Math 113: Linear Algebra and Matrix Theory

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Homework 1

Due Wednesday, January 16 in class.

From Chapter 1 of the textbook: Exercises 3, 9, 14, 15.

Question 1. Let X be a set, and let W be a vector space over $\mathbb{F}$. Recall from class that W^X is the set of functions $f\colon X \to W$. Given functions $f, g \in W^X$ and $a \in \mathbb{F}$, we define $f + g \in W^X$ and $a \cdot f \in W^X$ by

$$(f+g)(x) = f(x) + g(x)$$

$$(a \cdot f)(x) = a \cdot (f(x))$$

Prove that W^X is a vector space over $\mathbb{F}$.

Question 2. Let U be the following subset of $\mathbb{F}^\infty$:

$$U = \big\{(v_1, v_2, v_3, \ldots) \in \mathbb{F}^\infty \,\big|\, v_{i+2} = v_i \text{ for all } i\big\}$$

Prove that U is a subspace of $\mathbb{F}^\infty$.

Question 3. Let $U_1, \ldots, U_m$ be subspaces of a vector space V. As we said in class, the set $U_1 + \cdots + U_m$ is defined to be

$$U_1 + \cdots + U_m = \big\{u_1 + \cdots + u_m \,\big|\, u_i \in U_i \text{ for each } i = 1, \ldots, m\big\}.$$

Prove that $U_1 + \cdots + U_m$ is a subspace of V.

Question 4. Let $U_1 = \{(a, 0, 0) \,|\, a \in \mathbb{F}\}$ and $U_2 = \{(b, b, 0) \,|\, b \in \mathbb{F}\}$. These are both subsets of $\mathbb{F}^3$.

a) Prove that U_1 and U_2 are subspaces of $\mathbb{F}^3$.

b) Prove that $U_1 + U_2 = \{(x, y, 0) | x, y \in \mathbb{F}\}$.

Question 5. Let $U_1 = \{(a, -a, 0) \mid a \in \mathbb{F}\}$, let $U_2 = \{(0, b, -b) \mid b \in \mathbb{F}\}$, and let $U_3 = \{(c, 0, -c) \mid c \in \mathbb{F}\}$. These are all subspaces of $\mathbb{F}^3$ (you may assume this without proof).

a) Describe the subspace $U_1 + U_2 + U_3$.

b) Let $W = U_1 + U_2 + U_3$. Is W the direct sum of U_1, U_2, and U_3? Prove or disprove.

Question 6. Let V be a vector space, and let $\mathcal{U}$ be a collection of subspaces of V (meaning that $\mathcal{U}$ is a set, each of whose elements $U \in \mathcal{U}$ is a subspace U of V). The *intersection* $\bigcap \mathcal{U}$ is by definition

$$\bigcap \mathcal{U} = \{v \in V \mid v \in U \text{ for every } U \in \mathcal{U}\}.$$

a) Prove that $\bigcap \mathcal{U}$ is a subspace of V.

b) Now let $W_1, \ldots, W_m$ be subspaces of V, and let $\mathcal{U}$ is the collection of subspaces containing each W_i: that is,

$$\mathcal{U} = \{U \subset V \mid U \text{ is a subspace of } V, \text{ and } W_i \subset U \text{ for each } i = 1, \ldots, m\}$$

Prove that in this case $\bigcap \mathcal{U} = W_1 + \cdots + W_m$.
(Hint for b: first prove that $\bigcap \mathcal{U} \subset W_1 + \cdots + W_m$, then prove that $W_1 + \cdots + W_m \subset \bigcap \mathcal{U}$.)