Math 113-40, Mr. Church, Homework 2

Please staple your homework.

Part A: Due at the beginning of class on Friday, January 22.

1. For this question we will work in D_{10}, the group of symmetries of the pentagon. The standard form for these symmetries that we established in class is given in the following list:
$$I, R, R^2, R^3, R^4, F, FR, FR^2, FR^3, FR^4$$
Use the relations $R^5 = I$, $F^2 = I$, and $RF = FR^4$ to write each of the following in one of the standard forms.
   (a) FRF
   (b) $(R^2)(R^4)(R^3)$
   (c) $(FR)(R^2)(F)(R)(FR)$
   (d) $RFRF$

2. Recall that $\mathbb{Z}_6$ is the cyclic group with elements I, R, R^2, R^3, R^4, R^5 with the single relation $R^6 = I$. The element R clearly generates this group (meaning every element is obtained by combining R with itself).
   (a) Does R^2 generate the group?
   (b) Does R^3 generate the group?
   (c) Does R^5 generate the group?
   In each case, explain why or why not.

3. How do we know that $\mathbb{Z}_4$ and $\mathbb{Z}_6$ are different groups? That's easy: the former has four elements and the latter has six elements, so they much be different. The Klein 4-group K_4 is the group of symmetries of a long skinny rectangle; it has four elements (the identity, the horizontal and vertical flips, and the 180-degree rotation).
   Question: The cylic group $\mathbb{Z}_4$ also has four elements. So how can we be sure that these are really different groups? Try to find some property that K_4 has but $\mathbb{Z}_4$ could never have, no matter how you relabel the elements (or vice versa).
   [Note that this is a vaguer question than most, so even if you don't get it I'd like to hear any ideas or attempts you have.]

Part B on next page.

Part B: due at the beginning of class on Monday, January 25.

4. (a) If f is the permutation $(1\,2\,3\,4)$, what is the permutation $f \circ f$?
   (b) If g is the permutation $(1\,2\,3\,4\,5)$, what is the permutation $g \circ g$?
   (c) If h is the permutation $(1\,2\,3)(4\,5)$, what is the permutation $h \circ h$?

5. Recall that in a group where the identity ("do nothing") element is called I, the *inverse* of an element f is some other element g so that $f \circ g = I$.
   (a) What is the inverse of the permutation $(1\,2\,3)$?
   (b) What is the inverse of the permutation $(1\,2\,3\,4)$?

6. Recall that inside the group of symmetries of a square, we had a subgroup whose elements were $\{I, H, V, R^2\}$: the identity, a horizontal flip, a vertical flip, and a 180° rotation.

   With the labeling of the square from class (it's also in the book):
   (a) For each of these four elements, draw the arrow diagram for the associated permutation (as we did on Wednesday in class).
   (b) For each of these four elements, write the permutation in cycle-decomposition form.

7. [Difficult] If f is a permutation so that $f \circ f$ is the identity, what can you conclude about the cycle decomposition of f? Can you say anything about how many cycles f splits into? Or about the length of these cycles?

8. [Bonus question, very difficult, only solve if you want] A *adjacent transposition* is a permutation which switches two adjacent numbers: for example, $(1\,2)$ is an adjacent transposition, as are $(2\,3)$, $(3\,4)$, and $(7\,8)$. [1]

   Prove that S_n is generated by adjacent transpositions.
   [Hint: one possibility is to give some sort of pictorial proof.]

[1] There are $n-1$ adjacent transpositions, starting with $(1\,2)$, $(2\,3)$, $(3\,4)$, etc. up to $(n-1\; n)$.