Math 113: Linear Algebra and Matrix Theory

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Homework 8

Due **Wednesday, March 6** in class.

From Chapter 6 of the textbook: Exercises 2, 4, 5, 10, 15, 21, 24, 28, 29.

(In this chapter Axler has made the assumption that V is always a finite-dimensional inner product space over $\mathbb{R}$ or $\mathbb{C}$.)

[Edit Friday 3/1: For Exercises 2 and 4 you may assume that V is an inner product space over $\mathbb{R}$. Exercises 10, 15, and 29 use material we will cover on Monday, namely the Gram-Schmidt algorithm and the orthogonal complement, respectively. If you want to get started on these early, they are covered in the book on page 108 and page 111, respectively.]

Question 1. Fix an integer $n \geq 1$, and let $V = \mathbb{C}^n$ with the standard inner product. We let $R\colon V \to V$ be the operator defined by

$$R(a_1, \ldots, a_n) = (a_2, \ldots, a_n, a_1).$$

a) Prove that the characteristic polynomial of R is $\chi_R(x) = x^n - 1$.

This means that the eigenvalues of R are the roots of $x^n - 1$; since you might not be familiar with these awesome numbers (called "roots of unity"), here are the relevant facts.

Let $\lambda \in \mathbb{C}$ be the complex number $\cos(\frac{2\pi}{n}) + i\sin(\frac{2\pi}{n})$. Then $x^n - 1$ factors as

$$(x-1)(x-\lambda)(x-\lambda^2)\cdots(x-\lambda^{n-1})$$

All the roots $1, \lambda, \lambda^2, \ldots, \lambda^{n-1}$ are on the unit circle in $\mathbb{C}$ (meaning $z\overline{z} = 1$), and in fact they are equally spaced around the unit circle until you get back to $\lambda^n = 1$.

b) Since $\chi_R(x)$ has n distinct roots, we know that R is diagonalizable.

Diagonalize R by finding a basis of eigenvectors $v_1, \ldots, v_n$ for $\mathbb{C}^n$ satisfying

$$R(v_i) = \lambda^i \cdot v_i \quad \text{and} \quad ||v_i|| = 1.$$

c) Prove that if $\mu \in \mathbb{C}$ satisfies $\mu^n = 1$ but $\mu \neq 1$, then $1 + \mu + \mu^2 + \cdots + \mu^{n-1} = 0$.

[Hint: multiply by $\mu - 1$.]

d) Prove that your basis $v_1, \ldots, v_n$ is orthonormal.

e) If $v = (a_1, \ldots, a_n)$ is written as $v = b_1 v_1 + \cdots + b_n v_n$, give a formula for the coefficient b_i in terms of the coordinates $a_1, \ldots, a_n$. [Hint: use part d).]

f) If $v = (a_1, \ldots, a_n)$ is written as $v = b_1 v_1 + \cdots + b_n v_n$, give a formula for the coordinate a_i in terms of the coefficients $b_1, \ldots, b_n$. [Hint: this is easy.]

g) If $v = (a_1, \ldots, a_n)$ is written as $v = b_1 v_1 + \cdots + b_n v_n$, prove that the coordinates $a_1, \ldots, a_n$ and the coefficients $b_1, \ldots, b_n$ satisfy the relation

$$|a_1|^2 + \cdots + |a_n|^2 = |b_1|^2 + \cdots + |b_n|^2.$$

Historical Remark. The formula you found in e) is the Fourier transform, or rather a discretized version of it; the formula you found in f) is the inverse Fourier transform. The equality you proved in g) is a discrete version of the following famous theorem:

If $f\colon [-\pi, \pi] \to \mathbb{C}$ is a continuous function with $f(-\pi) = f(\pi)$, let $b_k \in \mathbb{C}$ be the sequence defined (for $k \in \mathbb{Z}$) by

$$b_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx.$$

Then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2\, dx = \sum_{k=-\infty}^{\infty} |b_k|^2.$$

Question 2. Let V be a finite dimensional inner product space over $\mathbb{F}$ (either $\mathbb{R}$ or $\mathbb{C}$). If we are given a basis $v_1, \ldots, v_n$ for V, let $g_1, \ldots, g_n$ in V^* be the functions $g_i\colon V \to \mathbb{F}$ defined by $g_i(v) = \langle v, v_i \rangle$.

a) Prove that g_i is a basis for V^*.

For the next part, recall that the dual basis $f_1, \ldots, f_n$ of V^* is given by $f_i(v_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.

b) Prove that the basis $(f_1, \ldots, f_n)$ is equal to the basis $(g_1, \ldots, g_n)$ if and only if $v_1, \ldots, v_n$ is an *orthonormal* basis for V.