Math 113: Linear Algebra and Matrix Theory

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Homework 2

Due Wednesday, January 23 in class.

From Chapter 2 of the textbook: Exercises 1, 5, 8, 9, 11, 14, 16, 17.

For the next two questions, recall that if X is a set and W is a vector space, then W^X is the set of all functions from X to W. You proved on HW1 that W^X is a vector space.

Question 1. If V is a vector space over the field $\mathbb{F}$, the *dual vector space* V^* is the set of all functions $f\colon V \to \mathbb{F}$ that satisfy

$$f(v+w) = f(v) + f(w) \qquad \forall v, w \in V$$
$$f(a \cdot v) = a \cdot f(v) \qquad \forall v \in V,\ a \in \mathbb{F}$$

a) By definition, V^* is a subset of the vector space $\mathbb{F}^V$ of functions from V to $\mathbb{F}$. Prove that V^* is a subspace of $\mathbb{F}^V$.

b) Assume that $\dim V = n$, and that $(v_1, \ldots, v_n)$ is a basis for V. Find a basis for V^*. What is $\dim V^*$?

Question 2. If V and W are both vector spaces over $\mathbb{F}$, we say that a function $T\colon V \to W$ is *linear* if it satisfies the two properties:

$$LA: \quad T(v+w) = T(v) + T(w) \qquad \forall v, w \in V$$
$$LM: \quad T(a \cdot v) = a \cdot T(v) \qquad \forall v \in V,\ a \in \mathbb{F}$$

We define $\mathcal{L}(V, W)$ to be the subset of W^V consisting of all linear functions $T\colon V \to W$.

a) Prove that $\mathcal{L}(V, W)$ is a subspace of W^V.

b) Assume that $\dim V = 3$ and $\dim W = 2$, and furthermore assume that (v_1, v_2, v_3) is a basis for V and (w_1, w_2) is a basis for W. Find a basis for $\mathcal{L}(V, W)$. What is $\dim \mathcal{L}(V, W)$?

Question 3. Recall that $\mathbb{R}^\infty$ is the vector space whose elements are infinite sequences of real numbers $(v_1, v_2, \ldots)$, where each v_i is a real number $v_i \in \mathbb{R}$.

Let U be the subset of $\mathbb{R}^\infty$ consisting of all sequences that satisfy

$$v_i + v_{i+2} = v_{i+1} \qquad \text{for all } i.$$

a) Prove that U is a subspace of $\mathbb{R}^\infty$.

b) Let $x, y \in U$ be the elements

$$x = (0, 1, 1, 0, -1, -1, 0, 1, 1, \ldots)$$
$$y = (1, 0, -1, -1, 0, 1, 1, 0, -1, \ldots)$$

Prove that (x, y) is a linearly independent set.

c) Prove that (x, y) is a basis for U.

d) Let W be the subspace of $\mathbb{R}^\infty$ consisting of all sequences with $v_1 = 0$ and $v_2 = 0$. (You do not have to prove that W is a subspace.) Prove that $\mathbb{R}^\infty = U \oplus W$.