Math 113 – Winter 2013 – Prof. Church
Midterm Solutions

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Question 1 (20 points)**.** Let V be a finite-dimensional vector space, and let $T \in \mathcal{L}(V, W)$. Assume that $v_1, \ldots, v_n$ is a basis for V. (For this question only, do not use the Rank-Nullity Theorem.)

a) Prove that T is injective if and only if $T(v_1), \ldots, T(v_n)$ are linearly independent in W.

Proof. ($\Longrightarrow$) Assume that T is injective. Consider a linear dependence $a_1 T(v_1) + \cdots + a_n T(v_n) = 0$. If we set $v = a_1 v_1 + \cdots + a_n v_n$, we have $T(v) = T(a_1 v_1 + \cdots + a_n v_n) = a_1 T(v_1) + \cdots + a_n T(v_n)$, so our assumption says that $T(v) = 0$. Since T is injective, this implies that $v = 0$. But since $v_1, \ldots, v_n$ is linearly independent (since it is a basis), the only way we can have $a_1 v_1 + \cdots + a_n v_n = 0$ is if $a_1 = 0, \ldots, a_n = 0$. This shows that $a_1 T(v_1) + \cdots + a_n T(v_n) = 0$ implies $a_1 = 0, \ldots, a_n = 0$, which is the definition of linear independence.

($\Longleftarrow$) Assume that $T(v_1), \ldots, T(v_n)$ are linearly independent. Consider $u \in \ker T$, so that $T(u) = 0$. Since $v_1, \ldots, v_n$ spans V (it is a basis), we can write $u = b_1 v_1 + \cdots + b_n v_n$. Therefore

$$0 = T(u) = T(b_1 v_1 + \cdots + b_n v_n) = b_1 T(v_1) + \cdots + b_n T(v_n).$$

Since $T(v_1), \ldots, T(v_n)$ are linearly independent, this is only possible if $b_1 = 0, \ldots, b_n = 0$. Therefore

$$u = b_1 v_1 + \cdots + b_n v_n = 0 \cdot v_1 + \cdots + 0 \cdot v_n = 0.$$

Therefore $u \in \ker T \implies u = 0$, or in other words $\ker T = \{0\}$. By Proposition 3.2, this is equivalent to injectivity of T. □

b) Prove that T is surjective if and only if $T(v_1), \ldots, T(v_n)$ spans W.

Proof. ($\Longrightarrow$) Assume that T is surjective. Therefore for any $w \in W$ there exists $v \in V$ such that $T(v) = w$. Since $v_1, \ldots, v_n$ is a basis, we can write $v = a_1 v_1 + \cdots + a_n v_n$. Then

$$w = T(v) = T(a_1 v_1 + \cdots + a_n v_n) = a_1 T(v_1) + \cdots + a_n T(v_n).$$

This shows that $w \in \operatorname{span}\big(T(v_1), \ldots, T(v_n)\big)$. Since this holds for any $w \in W$, we conclude that $\operatorname{span}\big(T(v_1), \ldots, T(v_n)\big) = W$ as desired.

($\Longleftarrow$) Assume that $T(v_1), \ldots, T(v_n)$ spans W. Then for any $w \in W$ there exist $a_1, \ldots, a_n$ such that $w = a_1 T(v_1) + \cdots + a_n T(v_n)$. Then if we set $v = a_1 v_1 + \cdots + a_n v_n$ we have $T(v) = w$. Therefore every $w \in W$ is in the image of T, and T is surjective. □

Question 2 (20 points)**.** We consider a linear transformation $T \in \mathcal{L}\big(P_{\leq 2}(\mathbb{R}), P_{\leq 3}(\mathbb{R})\big)$. Assume that we are given partial data about T:

$$\begin{aligned} T(x^2+1) &= x^2 - x \\ T(1) &= 2x+1 \end{aligned}$$

Given this partial data, answer the following questions. Justify your answers.

a) Could T be injective?

Answer. Yes. For example, consider the transformation T defined by the formula

$$T(ax^2+bx+c) = ax^2 + (b-3a+2c)x + (c-a)$$

We check: $T(x^2+1) = x^2 + (-3+2)x + (1-1) = x^2 - x$ and $T(1) = 0 + 2x + 1 = 2x+1$, so this fits the partial data. This map is injective: if $ax^2+bx+c \in \ker T$, we must have

$$ax^2 + (b-3a+2c)x + (c-a)x = 0 \qquad \implies \qquad \begin{cases} a = 0 \\ b - 3a + 2c = 0 \\ c - a = 0 \end{cases}$$

The first equation implies $a = 0$; given this, the third becomes $c = 0$; given these, the second becomes $b = 0$. Therefore $\ker T = \{0\}$ and T is injective. $\square$

b) Could T be surjective?

Answer. No. We know that $\dim P_{\leq 2}(\mathbb{R}) = 3$ and $\dim P_{\leq 3}(\mathbb{R}) = 4$. However Corollary 3.6 states that $T \colon V \to W$ cannot be surjective if $\dim V < \dim W$. $\square$

c) Can we determine $T(x^2+x+1)$ from the given data?

Answer. No. For the T given in a) we compute $T(x^2+x+1) = x^2$. However we could also define

$$T(ax^2+bx+c) = bx^3 + ax^2 + (-3a+2c)x + (c-a),$$

(again we can check that $T(x^2+1) = x^2 - x$ and $T(1) = 2x+1$), in which case $T(x^2+x+1) = x^3 + x^2 - x$. Therefore $T(x^2+x+1)$ cannot be definitively determined from the given data. $\square$

d) Can we determine whether $x^2+x+1 \in \text{Image}(T)$ from the given data?

Answer. Yes, and it is indeed in the image. We have $T(x^2+2) = T(x^2+1) + T(1) = (x^2 - x) + (2x+1) = x^2 + x + 1$, so $x^2+x+1 \in \text{Image}(T)$. $\square$

Question 3 (20 points)**.** Let V be a finite-dimensional vector space, and let $T \in \mathcal{L}(V)$. Assume that

$$\text{Image}(T) \neq \text{Image}(T^2).$$

a) Prove that T is not diagonalizable.

Proof. If T is diagonalizable, then there exists a basis $v_1, \ldots, v_n$ for V such that $T(v_i) = \lambda_i v_i$ for all $i = 1, \ldots, n$. For each i, let[1]

$$c_i = \begin{cases} \frac{1}{\lambda_i} & \text{if } \lambda_i \neq 0 \\ 0 & \text{if } \lambda_i = 0 \end{cases}$$

Note that in either case we have $c_i \cdot \lambda_i^2 = \lambda_i$ (in the first case $\frac{1}{\lambda_i}\lambda_i^2 = \lambda_i$, in the second case $0 \cdot 0^2 = 0$).

We know that $\text{Image}(T^2) \subset \text{Image}(T)$ (since $\text{Image}\, TS \subset \text{Image}\, T$ for any $S \in \mathcal{L}(V)$, including $S = T$). We will prove that $\text{Image}(T) \subset \text{Image}(T^2)$ (for a contradiction). Assume that $w \in \text{Image}(T)$, so we can write $w = T(v)$ for some $v \in V$. Since $v_1, \ldots, v_n$ is a basis for V, we can write $v = a_1 v_1 + \cdots + a_n v_n$. We can then calculate

$$\begin{aligned} w = T(v) &= T(a_1 v_1 + \cdots + a_n v_n) \\ &= a_1 T(v_1) + \cdots + a_n T(v_n) \\ &= a_1 \lambda_1 v_1 + \cdots + a_n \lambda_n v_n \end{aligned}$$

Now define

$$u := a_1 c_1 v_1 + \cdots + a_n c_n v_n.$$

I claim that $T^2(u) = w$. Indeed,

$$\begin{aligned} T^2(u) &= T^2(a_1 c_1 v_1 + \cdots + a_n c_1 v_n) \\ &= a_1 c_1 T^2(v_1) + \cdots + a_n c_n T^2(v_n) \\ &= a_1 c_1 \lambda_1^2 v_1 + \cdots + a_n c_n \lambda_n^2 v_n \\ &= a_1 \lambda_1 v_1 + \cdots + a_n \lambda_n v_n \\ &= w. \end{aligned}$$

Since $w = T^2(u)$, we conclude that $w \in \text{Image}(T^2)$. Since w was an arbitrary element of $\text{Image}\, T$, this shows that $\text{Image}(T) \subset \text{Image}(T^2)$. Combined with $\text{Image}(T^2) \subset \text{Image}(T)$ this implies that $\text{Image}(T) = \text{Image}(T^2)$, contradicting the hypothesis of the question. Therefore T must not be diagonalizable. □

[1] Many students forgot to consider the case $\lambda_i = 0$. Since part b tells us that T *must* have 0 as an eigenvalue, this is an important case!

b) Which of the following is true?

(I) T must be invertible.

(II) T must be non-invertible.

(III) T could be invertible or non-invertible.

Prove your answer.

Answer. (II) is correct. If T is invertible, then T is surjective, so $\text{Image}(T) = V$. Separately, if T is invertible, then so is T^2. (Its inverse is given by $(T^{-1})^2$, as we can check by

$$T^2(T^{-1})^2 = T \cdot T \cdot T^{-1} \cdot T^{-1} = T \cdot I \cdot T^{-1} = T \cdot T^{-1} = I.)$$

But if T^2 is invertible, then it is surjective, and so $\text{Image}(T^2) = V$ as well. This contradicts the hypothesis that $\text{Image}(T) \neq \text{Image}(T^2)$. □

Question 4 (20 points)**.** Let V be a finite-dimensional vector space over $\mathbb{C}$, and let $T \in \mathcal{L}(V)$. Let U and W be subspaces such that $V = U \oplus W$. Assume that U and W are invariant under T.

(Recall that when U is an invariant subspace, $T|_U : U \to U$ is the restriction of T to U.)

a) Prove that:
if the minimal polynomial of $T|_U$ is $x - 2$ and the minimal polynomial of $T|_W$ is $(x-3)^2$, then the minimal polynomial of T is $(x-2)(x-3)^2$.

Proof. Let $p(x) = (x-2)(x-3)^2$. We first check that $p(T) = 0$ on all of V. Since $m_{T|_U}(x) = x - 2$ we know that

$$(T - 2I)(u) = (T|_U - 2I)(u) = 0$$

for all $u \in U$, and similarly since $m_{T|_W}(x) = (x-3)^2$ we know that $(T-3I)^2(w) = (T|_W - 3I)^2(w) = 0$ for all $w \in W$. Since $V = U \oplus W$, we can write any $v \in V$ as $v = u + w$ for some $u \in U$ and $w \in W$. Therefore

$$\begin{aligned} p(T)(v) &= p(T)(u+w) \\ &= p(T)(u) + p(T)(w) \\ &= (T-3I)^2(T-2I)(u) + (T-2I)(T-3I)^2(w) \\ &= (T-3I)^2(0) + (T-2I)(0) = 0. \end{aligned}$$

This shows that $p(T) = 0$. We need to show that $p(x)$ is the minimal such polynomial.

Since $m_{T|_U}(x) = x - 2$, we know that 2 is the only eigenvalue of $T|_U$, and in fact $T|_U = 2I$ when restricted to U! Therefore for any $u \in U$ we have $T(u) = T|_U(u) = 2u$; in particular, this shows that 2 is an eigenvalue of T.

Similarly, $m_{T|_W}(x) = (x-3)^2$ implies that 3 is the only eigenvalue of T on W. This gives three things: first, there exists a nonzero $w \in W$ such that $T|_W(w) = 3w$, so that 3 is an eigenvalue of T. Second, $\ker T|_W - 2I = \{0\}$ (since 2 is not an eigenvalue of $T|_W$), so $T|_W - 2I$ is invertible as an operator on W. Third, there exists some $w' \in W$ so that $T(w') \neq 3w'$, since if $T(w') = 3w'$ were true for all $w' \in W$ then $T|_W$ would have minimal polynomial $x - 3$.

Since 2 and 3 are eigenvalues of T, they must be roots of $m_T(x)$. Assume for a contradiction that the degree of $m_T(x)$ is < 3. Since $m_T(x)$ has two roots, its degree must be ≥ 2. But the only quadratic polynomial with 2 and 3 as roots is $(x-2)(x-3)$. Therefore it suffices to prove that $(T-2I)(T-3I) \neq 0$. Consider the $w' \in W$ from above with $T(w') \neq 3w'$. Let $w'' = (T-3I)(w') \neq 0$. Since $(T|_W - 2I)$ is invertible, we have $(T-2I)(w) \neq 0 \iff w \neq 0$ for $w \in W$. Applying this to w'', we conclude that $(T-2I)(T-3I)(w') \neq 0$. Therefore $(x-2)(x-3)$ cannot be the minimal polynomial of T. Therefore the minimal polynomial has degree 3, and therefore must be $p(x) = (x-2)(x-3)^2$. □

b) Prove or give a counterexample to the following statement:
if the minimal polynomial of $T|_U$ is $f(x)$ and the minimal polynomial of $T|_W$ is $g(x)$, then the minimal polynomial of T is $f(x)g(x)$.

Counterexample. The statement is false. For a counter-example, let $V = \mathbb{R}^2$, and let $U = \{(x, 0)\}$ and $W = \{(0, y)\}$; we have seen before that $V = U \oplus W$.

Let $T = I \in \mathcal{L}(V)$. Every subspace is invariant under I, so this fits the setup of the question. We have $T|_U = I$ and $T|_V = I$. Note that the minimal polynomial of the identity is $m_I(x) = x - 1$, no matter what vector space we work on. (Proof: plugging in I to $x - 1$ gives $I - I = 0$. Since the minimal polynomial of I cannot be constant, $x - 1$ must be the minimal polynomial.)

Therefore we have $f(x) = m_{T|_U}(x) = x - 1$ and $g(x) = m_{T|_W}(x) = x - 1$. However we also have $m_T(x) = x - 1$, showing that

$$m_T(x) = (x - 1) \neq (x - 1)^2 = f(x)g(x). \qquad \square$$

Question 5 (20 points)**.** Let $V = \mathbb{R}^2$ and $T \in \mathcal{L}(V)$. Prove that if $T^3 = 0$, then $T^2 = 0$.

Proof. [There are a number of different ways to prove this; here's one that arises naturally by splitting up the possibilities case-by-case.]

Since $\dim V = 2$, we know that $\operatorname{rank} T = 0$, 1, or 2; we consider these cases one at a time. If $\operatorname{rank} T = 0$ we have $T = 0$, which certainly implies $T^2 = 0$. If $\operatorname{rank} T = 2$ we have $\operatorname{Image} T = V$, so T is invertible. But then T^3 would be invertible (with inverse $(T^{-1})^3$); this contradicts the assumption that $T^3 = 0$, so we conclude that $\operatorname{rank} T \neq 2$. It remains to consider the case $\operatorname{rank} T = 1$.

If $\dim \operatorname{Image} T = 0$, the intersection $\operatorname{Image} T \cap \ker T$ either has dimension 0 or 1; we consider each case separately.

In the first case $\operatorname{Image} T \cap \ker T = \{0\}$. Choose a nonzero $v \in \operatorname{Image} T$. Since $v \notin \ker T$ we have $T(v) \neq 0$. But of course $T(v)$ lies in $\operatorname{Image} T$. Since $\operatorname{Image} T$ is 1-dimensional we must have $T(v) = \lambda v$ for some nonzero λ. But then $T^3(v) = \lambda^3 v \neq 0$, contradicting the assumption that $T^3 = 0$.

In the second case $\operatorname{Image} T \cap \ker T = \operatorname{Image} T$, which means that $\operatorname{Image} T \subset \ker T$. Therefore for any $v \in V$ the element $T(v) \in \operatorname{Image} T$ lies in $\ker T$. This means precisely that $T^2(v) = 0$ for all $v \in V$, or in other words $T^2 = 0$, as desired. □