Math 113: Linear Algebra and Matrix Theory
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Homework 6 (complete)

Due **Wednesday, February 20** in class.

Questions A, B, and C added, covering the wedge vector space $\bigwedge^k V$ and determinants.

Recall from class on Friday that the determinant $\det(T)$ of $T \in \mathcal{L}(V)$ is defined by

$$T(v_1) \wedge \cdots \wedge T(v_n) = \det(T) \cdot v_1 \wedge \cdots \wedge v_n \qquad (*)$$

where $v_1, \ldots, v_n$ is any basis for V.

At the very end of class on Friday, I showed the following. Assume that $\dim V = 2$ and v_1, v_2 is a basis for V, and that $T \in \mathcal{L}(V)$ satisfies $T(v_1) = av_1 + cv_2$ and $T(v_2) = bv_1 + dv_2$. (The point of this labeling is that the matrix of T w.r.t the basis v_1, v_2 is $M(T) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.) Then:

$$\begin{aligned} T(v_1) \wedge T(v_2) &= (av_1 + cv_2) \wedge (bv_1 + dv_2) \\ &= ab \cdot v_1 \wedge v_1 + ad \cdot v_1 \wedge v_2 + bc \cdot v_2 \wedge v_1 + ad \cdot v_2 \wedge v_2 \\ &= ad \cdot v_1 \wedge v_2 + bc \cdot v_2 \wedge v_1 \\ &= ad \cdot v_1 \wedge v_2 - bc \cdot v_1 \wedge v_2 \\ &= (ad - bc) v_1 \wedge v_2 \end{aligned}$$

By the defining property $(*)$ of the determinant this implies $\det(T) = ad - bc$.

Question A. Assume now that $\dim V = 3$ and v_1, v_2, v_3 is a basis for V. Let $T \in \mathcal{L}(V)$ be an operator defined by

$$T(v_1) = av_1 + dv_2 + gv_3$$

$$T(v_2) = bv_1 + ev_2 + hv_3 \qquad M(T) = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$T(v_3) = cv_1 + fv_2 + iv_3$$

Derive a formula for $\det(T)$ in terms of a, b, c, d, e, f, g, h, and i. (Hint: your formula should have six terms. It's easy to find the formula online, if you want to check that you got the right answer — but you need to derive it using wedge vectors, just as I did above.)

Question B. Prove that if $v_1, \ldots, v_k$ are linearly dependent, then $v_1 \wedge \cdots \wedge v_k = 0$ in $\bigwedge^k V$.

(This is one direction of the Wedge Dependence Lemma I stated in class; for the other direction, we'll need a theorem that I'll cover on Wednesday.)

Question C. Assume that $\dim V = n$, and that $v_1, \ldots, v_n$ is a basis for V. Prove that $\dim \bigwedge^n V \leq 1$ by finding a single wedge vector that spans $\bigwedge^n V$. (As I mentioned in class on Friday, in fact $\dim \bigwedge^n V = 1$; we'll see this on Wednesday as well.)

A vector $v = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$ is called a *probability vector*[1] if each entry v_i is ≥ 0, and $v_1 + \cdots + v_n = 1$.

Question 1. If you flip a quarter and a penny, what are the possible outcomes? Listing the quarter first, we could have HH (both heads), HT (quarter heads), TH (penny heads), or TT (both tails). If both coins are fair (probability vector $= (\frac{1}{2}, \frac{1}{2})$ for both), the resulting probabilities are:

	H	T
H	$\frac{1}{4}$	$\frac{1}{4}$
T	$\frac{1}{4}$	$\frac{1}{4}$

We can write this as the probability vector $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) \in \mathbb{R}^4$.

However, if both coins are weighted to come up Heads two-thirds of the time (probability vector $= (\frac{2}{3}, \frac{1}{3})$ for both), then the resulting probabilities are:

	H	T
H	$\frac{4}{9}$	$\frac{2}{9}$
T	$\frac{2}{9}$	$\frac{1}{9}$

We can write this as the probability vector $(\frac{4}{9}, \frac{2}{9}, \frac{2}{9}, \frac{1}{9}) \in \mathbb{R}^4$.

a) If the quarter is weighted to come up Heads four-fifths of the time, and the penny is weighted to come up Heads three-sevenths of the time, what is the resulting probability vector? $(HH, HT, TH, TT) = (?, ?, ?, ?)$ You do not have to justify your answer.

[1] These vectors are used to model situations where n different possibilities occur with various probabilities. For example, for a fair coin, which comes up Heads half the time and Tails half the time (2 possibilities), we would take $n = 2$, and use the vector $v = (\frac{1}{2}, \frac{1}{2})$. If your coin was weighted to come up Heads two-thirds of the time (cheater!), you would use the vector $v = (\frac{2}{3}, \frac{1}{3})$. Probability vectors are also used to model proportions in a large population: for example, if 0.36% of the population has HIV (2008 CDC statistics), we could denote this by $(0.9964, 0.0036)$.

b) If $v = (v_1, v_2) \in \mathbb{R}^2$ represents the probability vector for the quarter, and $w = (w_1, w_2) \in \mathbb{R}^2$ represents the probability vector for the penny, let $J(v, w) \in \mathbb{R}^4$ denote the probability vector for both coins (J stands for the *joint distribution*). Give a formula for $J(v, w)$ in terms of v_1, v_2, w_1, and w_2. Is J a linear transformation?

c) Prove that $(0, \frac{1}{2}, \frac{1}{2}, 0)$ is not in the image of J: there does not exist any $v, w \in \mathbb{R}^2$ such that $J(v, w) = (0, \frac{1}{2}, \frac{1}{2}, 0)$.

d) Is $u = (1, 0, 0, 0)$ in the image of J? How about $u' = (\frac{1}{4}, \frac{1}{12}, \frac{1}{2}, \frac{1}{6})$? How about $u'' = (\frac{3}{10}, \frac{2}{10}, \frac{2}{10}, \frac{3}{10})$?

Question 2. Let V be a finite-dimensional vector space over $\mathbb{C}$, and let $T \in \mathcal{L}(V)$. Let U and W be nonzero subspaces such that $V = U \oplus W$.

Assume that U and W are invariant under T, so we can *restrict* the operator $T \colon V \to V$ to an operator $T|_U \colon U \to U$, and similarly we can restrict T to an operator $T|_W \colon W \to W$.

a) Prove that if $\lambda \in \mathbb{C}$ is an eigenvalue of T, then either λ is an eigenvalue of $T|_U$ or λ is an eigenvalue of $T|_W$ (or both).

[Hint: start with a nonzero eigenvector $v \in V$ such that $T(v) = \lambda v$, and somehow construct either an eigenvector $u \in U$ such that $T(u) = \lambda u$, or an eigenvector $w \in W$ such that $T(w) = \lambda w$.]

Let $f(x)$ be the minimal polynomial of $T|_U$, and let $g(x)$ be the minimal polynomial of $T|_W$.

b) Prove that $f(T)g(T) = 0$ in $\mathcal{L}(V)$.

c) Prove that if $f(x)$ and $g(x)$ have no shared roots (meaning no $\lambda \in \mathbb{C}$ is a root of both $f(x)$ and $g(x)$), then $f(x)g(x)$ is the minimal polynomial of T.

d) Prove that if $f(x)$ and $g(x)$ have a shared root $\lambda \in \mathbb{C}$, then $f(x)g(x)$ is **not** the minimal polynomial of T.

Question 3. Let X be a set. If Y_1 and Y_2 are two subsets of X, their *exclusive union* is the subset defined by:

$$Y_1 \,\Delta\, Y_2 = \left\{ \; x \in X \;\middle|\; x \text{ lies either in } Y_1 \text{ or } Y_2 \textbf{ but not both } \right\}$$

For example, $\{A, B, C, D\} \,\Delta\, \{C, D, E, F\} = \{A, B, E, F\}$.

Let V_X be the collection of all subsets of X:

$$V_X = \left\{ \; Y \;\middle|\; Y \subset X \; \right\}$$

We can make V_X into a vector space over[2] the field $\mathbb{F}_2 = \{\mathbb{0}, \mathbb{1}\}$, by defining addition of $Y_1 \in V_X$ and $Y_2 \in V_X$ to be $Y_1 \,\Delta\, Y_2$ and defining scalar multiplication by:

$$\mathbb{0} * Y = \emptyset \qquad\qquad \mathbb{1} * Y = Y$$

This makes V_X into a vector space (you may assume this without proof).

a) What is the additive identity in the vector space V_X? Prove your answer.

b) Let Y be an element of V_X (in other words, Y is a subset of X).
What is the additive inverse of Y? Prove your answer.

For the remaining two parts, let X be a set with three elements: $X = \{A, B, C\}$.
Note that V_X has eight elements:

$$V_X = \Big\{ \;\; \{\}, \;\; \{A\}, \;\; \{B\}, \;\; \{C\}, \;\; \{A, B\}, \;\; \{A, C\}, \;\; \{B, C\}, \;\; \{A, B, C\} \;\; \Big\}$$

c) Let $Y_1 = \{A, B\}$, $Y_2 = \{A, C\}$, and $Y_3 = \{B, C\}$. Are Y_1, Y_2, and Y_3 linearly independent? Prove your answer.

d) Find a basis for V_X in this case. What is the dimension of V_X? Prove your answer.

[2] Recall that $\mathbb{F}_2$ is the field with two elements $\mathbb{F}_2 = \{\mathbb{0}, \mathbb{1}\}$ with operations defined by

$$\begin{aligned} \mathbb{0} + \mathbb{0} &= \mathbb{0} & \qquad \mathbb{0} \cdot \mathbb{0} &= \mathbb{0} \\ \mathbb{0} + \mathbb{1} &= \mathbb{1} & \qquad \mathbb{0} \cdot \mathbb{1} &= \mathbb{0} \\ \mathbb{1} + \mathbb{0} &= \mathbb{1} & \qquad \mathbb{1} \cdot \mathbb{0} &= \mathbb{0} \\ \mathbb{1} + \mathbb{1} &= \mathbb{0} & \qquad \mathbb{1} \cdot \mathbb{1} &= \mathbb{1} \end{aligned}$$