Math 113: Linear Algebra and Matrix Theory

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Homework 3

Due Wednesday, January 30 in class.

From Chapter 3 of the textbook: Exercises 4, 8, 9, 14, 22, 23, 24

Question 1. Assume that $T \in \mathcal{L}(V)$. Recall from class that T^2 denotes the composition $T \circ T$.

a) Give an example of a vector space V and a linear operator $T \in \mathcal{L}(V)$ such that $T^2 = T$ (an example other than $T = I$ or $T = 0$, those are too easy).

b) Prove that if $T^2 = T$, then $V = \operatorname{Null} T \oplus \operatorname{Null}(T - I)$.

c) Prove that if $V = \operatorname{Null} T + \operatorname{Null}(T - I)$, then $T^2 = T$.

d) Give an example of a vector space V and a linear operator $T \in \mathcal{L}(V)$ such that $T^2 = -I$.

Question 2. Let V and W be finite dimensional, and consider $T \in \mathcal{L}(V, W)$ and $S \in \mathcal{L}(W, U)$.

a) Prove that $\dim(\operatorname{Image} ST) \leq \dim(\operatorname{Image} T)$.

b) Prove that $\dim(\operatorname{Image} ST) = \dim(\operatorname{Image} T)$ if and only if

$$\operatorname{Image} T + \operatorname{Null} S = \operatorname{Image} T \oplus \operatorname{Null} S.$$

c) Prove that $\dim(\operatorname{Null} ST) \leq \dim(\operatorname{Null} S) + \dim(\operatorname{Null} T)$.

d) Bonus question: can you give some description (in terms of conditions on T, S, V, etc.) of when we get equality in c): $\dim(\operatorname{Null} ST) = \dim(\operatorname{Null} S) + \dim(\operatorname{Null} T)$?