that this does not hold for operators on inner product spaces over $\mathbb{C}$.)

Question 3. Fix an integer $n \geq 1$, and let $V = \mathbb{C}^n$ with the standard inner product. We let $R: V \to V$ be the operator defined by

$$R(a_1, \ldots, a_n) = (a_2, \ldots, a_n, a_1).$$

(a) Set $p(x) = x^n - 1$. Prove that $p(R) = 0$.

(b*) Convince yourself that $p(x) = x^n - 1$ is in fact the minimal polynomial of R. (Hint: choose a small n, write the matrices for $I, R, R^2, \ldots, R^{n-1}$ and see that they are linearly independent.) (You do not have to turn anything in for this part.)

This means that the eigenvalues of R are the roots of $x^n - 1$; since you might not be familiar with these awesome numbers (called "roots of unity"), here are the relevant facts.

Let $\omega \in \mathbb{C}$ be the complex number $\omega = \cos(\frac{2\pi}{n}) + i\sin(\frac{2\pi}{n})$. Then $x^n - 1$ factors as

$$(x-1)(x-\omega)(x-\omega^2)\cdots(x-\omega^{n-1})$$

All the roots $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are on the unit circle in $\mathbb{C}$ (meaning $z\overline{z} = 1$), and in fact they are equally spaced around the unit circle until you get back to $\omega^n = 1$.

(c) Since $p(x)$ has n distinct roots, we know that R is diagonalizable.

Diagonalize R by finding a basis of eigenvectors $v_1, \ldots, v_n$ for $\mathbb{C}^n$ satisfying

$$R(v_k) = \omega^k \cdot v_k \quad \text{and} \quad ||v_k|| = 1.$$

(d) Prove that if $\mu \in \mathbb{C}$ satisfies $\mu^n = 1$ but $\mu \neq 1$, then $1 + \mu + \mu^2 + \cdots + \mu^{n-1} = 0$.

[Hint: multiply by $\mu - 1$.]

(e) Prove that your basis $v_1, \ldots, v_n$ is orthonormal.

(f) If $v = (a_1, \ldots, a_n)$ is written as $v = b_1 v_1 + \cdots + b_n v_n$, give a formula for the coefficient b_i in terms of the coordinates $a_1, \ldots, a_n$. [Hint: use part (e).]

(g) If $v = (a_1, \ldots, a_n)$ is written as $v = b_1 v_1 + \cdots + b_n v_n$, give a formula for the coordinate a_i in terms of the coefficients $b_1, \ldots, b_n$. [Hint: this is easy.]

(h) If $v = (a_1, \ldots, a_n)$ is written as $v = b_1 v_1 + \cdots + b_n v_n$, prove that the coordinates $a_1, \ldots, a_n$ and the coefficients $b_1, \ldots, b_n$ satisfy the relation

$$|a_1|^2 + \cdots + |a_n|^2 = |b_1|^2 + \cdots + |b_n|^2.$$

Remark: The formula you found in (f) is the Fourier transform, or rather a discretized version of it; the formula you found in (g) is the inverse Fourier transform.

The equality you proved in (h) is a discrete version of the famous Plancherel theorem (also known as Rayleigh's energy theorem): If $f\colon [-\pi, \pi] \to \mathbb{C}$ is a continuous function with $f(-\pi) = f(\pi)$, we saw in class that

$$\text{energy}(f)^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx.$$

Let the Fourier coefficients $b_k \in \mathbb{C}$ be the sequence defined for $k \in \mathbb{Z}$ by

$$b_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx.$$

Then $\text{energy}(f)^2$ can be computed by either side of:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx \quad = \quad \sum_{k=-\infty}^{\infty} |b_k|^2.$$