# Math 113: Linear Algebra and Matrix Theory

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# Homework 7

Due **Wednesday, February 27** in class.

**Question 1.** Let $V$ be a vector space with $\dim V = n$. Let $U$ be a subspace of $V$ with $\dim U = k$, and assume that $u_1, \ldots, u_k$ is a basis for $U$.

a) Prove that if $w_1, \ldots, w_k$ is another basis for $U$, then

$$w_1 \wedge \cdots \wedge w_k = a \cdot u_1 \wedge \cdots \wedge u_k \qquad \text{for some nonzero } a \in \mathbb{F}.$$

b) Let $W$ be another subspace of $V$, and assume that $w_1, \ldots, w_k$ is a basis for $W$. (We have dropped the assumption from part a) that $w_i \in U$.)

Prove that if

$$w_1 \wedge \cdots \wedge w_k = a \cdot u_1 \wedge \cdots \wedge u_k \qquad \text{for some nonzero } a \in \mathbb{F},$$

then $U = W$.

[Hint: start with a basis $v_1, \ldots, v_\ell$ for $U \cap W$, then extend it to a basis $v_1, \ldots, v_n$ for $V$.]

**Question 2.** Let $v_1, \ldots, v_n$ be a basis for $V$. We say that an operator $T \in \mathcal{L}(V)$ is "upper-triangular with respect to the basis $v_1, \ldots, v_n$" if

$$T(v_i) \in \operatorname{span}(v_1, \ldots, v_i) \quad \text{for all } i = 1, \ldots, n.$$

Assume that $T$ is upper-triangular w.r.t. the basis $v_1, \ldots, v_n$, so for each $i$ we can write

$$T(v_i) = d_i \cdot v_i + w_i \qquad \text{for some } d_i \in \mathbb{F} \text{ and } w_i \in \operatorname{span}(v_1, \ldots, v_{i-1}).$$

a) Prove that $\det(T) = d_1 \cdot d_2 \cdot \cdots \cdot d_n$.

b) Prove that each number $d_i$ is an eigenvalue of $T$. Note that the vectors $v_i$ are almost certainly **not** eigenvectors of T!

[Hint: I do not think a direct approach is best here. First think about how you would prove it when $d_i = 0$, then reduce the general case to this.]

**Question 3.** Let $V$ and $W$ be finite-dimensional vector spaces, and let $S\colon V \to W$ be a linear transformation. Let $S^\top\colon W^* \to V^*$ (pronounced "$S$-transpose") be defined as follows.[1] If $f \in W^*$ is a linear transformation $f\colon W \to \mathbb{F}$, then $S^\top(f) \in V^*$ is the linear transformation $V \to \mathbb{F}$ defined by

$$S^\top(f)(v) = f(\,S(v)\,).$$

(You do not need to prove that $S^\top(f)\colon V \to \mathbb{F}$ is linear, though you should understand why this is true.)

a) Prove that $S^\top$ is a linear transformation from $W^*$ to $V^*$.

b) Let Transpose: $\mathcal{L}(V, W) \to \mathcal{L}(W^*, V^*)$ be the function defined by

$$\text{Transpose}(S) = S^\top.$$

Prove that Transpose is a linear transformation from $\mathcal{L}(V, W)$ to $\mathcal{L}(V^*, W^*)$.

c) Prove that $0^\top = 0$ and $I^\top = I$ (this should not be difficult).

d) If $S \in \mathcal{L}(V, W)$ and $R \in \mathcal{L}(W, U)$, prove that

$$(R \circ S)^\top = S^\top \circ R^\top.$$

**Question 4.** Let $V$ be a finite-dimensional vector space. Given an operator $S \in \mathcal{L}(V)$, we have the operator $S^\top \in \mathcal{L}(V^*)$ defined in Question 2.

a) Prove that $\det(S^\top) = \det(S)$. [Hint: choose a basis $v_1, \ldots, v_n$ for $V$, and let $f_1, \ldots, f_n$ be the dual basis for $V^*$ defined by $f_i(v_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ that you found on HW2, Question 1b.]

b) Prove that $S^\top$ has the same minimal polynomial as $S$: i.e. prove that $m_{S^\top}(x) = m_S(x)$. (Note that this implies that $S$ and $S^\top$ have the same eigenvalues!)

[1] Recall that $V^* = \mathcal{L}(V, \mathbb{F})$.

**Question 5.** Recall from HW6 that a vector $v = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$ is called a *probability vector* if each entry $v_i$ is $\geq 0$, and $v_1 + \cdots + v_n = 1$. A matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ is called a *probability matrix* if each column of $A$ is a probability vector.

a) Prove that if $A$ and $B$ in $\mathrm{Mat}_{n\times n}(\mathbb{R})$ are both probability matrices, then their product $AB$ is also a probability matrix. [Hint: there is a smarter solution than just multiplying out the matrices.]

b) Let $T \in \mathcal{L}(\mathbb{R}^n)$, and let $A$ be its matrix (w.r.t. the standard basis $e_1, \ldots, e_n$). Prove that if $A$ is a probability matrix, then 1 is an eigenvalue of $T$.

c) Bonus question, for no points: prove that 1 is the *largest* eigenvalue of $T$.