Math 113: Linear Algebra and Matrix Theory
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Homework 5

Due **Wednesday, February 13** in class.

These questions address material covered on the midterm.

From Chapter 5 of the textbook: 7, 11.

Question 1.

a) Give an example of an operator T on $V = \mathbb{C}^3$ whose minimal polynomial is $(x-1)^2$.

b) Give an example of an operator S on $W = \mathbb{C}^4$ whose minimal polynomial is $(x-2)(x-3)^2$.

Question 2. Let V be a finite-dimensional vector space over $\mathbb{F}$. Recall that the *dual vector space* V^* is the vector space $V^* = \mathcal{L}(V, \mathbb{F})$ of linear functions $f \colon V \to \mathbb{F}$.

a) For a fixed vector $v \in V$, let $\text{eval}_v \colon V^* \to \mathbb{F}$ be the function defined by

$$\text{eval}_v(f) = f(v).$$

(Note that eval_v is a function from V^* to $\mathbb{F}$—**not** a function from V to $\mathbb{F}$.)

Prove that eval_v is linear.

b) In part a) you proved that for any $v \in V$, we have $\text{eval}_v \in \mathcal{L}(V^*, \mathbb{F})$. Let $E \colon V \to \mathcal{L}(V^*, \mathbb{F})$ be the function defined by

$$E(v) = \text{eval}_v\,.$$

Prove that E is a linear transformation.

c) Prove that E is injective.

d) Prove that E is surjective.

Together, parts c) and d) show that E is an isomorphism between V and $\mathcal{L}(V^*, \mathbb{F})$.