Math 113 Homework 6 Solutions

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Exercises from the book.

Exercise 6.C.4 Suppose U is the subspace of $\mathbb{R}^4$ defined by

$$U = \operatorname{span}((1,2,3,-4),(-5,4,3,2))$$

Find an orthonormal basis of U and an orthonormal basis of $U^\perp$

Answer. Notice that the list $(1,2,3,-4)$ and $(-5,4,3,2)$ is linearly independent since neither vector is the scalar multiple of the other. Thus we will extend the list $(1,2,3,-4),(-5,4,3,2)$ to a basis

$$(1,2,3,-4),(-5,4,3,2),w_1,w_2$$

of $\mathbb{R}^4$ and then apply the Gram-Schmidt Procedure.

To extend $(1,2,3,-4),(-5,4,3,2)$ to a basis of $\mathbb{R}^4$, we follow the idea of the proof of 2.33. Thus we start with the list

$$(1,2,3,-4),(-5,4,3,2),(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)$$

which spans $\mathbb{R}^4$. We need to apply the Gram-Schmidt Procedure anyway, and thus in this case the easiest thing to do is to start the Gram-Schmidt Procedure and throw out any vectors that would lead to division by 0(indicating linear independence), or stop when we reach a list of length four.

To get started, we have

$$e_1 = \frac{(1,2,3,-4)}{\|(1,2,3,-4)\|} = \left(\frac{1}{\sqrt{30}}, \sqrt{\frac{2}{15}}, \sqrt{\frac{3}{10}}, -2\sqrt{\frac{2}{15}}\right).$$

Next,

$$\begin{aligned} e_2 &= \frac{(-5,4,3,2) - \langle(-5,4,3,2),e_1\rangle e_1}{\|(-5,4,3,2) - \langle(-5,4,3,2),e_1\rangle e_1\|} \\ &= \left(-\frac{77}{\sqrt{12030}}, 28\sqrt{\frac{2}{6015}}, 13\sqrt{\frac{3}{4010}}, 19\sqrt{\frac{2}{6015}}\right). \end{aligned}$$

Next,

$$\begin{aligned} e_3 &= \frac{(1,0,0,0) - \langle(1,0,0,0),e_1\rangle e_1 - \langle(1,0,0,0),e_2\rangle e_2}{\|(1,0,0,0) - \langle(1,0,0,0),e_1\rangle e_1 - \langle(1,0,0,0),e_2\rangle e_2\|} \\ &= \left(\sqrt{\frac{190}{401}}, \frac{117}{\sqrt{76190}}, 6\sqrt{\frac{10}{7619}}, \frac{151}{\sqrt{76190}}\right). \end{aligned}$$

There is no division by 0 here, and no linear dependence yet.

Next,

$$\begin{aligned} e_4 &= \frac{(0,1,0,0) - \langle(0,1,0,0),e_1\rangle e_1 - \langle(0,1,0,0),e_2\rangle e_2 - \langle(0,1,0,0),e_3\rangle e_3}{\|(0,1,0,0) - \langle(0,1,0,0),e_1\rangle e_1 - \langle(0,1,0,0),e_2\rangle e_2 - \langle(0,1,0,0),e_3\rangle e_3\|} \\ &= \left(0, \frac{9}{\sqrt{190}}, -\sqrt{\frac{10}{19}}, -\frac{3}{\sqrt{190}}\right). \end{aligned}$$

Again there is no division by 0 here, and thus no linear dependence yet.

Since $\mathbb{R}^4$ has dimension 4, we know that e_1,e_2,e_3,e_4 is a basis of $\mathbb{R}^4$, and there is no need to continue the process further.

Thus, by the previous exercise,

$$\left(\frac{1}{\sqrt{30}}, \sqrt{\frac{2}{15}}, \sqrt{\frac{3}{10}}, -2\sqrt{\frac{2}{15}}\right), \left(-\frac{77}{\sqrt{12030}}, 28\sqrt{\frac{2}{6015}}, 13\sqrt{\frac{3}{4010}}, 19\sqrt{\frac{2}{6015}}\right)$$

is an orthonormal basis of U and

$$\left(\sqrt{\frac{190}{401}}, \frac{117}{\sqrt{76190}}, 6\sqrt{\frac{10}{7619}}, \frac{151}{\sqrt{76190}}\right), \left(0, \frac{9}{\sqrt{190}}, -\sqrt{\frac{10}{19}}, -\frac{3}{\sqrt{190}}\right)$$

is an orthonormal basis of $U^{\perp}$ □

Exercise 6.C.6 Suppose U and W are finite-dimensional subspaces of V. Prove that $P_U P_W = 0$ if and only if $\langle u, w\rangle = 0$ for all $u \in U$ and all $w \in W$.

Proof. First suppose $P_U P_W = 0$. Suppose $w \in W$. Then

$$\begin{aligned} 0 &= P_U P_W w \\ &= P_U w \end{aligned}$$

Hence $w \in \text{null} P_U$. Now 6.55(e) shows that $w \in U^{\perp}$. Thus $\langle u, w\rangle = 0$ for all $u \in U$, completing one direction of the proof.

To prove the other direction, now suppose that $\langle u, w\rangle = 0$ for all $u \in U$ and all $w \in W$. Thus $U \subset W^{\perp}$ and $W \subset U^{\perp}$. If $w \in W$, then

$$(P_U P_W)(w) = P_U(P_W w) = P_U w = 0$$

where the last equality holds because $w \in U^{\perp}$. If $v \in W^{\perp}$, then

$$(P_U P_W)(v) = P_U(P_W v) = P_U 0 = 0.$$

Since every element in V can be written as the sum of a vector in W and a vector in $W^{\perp}$ (by 6.47), the last two equations imply that $P_U P_W = 0$, as desired. □

Exercise 6.C.11 In $\mathbb{R}^4$, let

$$U = \text{span}\big((1,1,0,0),(1,1,1,2)\big)$$

Find $u \in U$ s.t. $||u - (1,2,3,4)||$ is as small as possible.

Proof. First, we find an orthogonal basis for U. (So we won't bother to make the vectors have norm 1.) We keep $u_1 = (1,1,0,0)$. Then we subtract off u_1 from $v_2 = (1,1,1,2)$. We have that $v_2 - u_1 = (0,0,1,2)$ is perpendicular to u_1. So we set $u_2 = (0,0,1,2)$. Now u_1, u_2 form a basis for U. Using this basis, we see that elements of U are vectors of the form $(x,x,y,2y)$ for $x, y \in \mathbb{R}$.

So we want to find x and y s.t. the vector $(x,x,y,2y) - (1,2,3,4)$ has the least norm. Noting that $(x,x,y,2y) - (1,2,3,4) = (x-1, x-2, y-3, 2y-4)$, we compute

$$\begin{aligned} ||(x-1, x-2, y-3, 2y-4)||^2 &= (x-1)^2 + (x-2)^2 + (y-3)^2 + (2y-4)^2 \\ &= 2x^2 - 6x + 5 + 5y^2 - 22y + 16 \\ &= 2x^2 - 6x + 5y^2 - 22y + 21 \end{aligned}$$

This is minimized when $p(x) = 2x^2 - 6x$ and $q(y) = 5y^2 - 22y$ are both minimized. As their leading coefficients are positive, both of these quadratics go to infinity as x

and y go to infinity, respectively. Thus their local critical points are their respective minima. Taking derivatives, we get that

$$p'(x) = 4x - 6 \text{ and } q'(y) = 10y - 22$$

So their minima are at $x = \frac{3}{2}$ and $y = \frac{11}{5}$, respectively. Therefore the vector $u \in U$ s.t. $||u - (1,2,3,4)||$ is smallest is $u = (\frac{3}{2}, \frac{3}{2}, \frac{11}{5}, \frac{22}{5})$.

Here is another way to do this problem:

The vector $v_3 = (0,1,1,0)$ is not in U because it is not of the correct form. Note that $(1,2,3,4) = u_1 + 2u_2 + v_3$ so $(1,2,3,4)$ is in the span of u_1, u_2 and v_3. We want to find a vector u_3 in the span of u_1, u_2 and v_3 s.t. u_3 is orthogonal to u_1 and u_2. We have that

$$v_3 \cdot u_1 = 1 \text{ and } v_3 \cdot u_2 = 1$$

We also have

$$u_1 \cdot u_1 = 2 \text{ and } u_2 \cdot u_2 = 5$$

Thus $u_3 = v_3 - \frac{1}{2}u_1 - \frac{1}{5}u_2$ is orthogonal to u_1 and u_2. We can see this directly by writing $u_3 = (-\frac{1}{2}, \frac{1}{2}, \frac{4}{5}, -\frac{2}{5})$. Since $(1,2,3,4) = u_1 + 2u_2 + v_3$ and $v_3 = u_3 + \frac{1}{2}u_1 + \frac{1}{5}u_2$, we get that

$$\begin{aligned}(1,2,3,4) &= \frac{3}{2}u_1 + \frac{11}{5}u_2 + u_3 \\ &= \frac{3}{2}(1,1,0,0) + \frac{11}{5}(0,0,1,2) + (-\frac{1}{2}, \frac{1}{2}, \frac{4}{5}, -\frac{2}{5})\end{aligned}$$

Now suppose $u \in U$ is the vector s.t. $||u - (1,2,3,4)||$ is minimal. Since $u \in U$, we can write $u = a_1u_1 + a_2u_2$ for some $a_1, a_2 \in \mathbb{R}$. Thus,

$$\begin{aligned}||u - (1,2,3,4)|| &= ||a_1u_1 + a_2u_2 - \frac{3}{2}u_1 + \frac{11}{5}u_2 + u_3|| \\ &= ||(a_1 - \frac{3}{2})u_1 + (a_2 - \frac{11}{5})u_2 - u_3|| \\ &= (a_1 - \frac{3}{2})^2||u_1||^2 + (a_2 - \frac{11}{5})^2||u_2||^2 + ||u_3||^2\end{aligned}$$

because u_1, u_2, u_3 are orthogonal. This quantity is minimized when $a_1 = \frac{3}{2}$ and $a_2 = \frac{11}{5}$. Thus the $u \in U$ that is closest to $(1,2,3,4)$ is $\frac{3}{2}u_1 + \frac{11}{5}u_2 = (\frac{3}{2}, \frac{3}{2}, \frac{11}{5}, \frac{22}{5})$. □

Exercise 7.A.1. Suppose n is a positive integer. Define $T \in \mathcal{L}(\mathbb{F}^n)$ by

$$T(z_1, ..., z_n) = (0, z_1, ..., z_{n-1})$$

Find a formula for $T^*(z_1, ..., z_n)$

Proof. Fix $(z_1, ..., z_n) \in \mathbb{F}^n$. Then for every $(w_1, ..., w_n) \in \mathbb{F}^n$, we have

$$\begin{aligned}\langle (w_1, ..., w_n), T^*(z_1, ..., z_n)\rangle &= \langle T(w_1, ..., w_n), (z_1, ..., z_n)\rangle \\ &= \langle (0, w_1, ..., w_{n-1}), (z_1, ..., z_n)\rangle \\ &= w_1\overline{z_2} + ... + w_{n-1}\overline{z_n} \\ &= \langle (w_1, ..., w_n), (z_2, ..., z_n, 0)\rangle.\end{aligned}$$

Thus

$$T^*(z_1, ..., z_n) = (z_2, ..., z_n, 0).$$

□

Exercise 7.A.2 Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. Prove that λ is an eigenvalue of T iff $\bar{\lambda}$ is an eigenvalue of T^*.

Proof. Suppose λ is an eigenvalue of T. Then there is some $v \neq 0$ s.t. $Tv = \lambda v$. Thus,

$$\langle Tv, w\rangle = \langle \lambda v, w\rangle$$

for each $w \in V$. Note that $\langle \lambda v, w\rangle = \lambda\langle v, w\rangle = \langle v, \bar{\lambda} w\rangle$ by linearity of the inner product. If T^* is the adjoint of T, then $\langle T(v), w\rangle = \langle v, T^* w\rangle$, so we have

$$\langle v, \bar{\lambda} w\rangle = \langle v, T^* w\rangle$$

for each w in W. By linearity of the inner product, this means that

$$\langle v, T^* w - \bar{\lambda} w\rangle = 0$$

for each $w \in W$. Thus, v is perpendicular to all vectors of the form $T^* w - \bar{\lambda} w$.

Let $S = T^* - \bar{\lambda} I$. The image of S is all vectors of the form $T^* w - \bar{\lambda} w$ so v is perpendicular to all vectors in the image of S. However, since v is nonzero, it cannot be perpendicular to itself ($\langle v, v\rangle > 0$ is an axiom of inner products), so $v \notin \text{Image}S$. This shows that $\text{Image}S \neq V$, so $\dim \text{Image}S$ must be strictly less than the dimension of the image of V. By Rank-Nullity, this implies that $\dim \text{Null}S > 0$. Therefore, there is some non-zero element in the null space of S. So there is some w s.t. $T^* w = \bar{\lambda} w$, meaning $\bar{\lambda}$ is an eigenvalue of T^*.

Since $(T^*)^* = T$, this also shows that any eigenvalue of T^* is also the conjugate of an eigenvalue of T. Therefore λ is an eigenvalue of T iff $\bar{\lambda}$ is an eigenvalue of T^*. □

Exercise 7.A.4 Suppose $T \in \mathcal{L}(V, W)$. Prove that

(a) T is injective if and only if T^* is surjective;
(b) T is surjective if and only if T^* is injective.

Proof. First we prove (a)

$$\begin{aligned} T \text{ is injective} &\iff \text{Null T} = 0 \\ &\iff (\text{Range}T^*)^\perp = 0 \\ &\iff \text{Range}T^* = W \\ &\iff T \text{ is surjective} \end{aligned}$$

Where the second line comes from $7.7(c)$.

Note that (a) has been proved, (b) follows immediately by replacing T with T^* in (a). □

Question 1. Suppose $(e_1, \dots, e_m)$ is an orthonormal list of vectors in V. Let $v \in V$. Prove that

$$||v||^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2$$

if and only if $v \in \text{span}(e_1, \dots, e_m)$.

Proof. Denote span$(e_1, \dots, e_m)$ by U. Then we can write every vector v in V as $u + w$ with $u \in U$ and $w \in U^{\perp}$. So we have

$$||v||^2 = \langle u + w, u + w\rangle = \langle u, u\rangle + \langle w, w\rangle = ||u||^2 + ||w||^2$$

The second equality holds since $(u, w) = (w, u) = 0$.

Then, since $(e_1, \dots, e_m)$ is an orthonormal basis of U. We can find $a_1, ..., a_m$ such that $u = a_1e_1 + \cdots + a_me_m$. Therefore we have

$$\begin{aligned} ||u||^2 &= \langle a_1e_1 + \cdots + a_me_m, a_1e_1 + \cdots + a_me_m\rangle \\ &= \sum_{i,j=1}^{m} \langle a_ie_i, a_je_j\rangle \\ &= \sum_{i,j=1}^{m} a_i\overline{a_j}\langle e_i, e_j\rangle \end{aligned}$$

Since e_i and e_j are orthogonal if $i \neq j$, and since $\langle e_i, e_i\rangle = 1$, we get

$$||u||^2 = |a_1|^2 + \cdots + |a_m|^2$$

On the other hand,

$$\begin{aligned} \langle v, e_i\rangle &= \langle u + w, e_i\rangle \\ &= \langle a_1e_1 + \cdots + a_me_m + w, e_i\rangle \\ &= a_i \end{aligned}$$

Since e_i and w are orthogonal for every $i \in \{1, 2..., m\}$, and since e_i and e_j are orthogonal if $i \neq j$.

So, $|\langle v, e_i\rangle|^2 = |a_i|^2$ meaning that

$$\begin{aligned} ||v||^2 &= ||u||^2 + ||w||^2 \\ &= |a_1|^2 + \cdots + |a_m|^2 + ||w||^2 \\ &= |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2 + ||w||^2 \end{aligned}$$

If $v \in$ span$(e_1, \dots, e_m)$, then $v = a_1e_1 + \cdots + a_me_m$. That is, $w = 0$. Thus $||v||^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2$.

If $||v||^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2$, then we have $||w||^2 = 0$, therefore $w = 0$, so we have $v = u + 0 = u \in U$. By definition, $U =$ span$(e_1, \dots, e_m)$, so $v \in$ span$(e_1, \dots, e_m)$.

□

Question 2. Let V be the vector space of infinite sequences of real numbers:

$$V = \{(a_1, a_2, \dots,) \mid a_i \in \mathbb{R}\}$$

This is an infinite dimensional vector space over $\mathbb{R}$. Let $T \in \mathcal{L}(V)$ be the forward shift defined by

$$T(a_1, a_2, \dots) = (0, a_1, a_2, \dots)$$

a) The operator $T+I$ is given by

$$(T+I)(a_1,a_2,a_3,\dots) = (a_1, a_1+a_2, a_2+a_3,\dots)$$

Find an inverse $(T+I)^{-1}$ for this operator.

b) For which values $\lambda \in \mathbb{R}$ is the operator $T-\lambda I$ non-invertible?

c) What are the eigenvalues of T?

d) Explain the discrepancy between your answers to 2 and 3.

Proof. a) The operator $T+I$ is given by

$$(T+I)(a_1,a_2,a_3,\dots) = (a_1, a_1+a_2, a_2+a_3,\dots)$$

Let

$$S(a_1,a_2,a_3,\dots) = (a_1, a_2-a_1, a_3-a_2+a_1, a_4-a_3+a_2-a_1,\dots)$$

This is the inverse of $T+I$. To see this, we compute $S(T+I)(a_1,a_2,\dots)$:

$$S(a_1,a_1+a_2,a_2+a_3,\dots) = (a_1,(a_1+a_2)-a_1,(a_2+a_3)-(a_1+a_2)+a_1,\dots)$$

Thus $S(T+I)v = v$ for all $v \in V$. We also need to compute $(T+I)S(a_1,a_2,\dots)$:

$$(T+I)(a_1,a_2-a_1,a_3-a_2+a_1,\dots) = (a_1,(a_2-a_1)+a_1,(a_3-a_2+a_1)+(a_2-a_1),\dots)$$

Thus $(T+I)Sv = v$ for all $v \in V$. Since $S(T+I) = (T+I)S = I$, we get that $S = (T+I)^{-1}$. □

b) The operator $T-\lambda I$ is given by

$$(T_\lambda I)(a_1,a_2,a_3,\dots) = (-\lambda a_1, a_1-\lambda a_2, a_2-\lambda a_3,\dots)$$

Let

$$S_\lambda = (-\frac{1}{\lambda}a_1, -\frac{1}{\lambda^2}a_1 - \frac{1}{\lambda}a_2, -\frac{1}{\lambda^3}a_1 - \frac{1}{\lambda^2}a_2 - \frac{1}{\lambda}a_3,\dots)$$

For $\lambda \neq 0$, we will show that $S_\lambda = (T-\lambda I)^{-1}$. Indeed, if we write $S_\lambda(a_1,a_2,\dots) = (b_1,b_2,\dots)$ then the n^{th} term of S_λ is $b_n = -\frac{1}{\lambda^n}a_1 - \frac{1}{\lambda^{n-1}}a_2 - \dots - \frac{1}{\lambda}a_n$, which is in fact $\frac{1}{\lambda}(b_{n-1}-a_n)$. We can see $b_1, b_2,\dots$ all as functions from V to $\mathbb{R}$.

We apply S_λ to $(T-\lambda I)(a_1,a_2,\dots)$. We have that $b_1(a_1,a_2,\dots)$ is $-\frac{1}{\lambda}(-\lambda a_1) = a_1$. The n^{th} term of $(T-\lambda I)(a_1,a_2,\dots)$ is $a_{n-1}-\lambda a_n$. Suppose $b_{n-1}(T-\lambda I)(a_1,a_2,\dots) = a_{n-1}$. Then

$$b_n(T-\lambda I)(a_1,a_2,\dots) = \frac{1}{\lambda}(b_{n-1}(T-\lambda I)(a_1,a_2,\dots) - (a_{n-1}-\lambda a_n))$$

(*because* $b_n(a_1,a_2,\dots) = b_{n-1}-a_n$)

$$= \frac{1}{\lambda}(a_{n-1} - (a_{n-1}-\lambda a_n))$$

(*since by assumption,* $b_{n-1}(T-\lambda I)(a_1,\dots) = a_n$)

$$= a_n$$

So by induction, $S_\lambda(T-\lambda I) = I$. This can be seen by direct computation for the first few terms:

$$S_\lambda(-\lambda a_1, a_1-\lambda a_2, a_2-\lambda a_3,\dots) =$$

$$(a_1, -\frac{1}{\lambda^2}(-\lambda a_1) - \frac{1}{\lambda}(a_1-\lambda a_2), -\frac{1}{\lambda^3}(-\lambda a_1) - \frac{1}{\lambda^2}(a_1-\lambda a_2) - \frac{1}{\lambda}(a_2-\lambda a_3),\dots)$$

$$= (a_1, a_2, a_3,\dots)$$

Next, we must show that $(T-\lambda I)S_\lambda(a_1,a_2,\dots)=(a_1,a_2,\dots)$. Once again, we use that the n^{th} term of $S_\lambda(a_1,a_2,\dots)$ is $b_n=\frac{1}{\lambda}(b_{n-1}-a_n)$, and that the n^{th} term of $(T-\lambda I)(a_1,a_2,\dots)$ is $a_{n-1}-\lambda a_n$. Thus,

$$\begin{aligned}(T-\lambda I)(S_\lambda(a_1,a_2,\dots)) &= (T-\lambda I)(b_1,b_2,\dots)\\ &= (-\lambda b_1, b_1-\lambda b_2,\dots,b_{n-1}-\lambda b_n,\dots)\\ &= (-\lambda(-\frac{1}{\lambda}a_1), b_1-\lambda(\frac{1}{\lambda}(b_1-a_n)),\dots,b_{n-1}-\lambda(\frac{1}{\lambda}(b_{n-1}-a_n)),\dots)\\ &= (a_1,a_2,\dots,a_n,\dots)\end{aligned}$$

Therefore $(T-\lambda I)S_\lambda=I$. Since $(T-\lambda I)S_\lambda=S_\lambda(T-\lambda I)=I$, $S_\lambda=(T-\lambda I)^{-1}$ for all $\lambda\neq 0$.

Thus $T-\lambda I$ is invertible for all $\lambda\neq 0$. However, for $\lambda=0$ we have $T-\lambda I=T-0I=T$, and T is not invertible. Indeed, the image of T is clearly contained in the subspace $\{(0,*,*,*,\dots)\}$ of sequences whose first entry is 0, so T is not surjective. Since T is not surjective, it cannot be bijective, so it cannot have an inverse even as a map of sets.

(However, note that if we let S be the backwards shift:

$$S(a_1,a_2,a_3,\dots)=(a_2,a_3,\dots)$$

Then applying S_0 to T, we get

$$\begin{aligned}S(T(a_1,a_2,\dots)) &= S(0,a_1,a_2,\dots)\\ &= (a_1,a_2,\dots)\end{aligned}$$

So $ST=I$, which might lead us to think that T is invertible.

However,

$$\begin{aligned}T(S(a_1,a_2,a_3,\dots)) &= T(a_2,a_3,\dots)\\ &= (0,a_2,a_3,\dots)\end{aligned}$$

So $TS\neq I$, and so we see that S is not an inverse for T.

So $T-\lambda I$ is not invertible only for $\lambda=0$. □

c) Suppose λ is an eigenvalue of T. Then $T(a_1,a_2,\dots)=\lambda(a_1,a_2,\dots)$ meaning that

$$(0,a_1,a_2,\dots)=(\lambda a_1,\lambda a_2,\dots)$$

This gives us that $\lambda a_1=0$, so either $\lambda=0$ or $a_1=0$. This equation also gives us $\lambda a_n=a_{n-1}$ for $n\geq 2$. If $\lambda=0$, then $a_1,a_2,\dots$ all equal zero. Thus λ is not an eigenvalue. If $a_1=0$ but $\lambda\neq 0$ then $\lambda a_2=a_1$ implies that $a_2=0$, and so on. So if $\lambda\neq 0$ then we also get that $a_1=a_2=\cdots=0$. Therefore T has no eigenvalues. □

d) The discrepancy is that $T-\lambda I$ is not invertible when $\lambda=0$, but 0 is not an eigenvalue of T. In the finite-dimensional case, when an operator is not invertible, it is also not injective by Rank-Nullity. If $T-\lambda I$ were not injective that would mean that $(T-\lambda I)\neq\{0\}$, so λ would be an eigenvalue. However, V is infinite-dimensional. In the infinite-dimensional case, an operator can be not invertible, and still be injective because Rank-Nullity no longer holds (nor does it make sense.) T is an example of such an operator that is injective but not invertible. That is why we have that T is not invertible, but 0 is not an eigenvalue of T. □