Math 113 – Winter 2013 – Prof. Church
Midterm Exam 2/11/2013

Name: ______________________________

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Question 1 (20 points)**.** Let V be a finite-dimensional vector space, and let $T \in \mathcal{L}(V, W)$. Assume that $v_1, \ldots, v_n$ is a basis for V. (For this question only, do not use the Rank-Nullity Theorem.)

a) Prove that T is injective if and only if $T(v_1), \ldots, T(v_n)$ are linearly independent in W.

b) Prove that T is surjective if and only if $T(v_1), \ldots, T(v_n)$ spans W.

Question 2 (20 points)**.** We consider a linear transformation $T \in \mathcal{L}\big(P_{\leq 2}(\mathbb{R}), P_{\leq 3}(\mathbb{R})\big)$. Assume that we are given partial data about T:

$$\begin{aligned} T(x^2+1) &= x^2 - x \\ T(1) &= 2x+1 \end{aligned}$$

Given this partial data, answer the following questions. Justify your answers.

a) Could T be injective?

b) Could T be surjective?

c) Can we determine $T(x^2+x+1)$ from the given data?

d) Can we determine whether $x^2+x+1 \in \text{Image}(T)$ from the given data?

Question 3 (20 points)**.** Let V be a finite-dimensional vector space, and let $T \in \mathcal{L}(V)$. Assume that

$$\text{Image}(T) \neq \text{Image}(T^2).$$

a) Prove that T is not diagonalizable.

b) Which of the following is true?

(I) T must be invertible.

(II) T must be non-invertible.

(III) T could be invertible or non-invertible.

Prove your answer.

Question 4 (20 points). Let V be a finite-dimensional vector space over $\mathbb{C}$, and let $T \in \mathcal{L}(V)$. Let U and W be subspaces such that $V = U \oplus W$. Assume that U and W are invariant under T.

(Recall that when U is an invariant subspace, $T|_U : U \to U$ is the restriction of T to U.)

a) Prove that:
if the minimal polynomial of $T|_U$ is $x - 2$ and the minimal polynomial of $T|_W$ is $(x-3)^2$, then the minimal polynomial of T is $(x-2)(x-3)^2$.

b) Prove or give a counterexample to the following statement:
if the minimal polynomial of $T|_U$ is $f(x)$ and the minimal polynomial of $T|_W$ is $g(x)$, then the minimal polynomial of T is $f(x)g(x)$.

Question 5 (20 points)**.** Let $V = \mathbb{R}^2$ and $T \in \mathcal{L}(V)$. Prove that if $T^3 = 0$, then $T^2 = 0$.

(This is a special case of the fact I mentioned in class that the degree of the minimal polynomial is $\leq \dim V$. However, we haven't gotten to that yet, so you can't quote me on that!)