Math 113 – Winter 2013 – Prof. Church
Final Exam: due Monday, March 18 at 3:15pm

Name: ________________________________

Student ID: ____________________________

Signature: _____________________________

Your exam should be turned in to me in my office, 383-Y (third floor of the math building). If I am not there, slide your exam under the door. Your exam **must** be handed in by 3:15pm or you will receive a zero.

This exam is open-book and open-notes, but closed-everything-else. (Needless to say, you should not discuss this exam with anyone.) In your proofs you may use any theorem from class; from Chapters 1–7 of Axler (plus Theorems 8.34 and 8.36); or from the notes on wedge vectors and determinants (available on my website). You may read the homework/midterm solutions if you like (also on my website), but you cannot quote them as a reference. When giving counterexamples, you may describe your operators either by a formula or by a matrix.

There are 5 questions worth 100 points total on this exam, plus a 10-point bonus question; you should finish the other questions before attempting the bonus question.

Questions? E-mail Prof. Church at `church@math.stanford.edu`.

1a	1b	2a	2b	3a	3b	4a	4b	4c	5a	5b	5c	5d	Bonus

Question 1 (20 points)**.** Let V be a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$, and let $S \in \mathcal{L}(V)$ and $T \in \mathcal{L}(V)$ be operators on V satisfying $ST = 0$.

1a) Prove that $\operatorname{Image} T \subset \operatorname{Null} S$.

Recall our assumptions: V is finite-dimensional;
$S \in \mathcal{L}(V)$, $T \in \mathcal{L}(V)$;
$ST = 0$.

1b) For each of the following assertions, either prove that it must hold, or give a counterexample.

I. Either $S = 0$ or $T = 0$.

II. $TS = 0$.

III. If $\det(S) = 6$, then $T = 0$.

IV. There exists a nonzero $v \in V$ such that $TS(v) = 0$.

Question 2 (15 points)**.** Let $V = \mathbb{R}^2$, and let $T \in \mathcal{L}(V)$ be an operator on V. Assume that $v \in V$ and $w \in V$ are two nonzero vectors satisfying

$$T(v) = 2v \qquad \text{and} \qquad T(w) = -w.$$

2a) Compute the determinant $\det(T^4 + T)$.

2b) Do we have enough information to determine the minimal polynomial $m_T(x)$? If so, find the minimal polynomial; if not, explain why not.

Question 3 (20 points)**.** Let V and W be finite-dimensional vector spaces over $\mathbb{R}$. Assume that $Q \in \mathcal{L}(V,W)$, $R \in \mathcal{L}(V,W)$, and $S \in \mathcal{L}(V,W)$ are each rank-1 transformations:

$$\text{rank}(Q) = 1 \qquad \text{rank}(R) = 1 \qquad \text{rank}(S) = 1$$

We'll be considering the transformation $Q + R + S \in \mathcal{L}(V,W)$, so let's give it a name: let $T \in \mathcal{L}(V,W)$ be the transformation

$$T = Q + R + S \in \mathcal{L}(V,W)$$

3a) Prove that $\text{rank}(Q + R + S) \leq 3$.

Recall our assumptions: V and W are finite-dimensional;
$Q,\ R,\ S \in \mathcal{L}(V, W)$
$\operatorname{rank}(Q) = 1$, $\operatorname{rank}(R) = 1$, $\operatorname{rank}(S) = 1$;
$T = Q + R + S$.

3b) Is the following assertion $(*)$ true?

$$\operatorname{rank}(Q + R + S) < 3 \quad \Longleftrightarrow \quad (Q,\ R,\ S) \text{ are linearly dependent in } \mathcal{L}(V, W) \qquad (*)$$

Prove or give a counterexample.

Question 4 (20 points)**.** Let V be a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$, and let $S \in \mathcal{L}(V)$ and $T \in L(V)$ be operators on V satisfying $S^2 = S$ and $T^2 = T$. Additionally, assume that

$$S + T = I.$$

4a) Prove that $ST = 0$.

Recall our assumptions: V is finite-dimensional; $S, T \in \mathcal{L}(V)$; $S^2 = S$ and $T^2 = T$; $S + T = I$.

Let $U = \operatorname{Image} S$ and $W = \operatorname{Image} T$.

4b) Prove that $W = \operatorname{Null} S$ and $U = \operatorname{Null} T$.

Recall our assumptions:	V is finite-dimensional;	$S, T \in \mathcal{L}(V)$;
	$S^2 = S$ and $T^2 = T$;	$S + T = I$;
	$U = \operatorname{Image} S$ and $W = \operatorname{Image} T$.	

4c) Prove that $V = U \oplus W$.

Question 5 (25 points)**.** If V is an inner product space over $\mathbb{R}$ or $\mathbb{C}$, we define an operator $S \in \mathcal{L}(V)$ to be *skew-self-adjoint* if it is equal to the *negative* of its adjoint:

$$S^* = -S$$

5a) Prove that *every* operator $R \in \mathcal{L}(V)$ can be written as a sum $R = T + S$ where $T \in \mathcal{L}(V)$ is self-adjoint and $S \in \mathcal{L}(V)$ is skew-self-adjoint.

For the remaining parts, assume that V is an inner product space over $\mathbb{R}$, and $S \in \mathcal{L}(V)$ is skew-self-adjoint.

5b) Prove that if S is injective, then S has no eigenvectors.

Recall our assumptions: V is an finite-dimensional inner product space over $\mathbb{R}$,
$S \in \mathcal{L}(V)$ is skew-self-adjoint.

5c) Prove that the operator $S^2 \in \mathcal{L}(V)$ is diagonalizable.

5d) Let $\mathrm{SSA}(V) \subset \mathcal{L}(V)$ be the subspace of skew-self-adjoint operators (you do not need to prove that this is a subspace).

Let V be a 3-dimensional inner product space over $\mathbb{R}$ with orthonormal basis v_1, v_2, v_3. Find an explicit basis for $\mathrm{SSA}(V)$. What is the dimension of $\mathrm{SSA}(V)$?

Question 6 (Bonus question, 10 points)**.** Let V be a finite-dimensional inner product space over $\mathbb{C}$, and let $S \in \mathcal{L}(V)$ and $T \in \mathcal{L}(V)$ be operators on V.

6a) Assume that S and T are self-adjoint operators. Prove that if $ST = TS$, then there exists an orthonormal basis $v_1, \ldots, v_n$ of V so that each basis vector v_i is both an eigenvector of S and an eigenvector of T.

Recall our assumption: V is a finite-dimensional inner product space over $\mathbb{C}$.

6b) If we only assume that S and T are *normal* operators satisfying $ST = TS$, is it true that there exists an orthonormal basis $v_1, \ldots, v_n$ of V so that each basis vector v_i is both an eigenvector of S and an eigenvector of T?

Either prove this or give a counterexample.