Math 196-47, Mr. Church, Homework 12

Due at the beginning of class on Monday, June 1.

Please staple your homework.

1. Exercise 8.1.2, duplicated here for your convenience.

   For each of the following functions, determine whether the functions is linear; you do not need to justify your answer. If the function is linear, find the standard matrix.

   (a) $T_1\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T_1(x, y) = (x, y^2)$.

   (b) $T_2\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T_2(x, y) = (x - y, y - x)$.

   (c) $T_3\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T_3(x, y) = (0, -xy)$.

   (d) $T_4\colon \mathbb{R}^2 \to \mathbb{R}^4$ defined by $T_4(x, y) = (0, 0, 0, x)$.

   (e) $T_5\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T_5(x, y, z) = (7x - 3, 2y)$.

   (f) $T_6\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T_6(x, y, z) = (\sin x, \cos y)$.

   (g) $T_7\colon \mathbb{R}^3 \to \mathbb{R}^4$ defined by $T_7(x, y, z) = (x, x, x, x)$.

2. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T(x, y, z) = (x-z, x+y+z)$, and $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $S(t, u) = (2t + u, t + u)$. Another way to write this is

$$T\colon \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x - z \\ x + y + z \end{pmatrix} \qquad\qquad S\colon \begin{pmatrix} t \\ u \end{pmatrix} \mapsto \begin{pmatrix} 2t + u \\ t + u \end{pmatrix}$$

   (a) Find the standard matrix of T and of S. (Recall that we use $[T]$ and $[S]$ to denote these matrices.)

   (b) Define $R\colon \mathbb{R}^3 \to \mathbb{R}^2$ by $R = S \circ T$, that is $R(x, y, z) = S\big(T(x, y, z)\big)$. Write down the explicit formula for $R(x, y, z)$ (like the definitions of T and S above).

   (c) Find the standard matrix of R.

   (d) Can you relate the three matrices $[R]$, $[S]$, and $[T]$?

3. (a) Prove (without reference to matrices) that if $T\colon \mathbb{R}^n \to \mathbb{R}^m$ and $S\colon \mathbb{R}^m \to \mathbb{R}^\ell$ are linear transformations, then $\ker T \subset \ker(S \circ T)$.

   (b) Check this for the explicit linear transformations in Question 2; that is, compute $\ker T$ and $\ker R$, and check that any element of $\ker T$ is also contained in $\ker R$.

4. [Will appear on HW 13, due Wednesday.] Consider $\mathbb{R}^3$ as 3-dimensional space in the usual way, with $e_1 = (1, 0, 0)$ the unit vector in the x direction, $e_2 = (0, 1, 0)$ the unit vector in the y direction, and $e_3 = (0, 0, 1)$.

   Let R be the counter-clockwise rotation by $45°$ around the z axis. This is a linear transformation (you don't have to prove this). Find the standard matrix of R.