Math 210A: Modern Algebra

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Homework 2

Due Thursday night, October 5 (technically 5am Oct. 6)

Note that Q3 says "optional, replaces Q2". This means that this question is optional (really!); however, if you write up and submit this optional question Q3, you do not have to submit Q2.

Recall that:

- a domain is a commutative ring in which $xy = 0 \implies (x = 0 \text{ or } y = 0)$;
- a prime ideal $P \subsetneq R$ is an ideal for which $xy \in P \implies (x \in P \text{ or } y \in P)$ (or equivalently, for which the quotient R/P is a domain); and
- a maximal ideal $\mathfrak{m} \subsetneq R$ is one for which the quotient ring $R/\mathfrak{m}$ is a field.

Question 1. Let R be a commutative ring, and let $I = \{r \in R \,|\, \exists k > 0 \text{ such that } r^k = 0\}$.

(a) Prove I is an ideal.

(b) Prove I is the intersection of all the prime ideals of R.
You may use without proof the following fact, a consequence of Zorn's lemma: if S is a subset of R satisfying $0 \notin S$ and $S \cdot S \subset S$, then the set of ideals $J \subset R$ for which $J \cap S = \emptyset$ has a maximal element.

Question 2. Let $R = C^0([0, 1])$ be the ring of real-valued continuous functions on the closed interval $[0, 1]$. For every point $p \in [0, 1]$, we obtain a maximal ideal $\mathfrak{m}_p = \{f \in R \,|\, f(p) = 0\}$.

Prove that *every* maximal ideal of R is of the form $\mathfrak{m}_p$ for a unique $p \in [0, 1]$.

(Hint: You may wish to recall that $[0, 1]$ is compact, which means that for any collection of open intervals covering it, there is some finite subcollection that still covers it.)

Note that this means that you can recover the set $[0, 1]$ just from the *ring* R.
(This actually works for any compact Hausdorff space, not just $[0, 1]$; the proof is the same.)

(Optional, to think about: can you also recover the *topology* on $[0, 1]$ from the ring R?)

(cont)

Question 3 (optional, replaces Q2). [This question is very hard, **100% optional**, and cannot be done without material from outside this course.]

Let $R = C^\infty(S^1;\mathbb{C})$ be the ring of complex-valued smooth functions on the circle S^1, which for concreteness I will realize as smooth 1-periodic functions on $\mathbb{R}$:

$$R \cong \{f \in C^\infty(\mathbb{R};\mathbb{C}) \,|\, f(x+1) = f(x)\}.$$

The proof of Q2 applies in exactly the same way to R, showing that every maximal ideal of R is of the form $\mathfrak{m}_p = \{f \in R \,|\, f(p) = 0\}$ for a unique $p \in [0,1) \approx S^1$; you do not have to prove this.
(The complex-valued vs real-valued is not an important point, it just simplifies the following.)

For any $f \in R$ we can define complex numbers $a_n \in \mathbb{C}$ for all $n \in \mathbb{Z}$ by $a_n = \int_0^1 f(x)e^{-2\pi inx}\,dx$. (Remark: It is a fact that these numbers decay rapidly as $n \to \infty$,
in the sense that for all $k \geq 0$ we have $n^k\,|a_n| \to 0$ and $n^k\,|a_{-n}| \to 0$ as $n \to +\infty$.)

Let $S \subset R$ be the subring consisting of those functions for which $a_{-1} = a_{-2} = \cdots = 0$, i.e.

$$S = \left\{ f \in R \,\middle|\, \int_0^1 f(x)e^{-2\pi inx}\,dx = 0 \text{ for all } n < 0 \right\}$$

(You do not have to prove that S is a subring of R, though you might benefit from thinking about why it is.)
For every $p \in [0,1)$, we still have a maximal ideal $\mathfrak{m}_p \subset S$ given by $\mathfrak{m}_p = \{f \in S \,|\, f(p) = 0\}$.

Exhibit a maximal ideal of S that is *not* of this form, and ideally exhibit *two* such maximal ideals. (If you really want a challenge: can you classify *all* maximal ideals of S? Warning: I do not know that this is possible using things you know. But you could at least come up with a guess, even if you can't completely prove it's correct.)

Question 4. Let $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ be coprime. Let C be any abelian group, and let

$$f\colon (\mathbb{Z}/a\mathbb{Z}) \times (\mathbb{Z}/b\mathbb{Z}) \to C$$

be a $\mathbb{Z}$-bilinear map. Prove that $f = 0$.

Question 5. Let N be a submodule of the R-module M. Prove that if N is finitely generated and M/N is finitely generated, then M is finitely generated.

Question 6. Let M be a finitely generated R-module. Let $\pi\colon M \twoheadrightarrow R^n$ be a surjective homomorphism, and let $K = \ker(\pi)$. Prove that the R-module K is finitely generated.

Localization of modules, which features in the next question, will be covered on Monday.

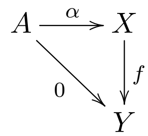
Question 7. Let R be a commutative ring, and let $S \subset R$ be a multiplicative set ($S \cdot S \subset S$). Let M be a finitely generated R-module. Prove that the localization $S^{-1}M$ satisfies

$$S^{-1}M = 0 \qquad \Longleftrightarrow \qquad \exists s \in S \text{ with } s \cdot M = 0.$$

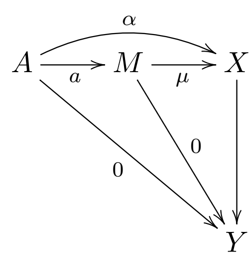
(cont)

Question 8. Let $f\colon X \to Y$ be a homomorphism of R-modules.

(a) Consider all pairs (A, α) of an R-module A and a homomorphism $\alpha\colon A \to X$ with $f \circ \alpha = 0$.

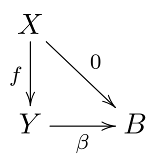


You will prove that these exists a "universal" such pair. Specifically, you must construct some $(M, \mu\colon M \to X)$ with $f \circ \mu = 0$ with the property that:
for any $(A, \alpha\colon A \to X)$ with $f \circ \alpha = 0$, there exists a unique $a\colon A \to M$ such that $\alpha = \mu \circ a$.

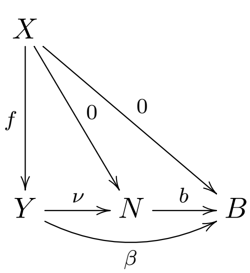


(We might abbreviate this property as saying roughly: "Every α with $f \circ \alpha = 0$ factors uniquely through M".)

(b) On the other side, consider all pairs $(B, \beta\colon Y \to B)$ with $\beta \circ f = 0$.



Prove that these exists a "universal" such pair, by constructing
some $(N, \nu\colon Y \to N)$ with $\nu \circ f = 0$ with the property that:
for any $(B, \beta\colon Y \to B)$ with $\beta \circ f = 0$, there exists a unique $b\colon N \to B$ such that $\beta = b \circ \nu$.



(We might abbreviate this property as saying roughly: "Every β with $\beta \circ f = 0$ factors uniquely through N".)