Math 153-37, Mr. Church, Homework 13
Due at the beginning of class on Friday, February 27.
Please staple your homework.

On the midterm, you were unable to apply the integral test to the series $\sum_{k=1}^{\infty} \frac{1}{k!}$ because we didn't know any continuous function $f(x)$ on the real numbers so that $f(k) = \frac{1}{k!}$. The problem is that we only know how to define the factorial function for natural numbres. For example, what should be the value of $\left(\frac{1}{2}\right)!$, or $\sqrt{3}!$, or $\pi!$, and so on? In the following problems, we answer these questions.

1. Show that the improper integral $\int_0^\infty e^{-t}\,dt = 1$.

2. By using integration by parts three times, show that $\int_0^\infty t^3 e^{-t}\,dt = 6$. You may find useful the following form of integration by parts for improper integrals:

$$\int_0^\infty u\,dv = \big[uv\big]_0^\infty - \int_0^\infty v\,du,$$

where by $\big[uv\big]_0^\infty$ we mean $\lim_{b\to\infty} \big(u(b)v(b) - u(0)v(0)\big)$.

Since $0! = 1$ and $3! = 3 \cdot 2 \cdot 1 = 6$, this suggests that $\int_0^\infty t^n e^{-t}\,dt = n!$ for any natural number n, and indeed this is true. Now we make a leap, and make the following definition:

$$\Pi(x) = \int_0^\infty t^x e^{-t}\,dt,$$

at least for those x where this improper integral converges. We have already observed that $\Pi(0) = 1$, that $\Pi(3) = 6$, and that in general $\Pi(n) = n!$ when n is a natural number.

We want to show that this definition works for other values of x, but we need to know that the improper integral in the definition converges. First we consider the question of convergence when $0 < x < 1$.

3. In this question, we assume that $0 < x < 1$.

   (a) Show that when $1 \le t$ we have $t^x e^{-t} < te^{-t}$, and use this to show that $\int_1^\infty t^x e^{-t}\,dt$ converges.

   (b) Conclude that $\int_0^\infty t^x e^{-t}\,dt$ converges. [Note that $\int_0^1 t^x e^{-t}\,dt$ is a definite integral (that is, not an improper integral) and thus it automatically converges.]

Now we have shown that the definition $\Pi(x)$ converges when $0 \le x \le 1$; how can we use this to show the same thing for other x? Answer: the same way that we related $\Pi(3)$ to $\Pi(0)$.

4. (a) Use integration by parts to show that

$$\int_0^\infty t^{\sqrt{2}} e^{-t}\, dt = \sqrt{2} \int_0^\infty t^{\sqrt{2}-1} e^{-t}\, dt.$$

[Hint: set $u = t^{\sqrt{2}}$, $dv = e^{-t}\, dt$.]

(b) In part (a) you showed that $\Pi(\sqrt{2}) = \sqrt{2} \cdot \Pi(\sqrt{2} - 1)$. Using the same method, show in general that $\Pi(x) = x \cdot \Pi(x-1)$ for any $x \neq 0$. Note that when x appears in an integral with respect to t, you can regard x as a constant.

By using the equation $\Pi(x) = x \cdot \Pi(x-1)$, we find that $\Pi(x)$ is defined for all real numbers except the negative integers. Here we will end the homework, finishing with some truly surprising facts about this function.

The whole point of defining $\Pi(x)$ was so we could extend the factorial function to real numbers other than whole numbers. So for the rest of this sheet, let us use the notation $x!$ to mean $\Pi(x)$.

- In the introduction we asked what $\left(\frac{1}{2}\right)!$ should be. It is possible to show directly from the definition that $\left(\frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2}$.
- For any non-integer x, we can multiply $x!$ by $(-x)!$. There is no reason to expect that we would get anything nice, but in fact Euler showed that

$$x! \cdot (-x)! = \frac{\pi x}{\sin(\pi x)}.$$

- Another equivalent definition of $\Pi(x)$ due to Euler is given by

$$x! = \lim_{n \to \infty} \frac{n!(n+1)^{x+1}}{(x+1)(x+2)\cdots(x+n+1)}.$$

(Each of these results is crazy, and I want to put exclamation points after each one of them, except that it would make things very confusing.)