Math 113: Linear Algebra and Matrix Theory

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Homework 4

Due Wednesday, February 6 in class.

From Chapter 5 of the textbook: Exercises 1, 2, 4, 5, 6, 8, 12.

Question 1. Let $C^\infty(\mathbb{R})$ denote the vector space (over $\mathbb{R}$) of infinitely-differentiable real-valued functions $f\colon \mathbb{R} \to \mathbb{R}$. [1]

a) Let U denote the subspace of $C^\infty(\mathbb{R})$ consisting of functions which vanish at 2 and at 7 (you do not have to prove that U is a subspace):

$$U = \{ f \in C^\infty(\mathbb{R}) \mid f(2) = 0, f(7) = 0 \}$$

Prove that the quotient vector space $C^\infty(\mathbb{R})/U$ is finite-dimensional. What is its dimension? (Note that $C^\infty(\mathbb{R})$ itself is very *infinite*-dimensional!)

b) Let W denote the subspace of $C^\infty(\mathbb{R})$ consisting of functions which "vanish to second order at 0":

$$W = \{ f \in C^\infty(\mathbb{R}) \mid f(0) = 0, f'(0) = 0, f''(0) = 0 \}$$

Prove that the quotient vector space $C^\infty(\mathbb{R})/W$ is finite-dimensional, and find a basis for $C^\infty(\mathbb{R})/W$.

Question 2. Let $C^\infty(\mathbb{R}, \mathbb{C})$ be the vector space (over $\mathbb{C}$) of complex-valued functions $f\colon \mathbb{R} \to \mathbb{C}$ that are infinitely differentiable. Let V be the space of functions $f \in C^\infty(\mathbb{R}, \mathbb{C})$ satisfying the differential equation $f'' = -f$:

$$V = \{ f \in C^\infty(\mathbb{R}, \mathbb{C}) \mid f'' = -f \}$$

a) Prove that V is a subspace of $C^\infty(\mathbb{R}, \mathbb{C})$.

[1] A function $f\colon \mathbb{R} \to \mathbb{R}$ is "infinitely differentiable" if f is continuous, its derivative f' exists and is continuous, *its* derivative f'' exists and is continuous, and so on. Almost all the standard functions you know are infinitely differentiable: for example, all polynomials, exponentials, sin, cos, etc. You do not need to prove that $C^\infty(\mathbb{R})$ or $C^\infty(\mathbb{R}, \mathbb{C})$ is a vector space; in fact, you shouldn't have to worry about the details of $C^\infty(\mathbb{R})$ at all.

b) If you take a course on differential equations, you'll learn how to prove that the space of solutions V is at most 2-dimensional, from the form of the differential equation $f'' = -f$. However, since this is a linear algebra course, just trust me on this, and assume without proof that $\dim V \leq 2$.

   Prove that the functions $\sin x$ and $\cos x$ both lie in V, and moreover that $(\sin x, \cos x)$ forms a basis for V. [2]

c) Let D be the operator on $C^\infty(\mathbb{R}, \mathbb{C})$ defined by $D(f) = f'$. Prove that V is an invariant subspace for D.

d) Now consider $D \in \mathcal{L}(V)$ as an operator on V (still defined by $D(f) = f'$). Find a basis for V consisting of eigenvectors for D. What are their eigenvalues?

[2] Remember your derivatives: $(\sin x)' = \cos x$, and $(\cos x)' = -\sin x$.