Math 113 – Fall 2015 – Prof. Church
Midterm Exam 10/26/2015

Name: ____________________________________

Student ID: ________________________________

Signature: _________________________________

This exam is closed-book and closed-notes. In your proofs you may use any theorem from class or from the sections of the book that are covered on the midterm (not including exercises or homework questions). You do not need to cite theorems by number; just give the statement of the theorem you wish to cite. When giving counterexamples, you may describe linear maps or operators either by a formula or by a matrix.

There are 5 questions worth 100 points total on this exam, plus a 10-point bonus question; you should finish all the other questions before attempting the bonus question.

Question 1 (20 points)**.** Let $T \in \mathcal{L}(V)$ be an operator on the vector space V.

(a) State clearly and precisely the definition of:

"v is an eigenvector of T with eigenvalue λ."

We continue to assume that $T \in \mathcal{L}(V)$ is an operator on the vector space V.
Let v_1 be an eigenvector of T with eigenvalue $\lambda_1 \in \mathbf{F}$, and
let v_2 be an eigenvector of T with eigenvalue $\lambda_2 \in \mathbf{F}$.

(b) Prove that if $\lambda_1 \neq \lambda_2$, then v_1 and v_2 are linearly independent.
(*On this question only, you cannot quote the theorem that says this.*)

We continue to assume that $T \in \mathcal{L}(V)$ is an operator on the vector space V,
v_1 is an eigenvector of T with eigenvalue $\lambda_1 \in \mathbf{F}$, and
v_2 is an eigenvector of T with eigenvalue $\lambda_2 \in \mathbf{F}$.

(c) Give two examples showing that if $\lambda_1 = \lambda_2$,
then v_1 and v_2 might be either linearly independent or linearly dependent.
(After specifying the operator T, you can just indicate the vectors v_1 and v_2;
as long as they really are eigenvectors, you do not have to prove that they are.)

Question 2 (20 points)**.** Let V be a finite-dimensional vector space with $\dim V = n \geq 1$, and let $T \in \mathcal{L}(V)$ and $S \in \mathcal{L}(V)$ be operators on V.

Assume that $ST = 0$.
Prove that there exists a nonzero vector $v \neq 0 \in V$ with $TS(v) = 0$.

Question 3 (20 points)**.** Let V, W, and U be finite-dimensional vector spaces.
Let $T\colon V \to W$ be a linear map from V to W, and
let $S\colon W \to U$ be a linear map from W to U.

(a) Prove that $\operatorname{range} ST \subseteq \operatorname{range} S$.

We continue to assume that V, W, and U are finite-dimensional vector spaces,
$T\colon V \to W$ is a linear map from V to W, and
$S\colon W \to U$ is a linear map from W to U.

(b) Assume that $\operatorname{range} ST = \operatorname{range} S$. Which of the following is true?

(I) T must be surjective.

(II) T must be non-surjective.

(III) T could be surjective or non-surjective.

Prove your answer.

Question 4 (20 points)**.** Let V be a finite-dimensional vector space, and let $T \in \mathcal{L}(V)$ be an operator on V.

Is the following statement true or not?

$$\text{If } T^3 = T^2, \quad \text{then } V = \operatorname{null} T \oplus \operatorname{null}(T - I). \tag{$*$}$$

Prove the statement $(*)$ or give a counterexample.

Question 5 (20 points)**.** Let V be a finite-dimensional vector space with $\dim V = n$.
Let $S \in \mathcal{L}(V)$ be an operator on V with n distinct eigenvalues, and
let $T \in \mathcal{L}(V)$ be another operator on V.

Prove that if $ST = TS$, then T is diagonalizable.

(*Hint: prove that an eigenbasis for S is also an eigenbasis for T.*)

Question 6 (Bonus question, 10 points)**.** Let V be a finite-dimensional vector space with $\dim V = n$, and let $S \in \mathcal{L}(V)$ and $T \in \mathcal{L}(V)$ be operators on V satisfying $ST = TS$.

Assume that S and T are diagonalizable. Prove that there exists a basis $v_1, \ldots, v_n$ for V which is *simultaneously* an eigenbasis for S and also an eigenbasis for T.

Total

1a	1b	1c	2	3a	3b	4	5	Bonus