Thomas Church (church@math.stanford.edu)
math.stanford.edu/~church/teaching/113/

Homework 9

Due **Wednesday, March 13** in class.

From Chapter 6 of the textbook: Exercises 13, 31, 32
From Chapter 7 of the textbook: Exercises 1, 3, 6, 8, 16

Question 1.

a) Give an example of two self-adjoint operators $S \in \mathcal{L}(\mathbb{R}^2)$ and $T \in \mathcal{L}(\mathbb{R}^2)$ whose product ST is not self-adjoint.

Let V be a finite-dimensional inner product space, and assume that $S \in \mathcal{L}(V)$ and $T \in \mathcal{L}(V)$ are self-adjoint.

b) Prove that $ST + TS$ is a self-adjoint operator.

c) Prove that ST is self-adjoint if and only if $ST = TS$.

Question 2. Let $V = \mathbb{C}^2$, and define three operators on V by

$$S_z(a, b) = (a, -b) \qquad S_x(a, b) = (b, a) \qquad S_y(a, b) = (-ib, ia).$$

The names are traditional, though these operators are usually described by giving their matrices w.r.t. the standard basis:

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

a) Prove by hand[1] that each of these operators is self-adjoint (w.r.t. the standard inner product on $\mathbb{C}^2$ given by $\langle (a, b), (c, d) \rangle = a\bar{c} + b\bar{d}$).

b) Prove that each of these operators has minimal polynomial $x^2 - 1$.

c) Find an orthonormal basis of eigenvectors v_1^z, v_2^z for S_z, and orthonormal basis of eigenvectors v_1^x, v_2^x for S_x, and an orthonormal basis of eigenvectors v_1^y, v_2^y for S_y.

[1] meaning don't just say "the matrices are conjugate-symmetric"

d) Fill out the following table by computing (the absolute values of) the inner products:

$\lvert\langle v_1^z, v_1^x\rangle\rvert =$	$\lvert\langle v_2^z, v_1^x\rangle\rvert =$	$\lvert\langle v_1^x, v_1^y\rangle\rvert =$
$\lvert\langle v_1^z, v_2^x\rangle\rvert =$	$\lvert\langle v_2^z, v_2^x\rangle\rvert =$	$\lvert\langle v_1^x, v_2^y\rangle\rvert =$
$\lvert\langle v_1^z, v_1^y\rangle\rvert =$	$\lvert\langle v_2^z, v_1^y\rangle\rvert =$	$\lvert\langle v_2^x, v_1^y\rangle\rvert =$
$\lvert\langle v_1^z, v_2^y\rangle\rvert =$	$\lvert\langle v_2^z, v_2^y\rangle\rvert =$	$\lvert\langle v_2^x, v_2^y\rangle\rvert =$

[Hint: this will be much easier than you think.]

Question 3. Let V be the vector space of infinite sequences of real numbers:

$$V = \left\{ (a_1, a_2, a_3, \ldots) \mid a_i \in \mathbb{R} \right\}$$

This is an infinite-dimensional vector space over $\mathbb{R}$.

On Homework 4 we considered the backward shift on V, so now let's consider the forwards shift: let $T \in \mathcal{L}(V)$ be the operator defined by

$$T(a_1, a_2, a_3, \ldots) = (0, a_1, a_2, \ldots).$$

a) The operator $T + I$ is given by

$$(T + I)(a_1, a_2, a_3, a_4, \ldots) = (a_1, a_1 + a_2, a_2 + a_3, a_3 + a_4, \ldots).$$

Find an inverse $(T + I)^{-1}$ for this operator.

b) For which values of $\lambda \in \mathbb{R}$ is the operator $T - \lambda I$ non-invertible? Try to prove your answer is correct; if you cannot prove it completely, give as much justification as you can.

c) What are the eigenvalues of T?

d) Explain the discrepancy between your answers to b) and c).