Math 113 Homework 9 Solutions

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Question 1. Let $V = \mathbb{R}^2$, and let $T \in \mathcal{L}(V)$ be an operator on V . Assume that $v \in V$ and $w \in V$ are two non-zero vectors satisfying

$$T(v) = 2v \text{ and } T(w) = -w$$

Compute the determinant $\det(T^4 + T)$.

Answer. Notice that $(T^4 + T)(w) = T^4w + Tw = w - w = 0$. Therefore $T^4 + T$ is not injective, and thus not invertible. Using Proposition 3.3, we know that $\det(T^4 + T)$=0. □

Question 2. On HW 5, you found the minimal polynomial of the operator $T \in \mathcal{L}(\mathbb{R}^4)$ with matrix

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Find the characteristic polynomial of T.

Answer. Recall that the characteristic polynomial $\chi_T(x)$ is the function defined by

$$\chi_T(x) = \det(xI - T)$$

The operator $xI - T \in \mathcal{L}(\mathbb{R}^4)$ has matrix

$$\begin{pmatrix} x-2 & 0 & 0 & 0 \\ 0 & x-3 & 0 & -1 \\ 0 & 0 & x-3 & 0 \\ 0 & 0 & 0 & x-3 \end{pmatrix}$$

which is upper-triangular. By proposition 3.7, the determinant of an upper-triangular operator is product of diagonal entries, so $\chi_T(x) = \det(xI-T) = (x-2)(x-3)^3$. □

Question 3. Let V be an n-dimensional vector space, and let $T \in \mathcal{L}(V)$ be an operator on V. Let $\chi_T(x)$ be the characteristic polynomial of T. Which of the following implications is true?

I. If $\chi_T(x)$ has n distinct roots, then T is diagonalizable.
II. If T is diagonalizable, then $\chi_T(x)$ has n distinct roots.
III. Both I and II are true.
IV. Neither I nor II is true.

Prove that your answer is correct, by either proving or giving a counterexample for I, and either proving or giving a counterexample for II.

Proof. Statement I is correct. It follows from Proposition 4.2 that the operator T has n distinct eigenvalues, then from Theorem 5.44 in our textbook, we know that T is diagonalizable.

Statement II is false. Now suppose $n = 2$, $\mathbb{F} = \mathbb{R}$ and let $T = I$, the identity operator on $\mathbb{R}^2$. Then T is diagonalizable. Meanwhile the operator $xI - T = xI - I = (x-1)I \in \mathcal{L}(\mathbb{R}^2)$ has matrix

$$\begin{pmatrix} x-1 & 0 \\ 0 & x-1 \end{pmatrix}$$

So we have $\chi_T(x) = \det(xI - T) = \det(xI - I) = \det((x-1)I) = (x-1)^2$, which has only one root 1. Therefore this is a counterexample of statement II. □

Question 4. Let V be a finite-dimensional complex inner product space, and let $T : V \to V$ be an operator on V. Prove that if T is an isometry, then $|\det T| = 1$.

Proof. We know from Thm 7.43 that there is an orthonormal basis of V consisting of eigenvectors of T whose corresponding eigenvalues all have absolute value 1. Let $e_1, \ldots, e_n$ be the eigenbasis and $\lambda_1, \ldots, \lambda_n$ be the corresponding eigenvalues. Then we just need to calculate $T(e_1) \wedge \cdots \wedge T(e_n)$

$$T(e_1) \wedge \cdots \wedge T(e_n) = \lambda_1 e_1 \wedge \cdots \wedge \lambda_n e_n = \lambda_1 \cdots \lambda_n \cdot e_1 \wedge \cdots \wedge e_n$$

Therefore $|\det T| = |\lambda_1 \cdots \lambda_n| = 1$ since $|\lambda_i| = 1$ for any $i \in \{1, \cdots, n\}$ □

Question 5. Let V be a finite-dimensional complex inner product space, and let $T : V \to V$ be an operator on V. Prove that

$$\det T^* = \overline{\det T}$$

Proof. We can first find a basis $v_1 \cdots v_n$ of V such that the matrix of T under this basis is upper-triangular. So we have $T(v_i) = d_i v_i + w_i$ for some $d_i \in \mathbb{F}$ and $w_i \in \operatorname{span}(v_1, \cdots, v_{i-1})$, then Proposition 3.7 in the lecture notes gives us that $\det T = d_1 d_2 \cdots d_n$. Proposition 7.10 says that the matrix of T^* under the basis $v_1 \cdots v_n$ is the conjugate transpose of the matrix of T under $v_1 \cdots v_n$, which is an lower-triangular matrix. So we have $T^*(v_i) = \overline{d}_i v_i + u_i$ for some $u_i \in \operatorname{span}(v_{i+1}, \cdots, v_n)$. Notice that if we reorder the basis as $\{v_n, v_{n-1}, \cdots, v_1\}$, then the matrix of T^* under the new basis is upper-triangular, with diagonal entries $\overline{d}_n, \overline{d}_{n-1}, \cdots, \overline{d}_1$. Thus we have $\det T^* = \overline{d_n \cdots d_1} = \overline{d_1 \cdots d_n} = \overline{\det T}$, as desired.

□