# Math 113 – Fall 2015 – Prof. Church
# Midterm Exam 10/26/2015

Name: ______________________________

Student ID: ______________________________

Signature: ______________________________

This exam is closed-book and closed-notes. In your proofs you may use any theorem from class or from the sections of the book that are covered on the midterm (not including exercises or homework questions). You do not need to cite theorems by number; just give the statement of the theorem you wish to cite. When giving counterexamples, you may describe linear maps or operators either by a formula or by a matrix.

There are 5 questions worth 100 points total on this exam, plus a 10-point bonus question; you should finish all the other questions before attempting the bonus question.

**Question 1** (20 points)**.** Let $T \in \mathcal{L}(V)$ be an operator on the vector space $V$.

(a) State clearly and precisely the definition of:

"$v$ is an eigenvector of $T$ with eigenvalue $\lambda$."

**Solution.** "$v$ is a nonzero vector in $V$, and $T(v) = \lambda v$."

We continue to assume that $T \in \mathcal{L}(V)$ is an operator on the vector space $V$.
Let $v_1$ be an eigenvector of $T$ with eigenvalue $\lambda_1 \in \mathbf{F}$, and
let $v_2$ be an eigenvector of $T$ with eigenvalue $\lambda_2 \in \mathbf{F}$.

(b) Prove that if $\lambda_1 \neq \lambda_2$, then $v_1$ and $v_2$ are linearly independent.
(*On this question only, you cannot quote the theorem that says this.*)

**Solution.** The definition of linear independence is that $c_1 v_1 + c_2 v_2 = 0$ implies $c_1 = 0$ and $c_2 = 0$.

Therefore assume that $c_1 \in \mathbf{F}$ and $c_2 \in \mathbf{F}$ satisfy

$$c_1 v_1 + c_2 v_2 = 0. \tag{$*$}$$

Our goal is to prove that $c_1 = 0$ and $c_2 = 0$.

Applying $T$ to the left side of $(*)$ yields

$$\begin{aligned} T(c_1 v_1 + c_2 v_2) &= c_1 T(v_1) + c_2 T(v_2) \\ &= c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 \end{aligned}$$

while applying $T$ to the right side of $(*)$ yields $T(0) = 0$. Therefore

$$c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 = 0. \tag{$**$}$$

Now subtract $\lambda_1$ times $(*)$ from $(**)$ to obtain

$$\begin{aligned} (c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2) - \lambda_1 (c_1 v_1 + c_2 v_2) &= 0 \\ (c_1 \lambda_1 - c_1 \lambda_1) v_1 + (c_2 \lambda_2 - c_2 \lambda_1) v_2 &= 0 \\ c_2 (\lambda_2 - \lambda_1) v_2 &= 0 \end{aligned}$$

Since $\lambda_1 \neq \lambda_2$ by assumption, we know that $(\lambda_2 - \lambda_1) \neq 0$, so we can multiply by $\frac{1}{\lambda_2 - \lambda_1}$ to obtain

$$c_2 v_2 = 0$$

We know that $v_2 \neq 0$ since $v_2$ is an eigenvector, so we must have $c_2 = 0$. Substituting $c_2 = 0$ into $(*)$ yields

$$c_1 v_1 = 0.$$

We know that $v_1 \neq 0$ since $v_1$ is an eigenvector, so we must have $c_1 = 0$.

We conclude that $(*)$ implies that $c_1 = 0$ and $c_2 = 0$. Therefore by definition of linear independence, this proves that $v_1$ and $v_2$ are linearly independent.

We continue to assume that $T \in \mathcal{L}(V)$ is an operator on the vector space $V$,
$v_1$ is an eigenvector of $T$ with eigenvalue $\lambda_1 \in \mathbf{F}$, and
$v_2$ is an eigenvector of $T$ with eigenvalue $\lambda_2 \in \mathbf{F}$.

(c) Give two examples showing that if $\lambda_1 = \lambda_2$, then $v_1$ and $v_2$ might be either linearly independent or linearly dependent.
(After specifying the operator $T$, you can just indicate the vectors $v_1$ and $v_2$; as long as they really are eigenvectors, you do not have to prove that they are.)

**Solution.** Let $V$ be any 2-dimensional vector space, with basis $w_1, w_2$ and let $T = I$. Note that any vector $v \in V$ satisfies $Tv = v = 1v$; therefore any *nonzero* vector $v \in V$ is an eigenvector of $T$ with eigenvalue 1.

**First example:** set $v_1 = w_1$ and $v_2 = w_2$.

**Second example:** set $v_1 = w_1$ and $v_2 = 77w_1$.

(The vectors $w_1$ and $w_2$ are nonzero, because they are part of a basis; $77w_1$ is nonzero as well, because it is a nonzero multiple of a nonzero vector. The vectors $w_1$ and $w_2$ are linearly independent because $w_1, w_2$ is a basis; the vectors $w_1$ and $77w_1$ are linearly dependent because $77v_1 - v_2 = 0$.(

**Question 2** (20 points)**.** Let $V$ be a finite-dimensional vector space with $\dim V = n \geq 1$, and let $T \in \mathcal{L}(V)$ and $S \in \mathcal{L}(V)$ be operators on $V$.

Assume that $ST = 0$.
Prove that there exists a nonzero vector $v \neq 0 \in V$ with $TS(v) = 0$.

**Solution.** Proof #1: We consider two cases: either $\operatorname{range} T = \{0\}$ or $\operatorname{range} T \neq \{0\}$. First, assume $\operatorname{range} T = \{0\}$. In this case we may choose any nonzero vector $v \neq 0$, and we find $TS(v) = T(Sv) \in \operatorname{range} T$. Since $\operatorname{range} T = \{0\}$, this implies $TS(v) = 0$, as desired.

Now assume that $\operatorname{range} T \neq \{0\}$. Choose a nonzero $v \neq 0 \in \operatorname{range} T$. By definition, there exists some $u \in V$ with $T(u) = v$ (this is what it means that $v \in \operatorname{range} T$). Then we can compute $TS(v) = TS(Tu) = T(ST(u))$. Since we have assumed $ST = 0$, we know that $ST(u) = 0$. Therefore $TS(v) = T(ST(u)) = T(0) = 0$, as desired. □

Proof #2: For any operators $S$ and $T$, we know that $\operatorname{null} S \subset \operatorname{null} TS$ and $\operatorname{range} TS \subset \operatorname{range} T$. (Proof not necessary, but if you wanted to give it: $v \in \operatorname{null} S \iff S(v) = 0 \implies TS(v) = 0 \iff v \in \operatorname{null} TS$, and $w \in \operatorname{range} TS \iff w = TS(v) \implies w = T(u) \iff w \in \operatorname{range} T$, taking $u = S(v)$ in the last implication.)

Moreover, the assumption that $ST = 0$ means precisely that $\operatorname{range} T \subset \operatorname{null} S$. (Proof not necessary, but if you wanted to give it: $w \in \operatorname{range} T \iff w = T(u) \implies S(w) = ST(u) \implies S(w) = 0 \iff w \in \operatorname{null} S$, where we used $ST = 0$ in the last implication.)

Together, these say that

$$\operatorname{range} TS \subset \operatorname{range} T \subset \operatorname{null} S \subset \operatorname{null} TS. \qquad (\star)$$

Now assume for a contradiction that there is no nonzero $v \neq 0 \in V$ with $TS(v) = 0$. In other words, $\operatorname{null} TS = \{0\}$. By $(\star)$, this means that $\operatorname{range} TS \subset \operatorname{null} TS = \{0\}$, so $\operatorname{range} TS = \{0\}$. The Fundamental Theorem of Linear Maps then tells us that

$$\dim V = \dim \operatorname{range} TS + \dim \operatorname{null} TS = 0 + 0 = 0.$$

This contradicts the assumption that $V$ has dimension $n \geq 1$. Therefore there must exist a nonzero $v \in V$ with $TS(v) = 0$. □

[Many other proofs are possible as well.]

**Question 3** (20 points)**.** Let $V$, $W$, and $U$ be finite-dimensional vector spaces.
Let $T\colon V \to W$ be a linear map from $V$ to $W$, and
let $S\colon W \to U$ be a linear map from $W$ to $U$.

(a) Prove that $\operatorname{range} ST \subseteq \operatorname{range} S$.

**Solution.** (a) Assume that $u \in U$ lies in $\operatorname{range} ST$. By definition, this means that there exists $v \in V$ such that $u = ST(v)$. Choose such a $v \in V$, and let $w = T(v)$. Then $S(w) = S(T(v)) = ST(v) = u$. This shows that $u$ can be written as $S(w)$ for this $w \in W$, so $u \in \operatorname{range} S$.

We have proved that every $u \in \operatorname{range} ST$ lies in $\operatorname{range} S$, so $\operatorname{range} ST \subseteq \operatorname{range} S$. □

We continue to assume that $V$, $W$, and $U$ are finite-dimensional vector spaces,
$T: V \to W$ is a linear map from $V$ to $W$, and
$S: W \to U$ is a linear map from $W$ to $U$.

(b) Assume that $\operatorname{range} ST = \operatorname{range} S$. Which of the following is true?

(I) $T$ must be surjective.

(II) $T$ must be non-surjective.

(III) $T$ could be surjective or non-surjective.

Prove your answer.

**Solution.** (b) The correct answer is (III): $T$ could be surjective or non-surjective.

First, an example where $T$ is surjective:

Let $V = W = U = \mathbf{F}$ and let $T = I$ and $S = I$ both be the identity map. Then $S = I$ and $ST = I \circ I = I$, so $\operatorname{range} S = \mathbf{F}$ and $\operatorname{range} ST = \mathbf{F}$. The identity map $T = I: \mathbf{F} \to \mathbf{F}$ is surjective.
(We could have taken any vector space here, as long as $T = I$ and $S = I$.)

Second, an example where $T$ is non-surjective:

Let $V = W = U = \mathbf{F}$ and let $T = 0$ and $S = 0$ both be the zero map. Then $S = 0$ and $ST = 0 \circ 0 = 0$, so $\operatorname{range} S = \{0\}$ and $\operatorname{range} ST = \{0\}$. Since $\operatorname{range} T = \{0\} \neq \mathbf{F}$, the map $T$ is not surjective.
(We could have taken any nonzero vector space here, as long as $T = 0$ and $S = 0$.) $\square$

[Of course, there are many other examples we could choose; these are just the simplest.]

**Question 4** (20 points)**.** Let $V$ be a finite-dimensional vector space, and let $T \in \mathcal{L}(V)$ be an operator on $V$.

Is the following statement true or not?

$$\text{If } T^3 = T^2, \quad \text{then } V = \operatorname{null} T \oplus \operatorname{null}(T - I). \tag{$*$}$$

Prove the statement $(*)$ or give a counterexample.

**Solution.** No, it is not true. For example, let $V = \mathbb{R}^2$ and let $T$ be the operator

$$T(x, y) = (y, 0)$$

with matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $T^2 = 0$, so $T^3 = 0$.

However $\operatorname{null} T = \{(x, 0) \mid x \in \mathbb{R}\}$ is 1-dimensional, and $\operatorname{null}(T - I) = \{0\}$. Since $\dim(U \oplus W) = \dim U + \dim W$, this implies

$$\dim(\operatorname{null} T \oplus \operatorname{null}(T - I)) = \dim \operatorname{null} T + \dim \operatorname{null}(T - I) = 0 + 1 = 1 \neq 2 = \dim V.$$

(To obtain this description of $\operatorname{null} T$: if $v = (x, y)$ lies in $\operatorname{null} T$, then $Tv = (0, 0)$. Since $Tv = (y, 0)$, this implies that $y = 0$. Conversely, if $v = (x, 0)$, then $Tv = (0, 0)$ and $v \in \operatorname{null} T$.

Similarly, to obtain this description of $\operatorname{null}(T-I)$, note that $(T-I)(x, y) = (y-x, -y)$. Therefore if $v = (x, y)$ is in $\operatorname{null}(T - I)$, we have $x = y - x$ and $y = -y$. The latter equation implies $y = 0$; substituting into the first yields $x = -x$, so $x = 0$. Therefore $v = (0, 0)$.) $\square$

**Question 5** (20 points)**.** Let $V$ be a finite-dimensional vector space with $\dim V = n$. Let $S \in \mathcal{L}(V)$ be an operator on $V$ with $n$ distinct eigenvalues, and let $T \in \mathcal{L}(V)$ be another operator on $V$.

Prove that if $ST = TS$, then $T$ is diagonalizable.

(*Hint: prove that an eigenbasis for $S$ is also an eigenbasis for $T$.*)

**Solution.** We first prove:
**Lemma 1:** If $v$ is an eigenvector for $S$ with eigenvalue $\lambda$, then $Tv$ is also an eigenvector for $S$ with eigenvalue $\lambda$ (or $Tv = 0$).

*Proof of Lemma 1.* The proof uses only the assumption that $ST = TS$. Set $w = Tv$; then

$$S(w) = S(Tv) = ST(v) = TS(v) = T(\lambda v) = \lambda Tv = \lambda w.$$

This shows that $w$ (that is, $Tv$) satisfies the eigenvector equation $Sw = \lambda w$; therefore if $w \neq 0$, it is an eigenvector of $S$ with eigenvalue $\lambda$. □

We now turn to the assumption that $S$ has $n$ distinct eigenvalues. Let $\lambda_1, \ldots, \lambda_n$ be the $n$ eigenvalues of $S$, and let $v_1, \ldots, v_n$ be the corresponding eigenvectors (so $S(v_k) = \lambda_k v_k$). The vectors $v_1, \ldots, v_n$ are linearly independent, since eigenvectors whose eigenvalues are distinct are linearly independent. Since $\dim V = n$, these $n$ vectors form a basis for $V$. We next prove:
**Lemma 2:** If $u \in V$ satisfies $S(u) = \lambda_k u$, then $u = cv_k$ for some $c \in \mathbf{F}$.

*Proof of Lemma 2.* Since $v_1, \ldots, v_n$ is a basis of $V$, any $u \in V$ can be written as

$$u = c_1 v_1 + \cdots c_n v_n.$$

We can compute $S(u)$ directly as

$$S(u) = S(c_1 v_1 + \cdots c_n v_n) = c_1 S(v_1) + \cdots + c_n S(v_n) = c_1 \lambda_1 v_1 + \cdots c_n \lambda_n v_n.$$

If we also assume that $S(u) = \lambda_k u$, then

$$S(u) = \lambda_k (c_1 v_1 + \cdots c_n v_n) = c_1 \lambda_k v_1 + \cdots c_n \lambda_k v_n.$$

Subtracting the latter equation from the former gives

$$\begin{aligned} S(u) - S(u) &= (c_1 \lambda_1 v_1 + \cdots c_n \lambda_n v_n) - (c_1 \lambda_k v_1 + \cdots c_n \lambda_k v_n) \\ 0 &= c_1(\lambda_1 - \lambda_k) v_1 + \cdots + c_k(\lambda_k - \lambda_k) c_k + \cdots + c_n(\lambda_n - \lambda_k) v_n. \end{aligned}$$

Since $v_1, \ldots, v_n$ is a basis, we know that there is only one way to write 0 as a linear combination of these basis vectors (namely with all coefficients 0). So we conclude that

$$c_1(\lambda_1 - \lambda_k) = 0, \ldots, \qquad c_k(\lambda_k - \lambda_k) = 0, \ldots, \qquad c_n(\lambda_n - \lambda_k) = 0.$$

For each $i$ other than $k$, we know that $\lambda_i - \lambda_k \neq 0$ (since $\lambda_i$ and $\lambda_k$ are distinct), so we conclude that $c_i = 0$ for all $i$ other than $k$. This shows that $u = 0v_1 + \cdots + c_k v_k + \cdots + 0v_n = c_k v_k$, as desired. This completes the proof of Lemma 2. □

We now prove the claim that $T$ is diagonalizable by showing that $v_1, \ldots, v_n$ is an eigenbasis for $T$. Consider a single vector $v_k$ from this basis, which is an eigenvector of $S$ with eigenvalue $v_k$. By Lemma 1, we know that $Tv_k$ is an eigenvector of $S$ with eigenvalue $\lambda_k$, or else $Tv = 0$; in either case, it satisfies $S(Tv_k) = \lambda_k T v_k$. Therefore by Lemma 2, we see that $Tv_k = cv_k$ for some $c \in \mathbf{F}$. In other words, $v_k$ is an eigenvector of $T$ (since it is nonzero). Therefore $v_1, \ldots, v_n$ is an eigenbasis for $T$, so $T$ is diagonalizable.

**Question 6** (Bonus question, 10 points)**.** Let $V$ be a finite-dimensional vector space with $\dim V = n$, and let $S \in \mathcal{L}(V)$ and $T \in \mathcal{L}(V)$ be operators on $V$ satisfying $ST = TS$.

Assume that $S$ and $T$ are diagonalizable. Prove that there exists a basis $v_1, \ldots, v_n$ for $V$ which is *simultaneously* an eigenbasis for $S$ and also an eigenbasis for $T$.

**Solution.** First, a lemma. For any subspace $U$ invariant under $T$, let $R \in \mathcal{L}(U)$ be the restriction $R = T|_U$.

**Lemma 3:** The minimal polynomial $p_R(x)$ of $R$ divides the minimal polynomial $p_T(x)$ of $T$.

*Proof of Lemma 3.* By definition $Ru = Tu$ for all $u \in U$ (this is the definition of the restriction $R = T|_U$), from which it follows that $R^2u = T^2u$, and in general $f(R)u = f(T)u$ for any polynomial $f(x)$. Applying this to the polynomial $p_T(x)$, we find that $p_T(R)u = p_T(T)u$ for all $u \in U$.

However $p_T(T) = 0 \in \mathcal{L}(V)$ by definition, so for all $v \in V$ (not just those in $U$), we have $p_T(T)v = 0$. Therefore the previous paragraph implies that $p_T(R)u = 0$ for all $u \in U$; in other words, $p_T(R) = 0 \in \mathcal{L}(U)$.

Whenver a polynomial $f(x)$ satisfies $f(R) = 0 \in \mathcal{L}(U)$, that polynomial is divisible by the minimal polynomial of $R$ (this is Prop 8.46, or you could prove it yourself; there are other approaches to this problem too). Therefore $p_T(x)$ is divisible by the minimal polynomial $p_R(x)$ of $R$. □

**Lemma 4:** For each eigenvalue $\lambda$ of $S$, we can choose a basis for $E(S, \lambda)$ consisting of eigenvectors for $T$.

*Proof of Lemma 4.* Since $T$ is diagonalizable, we know that the minimal polynomial $p_T(x)$ has no repeated roots.

Fix an eigenvalue $\lambda$ of $S$, and let $U = E(S, \lambda)$. What Lemma 1 from the solution of Question 5 says is that under the assumption that $ST = TS$, the eigenspaces $E(S, \lambda)$ of $S$ are invariant subspaces under $T$. Therefore we can let $R \in \mathcal{L}(U)$ be the restriction $R = T|_U$.

By Lemma 3, the minimal polynomial of $R$ divides $p_T(x)$; therefore it cannot have any repeated roots (it has even *fewer* roots than $p_T(x)$, where would you get a repeated root from?). Since the minimal polynomial of $R$ has no repeated roots, $R$ is diagonalizable. Therefore we may choose a basis of $U = E(S, \lambda)$ consisting of eigenvectors for $T$. □

Let $\lambda_1, \ldots, \lambda_k$ be the eigenvalues of $S$. Recall that the sum of the eigenspaces $E(S, \lambda_1) + \cdots + E(S, \lambda_k)$ is a direct sum $E(S, \lambda_1) \oplus \cdots \oplus E(S, \lambda_k)$.

If we concatenate the basis of $E(S, \lambda_1)$ from Lemma 4, with the basis of $E(S, \lambda_2)$ from Lemma 4, ...., with the basis of $E(S, \lambda_k)$ from Lemma 4, what we obtain is a basis for $E(S, \lambda_1) + \cdots + E(S, \lambda_k)$. Each vector in this basis is an eigenvector for $S$ (since each one is contained in $E(S, \lambda_i)$ for some $i$) and is also an eigenvector for $T$ (since we chose them that way in Lemma 4).

Finally, we use the assumption that $S$ is diagonalizable, which we have not used yet. Since $S$ is diagonalizable, we know that $V = E(S, \lambda_1) + \cdots + E(S, \lambda_k)$. Therefore the basis for $E(S, \lambda_1) + \cdots + E(S, \lambda_k)$ we obtained above is the desired basis for $V$. □

| Total |
| --- |
| |

| 1a | 1b | 1c | 2 | 3a | 3b | 4 | 5 | Bonus |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| | | | | | | | | |