Math 113 Homework 8 Solutions

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Exercises from the book.

Exercise 7.A.11 Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that there is a subspace U of V such that $P = P_U$ if and only if P is self-adjoint.

Proof. First suppose there is a subspace U of V such that $P = P_U$. Suppose $v_1, v_2 \in V$. Write

$$v_1 = u_1 + w_1,\ v_2 = u_2 + w_2,$$

where $u_1, u_2 \in U$ and $w_1, w_2 \in U^\perp$ (see 6.47). Now

$$\begin{aligned}\langle Pv_1, v_2\rangle &= \langle u_1, u_2 + w_2\rangle \\ &= \langle u_1, u_2\rangle + \langle u_1, w_2\rangle \\ &= \langle u_1, u_2\rangle \\ &= \langle u_1, u_2\rangle + \langle w_1, u_2\rangle \\ &= \langle v_1, u_2\rangle \\ &= \langle v_1, Pv_2\rangle\end{aligned}$$

Therefore $P = P^*$. Hence P is self-adjoint.

To prove the implication in the other direction, now suppose P is self-adjoint. Let $v \in V$, because $P(v - Pv) = Pv - P^2v = 0$, we have

$$v - Pv \in \operatorname{null} P = (\operatorname{range} P^*)^\perp = \operatorname{range} P^\perp,$$

where the first equality comes from 7.7(c). Writing

$$v = Pv + (v - Pv).$$

We have $Pv \in \operatorname{range} P$ and $v - Pv \in \operatorname{range} P^\perp$. Thus $Pv = P_{\operatorname{range} P}v$. Since this holds for all $v \in V$, we have $P = P_{\operatorname{range} P}$. □

Exercise 7.A.12 Suppose that T is a normal operator on V and that 3 and 4 are eigenvalues of T. Prove that there exists a vector $v \in V$ such that $||v|| = \sqrt{2}$ and $||Tv|| = 5$.

Proof. Let u and v be eigenvectors of T corresponding to the eigenvalues 3 and 4. Thus,

$$Tu = 3u \text{ and } Tw = 4w.$$

Replacing u with $\frac{u}{||u||}$ and w with $\frac{w}{||w||}$, we can assume that

$$||u|| = ||w|| = 1.$$

Because T is normal, 7.22 implies that u and w are orthogonal. Now the Pythagoream Theorem implies that

$$||u + w|| = \sqrt{||u||^2 + ||w||^2} = \sqrt{2}.$$

Using the Pythagoream Theorem again, we have

$$||T(u + w)|| = ||3u + 4w|| = \sqrt{9||u||^2 + 16||w||^2} = \sqrt{25} = 5.$$

Thus taking $v = u + w$, we have a vector v such that $||v|| = \sqrt{2}$ and $||Tv|| = 5$. □

Exercise 7.A.16 Prove that if $T \in L(V)$ is normal, then

$$\operatorname{range} T = \operatorname{range} T^*$$

Proof. By Prop 7.20 in the book, T is normal implies that $||Tv|| = ||T^*v||$ for all v. Thus, if $v \in \operatorname{null} T$ then $||Tv|| = 0$ implies that $||T^*v|| = 0$, thus $v \in \operatorname{null} T^*$. As $(T^*)^* = T$, this means that $v \in \operatorname{null} T$ iff $v \in \operatorname{null} T^*$. So the kernels of T and T^* are equal.

By Prop 7.7, $\operatorname{null} T^* = (\operatorname{range} T)^\perp$ and $\operatorname{null} T = (\operatorname{range} T^*)^\perp$. As $\operatorname{null} T = \operatorname{null} T^*$, this implies that

$$(\operatorname{range} T)^\perp = (\operatorname{range} T^*)^\perp$$

If U is a subspace of V, then $(U^\perp)^\perp = U$. Taking the orthogonal complement of both sides of the above equation give us $\operatorname{range} T = \operatorname{range} T^*$. □

Exercise 7.B.1. True or false (and give a proof of your answer): There exists $T \in \mathcal{L}(\mathbb{R}^3)$ such that T is not self-adjoint (with respect to the usual inner product) and such that there is a basis of $\mathbb{R}^3$ consisting of eigenvectors of T.

Proof. The statement above is true. To produce the desired example, note that $(1, 0, 0), (0, 1, 0), (1, 1, 1)$ is a basis of $\mathbb{R}^3$ and consider the operator $T \in \mathbb{R}^3$ such that

$$\begin{aligned} T(1, 0, 0) &= (0, 0, 0) \\ T(0, 1, 0) &= (0, 0, 0) \\ T(1, 1, 1) &= (1, 1, 1) \end{aligned}$$

here we have used 3.5 to guarantee the existence of an operator T with the properties above.

The vector $(1, 0, 0)$ and $(0, 1, 0)$ are eigenvectors of T with eigenvalue 0; the vector $(1, 1, 1)$ is an eigenvector of T with eigenvalue 1. Thus there is a basis of $\mathbb{R}^3$ consisting of eigenvectors of T.

However, 7.22 tells us that T is not normal (and thus not self-adjoint) because the eigenvectors $(1, 0, 0)$ and $(1, 1, 1)$ correspond to distinct eigenvalues but these eigenvectors are not orthogonal. □

Exercise 7.B.2 Suppose that T is a self-adjoint operator on a finite-dimensional inner product space and that 2 and 3 are the only eigenvalues of T. Prove that $T^2 - 5T + 6I = 0$

Proof. If v is an eigenvector of T with eigenvalue 2, then

$$\begin{aligned} (T^2 - 5T + 6I)v &= ((T - 3I)(T - 2I))v \\ &= (T - 3I)((T - 2I)v) \\ &= (T - 3I)0 \\ &= 0 \end{aligned}$$

Similarly, if v is an eigenvector of T with eigenvalue 3, then

$$\begin{aligned}(T^2 - 5T + 6I)v &= ((T - 2I)(T - 3I))v \\ &= (T - 2I)((T - 3I)v) \\ &= (T - 2I)0 \\ &= 0\end{aligned}$$

By the Complex Spectral Theorem, there is an orthonormal basis of the domain of T consisting of eigenvectors of T. The equations above show that $T^2 - 5T + 6I$ applied to any such basis vector equals 0. Since a linear map is determined by its values on a basis, $T^2 - 5T + 6I = 0$. □

Exercise 7.B.7 Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$ is a normal operator such that $T^9 = T^8$. Prove that T is self-adjoint and $T^2 = T$.

Proof. By the Complex Spectral Theorem(7.24), there is an orthonormal basis $e_1, ..., e_n$ of V consisting of eigenvectors of T. Let $\lambda_1, ..., \lambda_n$ be the corresponding eigenvalues. Thus

$$Te_j = \lambda_j e_j$$

for $j = 1, ..., n$. Applying T repeatedly both sides of the equation above, we get $T^9 e_j = \lambda_j^9 e_j$ and $T^8 e_j = \lambda_j^8 e_j$. Thus $\lambda_j^9 = \lambda_j^8$, which implies that λ_j equals 0 or 1. In particular, all the eigenvalues of T are real. The matrix of T with respect to the orthonormal basis $e_1, ..., e_n$ is the diagonal matrix with $\lambda_1, ..., \lambda_n$ on the diagonal. This matrix equals its conjugate transpose. Thus $T = T^*$. Hence T is self-adjoint, as desired. [Alternate argument: we know from class that "self-adjoint" is equivalent to "normal and all eigenvalues are real".]

Applying T to both sides of the equation above, we get

$$\begin{aligned}T^2 e_j &= \lambda_j^2 e_j \\ &= \lambda_j e_j \\ &= Te_j,\end{aligned}$$

where the second equality holds because λ_j equals 0 or 1. Because T^2 and T agree on a basis, they are equal. □

Question 1.

a) Given an example of two self-adjoint operators $S \in \mathcal{L}(\mathbb{R}^2)$ and $T \in \mathcal{L}(\mathbb{R}^2)$ whose product is not self-adjoint.

Let V be a finite-dimensional inner product space, and assume that $S, T \in \mathcal{L}(V)$ are self-adjoint.

b) Prove that $ST + TS$ is a self-adjoint operator.

c) Prove that ST is self-adjoint iff $ST = TS$.

Proof. a) Let $T, S : \mathbb{R}^2 \to \mathbb{R}^2$ s.t.

$$T(x, y) = (x + 2y, 2x) \quad \text{and} \quad S(x, y) = (y, x + y)$$

Their matrices with respect to the standard basis (which is orthonormal) are

$$M(T) = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad M(S) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

These operators are self-adjoint because the matrices are equal to their conjugate-transposes. The product of these matrices is

$$M(T)M(S) = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$$

This matrix is not equal to its conjugate transpose. As the standard basis is orthonormal, this implies that TS is not self-adjoint.

b) We expand the following expression, using the fact that S, T are self-adjoint:

$$\begin{aligned} \langle (ST + TS)v, w \rangle &= \langle STv, w \rangle + \langle TSv, w \rangle \\ &= \langle Tv, S^* w \rangle + \langle Sv, T^* w \rangle \\ &= \langle Tv, Sw \rangle + \langle Sv, Tw \rangle \\ &= \langle v, T^* Sw \rangle + \langle v, S^* Tw \rangle \\ &= \langle v, TSw \rangle + \langle v, STw \rangle \\ &= \langle v, (TS + ST)w \rangle \end{aligned}$$

Therefore, $\langle (ST + TS)v, w \rangle = \langle v, (TS + ST)w \rangle$ so $ST + TS$ is self-adjoint. □

c) If $ST = TS$, then $ST + TS = 2ST$. Since $2ST$ is self-adjoint, and 2 is a real number,

$$\begin{aligned} 2\langle STv, w \rangle &= \langle 2STv, w \rangle \\ &= \langle v, 2STw \rangle \\ &= 2\langle v, STw \rangle \end{aligned}$$

Since our field is either $\mathbb{R}$ or $\mathbb{C}$, we get that $\langle STv, w \rangle = \langle v, STw \rangle$, so ST is self-adjoint.

Suppose ST is self-adjoint. Then

$$\langle STv, w \rangle = \langle v, STw \rangle$$

and,

$$\begin{aligned} \langle STv, w \rangle &= \langle v, (ST)^* w \rangle \\ &= \langle v, T^* S^* w \rangle \\ &= \langle v, TSw \rangle \text{ because } T, S \text{ are self-adjoint.} \end{aligned}$$

Since

$$\begin{aligned} \langle v, STw \rangle &= \langle v, TSw \rangle \text{ for all } v, w \in V, \\ \langle v, (ST - TS)w \rangle &= 0 \text{ for all } v, w \in V \text{, so setting } v = (ST - TS)w, \\ ||(ST - TS)w||^2 &= 0 \text{ for all } w \in V \text{, therefore,} \\ ST - TS &= 0 \end{aligned}$$

So $ST = TS$. □