Math 210A: Modern Algebra

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Homework 1

Due Saturday, September 30 by 5:00pm

Do all the **unstarred questions** below. (Starred questions are important and you really should do them, but they will not be collected or graded.)

Reminder: when we just say "R-module" we mean "left R-module", and we allow R to not be commutative. But in these questions, it doesn't really matter whether R is commutative or not.

The material needed for this assignment (R-modules, homomorphisms, submodules) is covered in: Lang p117; or Atiyah–Macdonald pp17–18; or Dummit–Foote p337.

Question 1. Given an R-module M and a subset $S \subset M$, prove that the following are equivalent.

(A) "*Every element of M is an R-linear combination of elements of S*":
For all $m \in M$ there exist $r_1, \ldots, r_k \in R$ and $s_1, \ldots, s_k \in S$ such that $m = \sum_{i=1}^{k} r_i s_i$.

(B) "*Homomorphisms are determined by their value on S*":
For any R-module N and any homomorphisms $f\colon M \to N$ and $g\colon M \to N$,

$$f|_S = g|_S \quad \Longrightarrow \quad f = g.$$

(C) "*Any homomorphism whose image contains S is a surjection*":
For any R-module L and any homomorphism $h\colon L \to M$,

$$h(L) \supseteq S \quad \Longrightarrow \quad h(L) = M.$$

When these equivalent conditions hold, we say that S *generates* M (or S *spans* M, or M *is generated by* S, or M *is spanned by* S).

Question 2. We say an R-module M is *finitely generated* if there exists a finite set $S \subset M$ that generates M. Prove that if M is finitely generated, then it has the following property:

For any infinite chain $N_1 \subseteq N_2 \subseteq \cdots \subseteq M$ of submodules of M whose union $\bigcup_{i \in \mathbb{N}} N_i = M$ is equal to M, there exists $k \in \mathbb{N}$ such that $N_k = M$.

(You can think about whether this condition is *equivalent* to finite generation, but you do not have to prove it.)

Question 3. Prove that $\mathbb{Z}$-modules and abelian groups are the same thing. Specifically, prove that

(a*) Every abelian group A admits one and only one structure of a $\mathbb{Z}$-module (i.e. there is a unique "multiplication map" $\cdot\colon \mathbb{Z} \times A \to A$ making A into a $\mathbb{Z}$-module).

(b*) For any abelian groups A and B, the set of $\mathbb{Z}$-module homomorphisms $f\colon A \to B$ is exactly the set of abelian-group homomorphisms $f\colon A \to B$.

(c) Please write one sentence summarizing what makes (a) and (b) happen (i.e. what's the key idea of your proof?).

[Note (c) is the only part that you need to submit! You should work out (a)+(b) but don't need to write them up.]

Question 4. Let R be a ring, and let $\{M_i\}_{i\in I}$ be a family of R-modules indexed by some set I.

Define the *direct product* $\prod_{i\in I} M_i$ to be the Cartesian product, i.e. the set of families $(m_i)_{i\in I}$ with $m_i \in M_i$. This becomes an R-module with the component-wise addition and multiplication:

$$(m_i)_i + (n_i)_i = (m_i + n_i)_i \qquad\qquad r\cdot (m_i)_i = (r\cdot m_i)_i$$

Define the *direct sum* $\bigoplus_{i\in I} M_i$ to be the submodule consisting of elements where $m_i = 0$ for all but finitely many i. (You do not have to prove this is a submodule, but you should understand why it is.) Note that when I is finite $\bigoplus_{i\in I} M_i$ is the same as $\prod_{i\in I} M_i$, but in general it is a proper submodule.

For readability, let $P = \prod_{i\in I} M_i$ and $S = \bigoplus_{i\in I} M_i$.

(a) Show that

$$\text{“a map to } P = \prod_{i\in I} M_i \text{ is the same as a family of maps to } M_i\text{”},$$

by proving the following. Let $\pi_i\colon P \to M_i$ be the projection taking $(m_i)_{i\in I} \mapsto m_i$. Prove that for any R-module L, given homomorphisms $f_i\colon L \to M_i$ there exists a unique homomorphism $f\colon L \to P$ such that $f_i = \pi_i \circ f$ for all i.

(b) Show that

$$\text{“a map \textit{from} } S = \bigoplus_{i\in I} M_i \text{ is the same as a family of maps \textit{from} } M_i\text{”},$$

by **formulating** and **proving** a separate universal property along similar lines to (a).
(Hint: consider the inclusions $j_i\colon M_i \to S$ given by $m_i \mapsto (0,\dots,0,m_i,0,\dots,0)$ [0 except in the M_i factor].)