Math 210A: Modern Algebra

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Homework 3

Due Thursday night, October 12 (technically 5am Oct. 13)

Given R-module homomorphisms $A \xrightarrow{\alpha} B$ and $B \xrightarrow{\beta} C$ with $\beta \circ \alpha = 0$, we say they form a *short exact sequence*, and write $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$, if

(i) $\alpha\colon A \to B$ is injective;
(ii) $\beta\colon B \to C$ is surjective; and
(iii) $\mathrm{im}(\alpha) = \ker(\beta)$.

(The content is essentially the same as saying that $C \cong B/A$, except that it allows us the freedom to consider e.g. that A might not actually be a *subset* of B.) We might sometimes leave the labels α and β off, if we don't need names for the maps at the moment, but these maps are still an essential part of the SES.

Question 1. Consider a short exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$.
Prove that the following are equivalent.

(A) There exists a homomorphism $\sigma\colon C \to B$ such that $\beta \circ \sigma = \mathrm{id}_C$.

(B) There exists a homomorphism $\tau\colon B \to A$ such that $\tau \circ \alpha = \mathrm{id}_A$.

(C) There exists an isomorphism $\varphi\colon B \to A \oplus C$ under which α corresponds to the inclusion $A \hookrightarrow A \oplus C$ and β corresponds to the projection $A \oplus C \twoheadrightarrow C$.

When these equivalent conditions hold, we say the short exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *splits*.
We can also equivalently say "$\beta\colon B \to C$ splits" (since by (i) this only depends on β) or "$\alpha\colon A \to B$ splits" (by (ii)).

Question 2. Given an R-module, prove that the following are equivalent.

(A) Every short exact sequence $0 \to A \to B \to M \to 0$ splits.

(B) There exists some R-module N such that $M \oplus N$ is free.

When these equivalent conditions hold, we say that the R-module M is *projective*.

Given R-modules M and N, recall that $\mathrm{Hom}_R(M, N)$ is an R-module where $(rf+g)(m) = r \cdot f(m) + g(m)$. We can leave off the subscript if the ring R is unambiguous from context.

Question 3*. (Make sure you understand this, but don't write it up.)

Consider a short exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$.

(a) Prove for any X that $\mathrm{Hom}(C, X)$ can be identified with

$$\{f \in \mathrm{Hom}(B, X) \mid f \circ \alpha = 0\} \quad = \quad \ker \mathrm{Hom}(B, X) \xrightarrow{-\circ\alpha} \mathrm{Hom}(A, X).$$

(b) Prove for any Y that $\mathrm{Hom}(Y, A)$ can be identified with

$$\{f \in \mathrm{Hom}(Y, B) \mid \beta \circ f = 0\} \quad = \quad \ker \mathrm{Hom}(Y, B) \xrightarrow{\beta\circ-} \mathrm{Hom}(Y, C).$$

Question 4*. (NO LONGER ASSIGNED)
~~Let R be a commutative ring, and let M be a finitely generated R-module.~~
~~Prove that if $\alpha\colon M \to M$ is surjective, then it is an isomorphism.~~

~~(Note the following useful consequence: any n elements that generate R^n are actually a basis of R^n.)~~

To the equivalent definitions (A) $\iff$ (B) $\iff$ (C) of finite generation on HW1, we could add the following equivalent condition (you don't need to prove this).
An R-module M is finitely generated if and only if:

(D) There exists a short exact sequence $0 \to A \to F \to M \to 0$ where F is a finitely generated free module.

Question 5. Let M be a finitely generated R-module. Prove that the following are equivalent.

(A) There exists a short exact sequence $0 \to A \to F \to M \to 0$ where F is a finitely generated free module and A is finitely generated.

(B) For *every* short exact sequence $0 \to Q \to F \to M \to 0$ where F is a finitely generated free module, Q is finitely generated.

When these equivalent conditions hold, we say that the R-module M is *finitely presented*.

Question 6*. (Make sure you understand this, but don't write it up.) Let R be a commutative ring, and $S \subset R$ a multiplicative set. Consider a short exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$. Applying the localization functor, we obtain maps $S^{-1}A \xrightarrow{S^{-1}\alpha} S^{-1}B$ and $S^{-1}B \xrightarrow{S^{-1}\beta} S^{-1}C$.

Prove that $0 \to S^{-1}A \xrightarrow{S^{-1}\alpha} S^{-1}B \xrightarrow{S^{-1}\beta} S^{-1}C \to 0$ is a short exact sequence.
(Hint: this is essentially equivalent to $\ker(S^{-1}f) = S^{-1}\ker(f)$ and $\operatorname{im}(S^{-1}f) = S^{-1}\operatorname{im}(f)$, which we saw in class. So just make sure you understand why it holds.)

(cont)

Let R be a commutative ring, and $S \subset R$ a multiplicative set. Given an R-linear map $f\colon M \to N$, in class we defined[1] the $R[\frac{1}{S}]$-linear map $\frac{f}{S}\colon S^{-1}M \to S^{-1}N$ given by $\frac{f}{S}(\frac{m}{s}) = \frac{f(m)}{s}$.

A natural question is whether *every* $R[\frac{1}{S}]$-linear map from $S^{-1}M$ to $S^{-1}N$ is of this form. Thinking a moment shows this can't quite be true: for example, the map $g\colon \mathbb{Z}[\frac{1}{2}] \to \mathbb{Z}[\frac{1}{2}]$ given by $g(x) = \frac{1}{2^{10}} \cdot x$ can't come from a map $\mathbb{Z} \to \mathbb{Z}$.

So the real question is whether, for every $g\colon S^{-1}M \to S^{-1}N$, there exist $s \in S$ and $f\colon M \to N$ such that $s \cdot g = \frac{f}{S}$. In this question, you will show that this holds when M is finitely presented.

Question 7. If we set $L(f) = \frac{f}{S}$, this gives a set function

$$L\colon \operatorname{Hom}_R(M,N) \to \operatorname{Hom}_{R[\frac{1}{S}]}(S^{-1}M, S^{-1}N).$$

Observe that L is actually R-linear (you do not need to prove this).

Prove that if M is finitely presented, then L is the localization map of the R-module $\operatorname{Hom}_R(M,N)$. More precisely, for any M the universal property gives a map

$$L'\colon S^{-1}\operatorname{Hom}_R(M,N) \to \operatorname{Hom}_{R[\frac{1}{S}]}(S^{-1}M, S^{-1}N);$$

you must prove that if M is finitely presented, then L' is an isomorphism.

Question 8. Give a counterexample to Q7 when M is not finitely presented, by exhibiting some $g\colon S^{-1}M \to S^{-1}N$ for which there do not exist $s \in S$ and $f\colon M \to N$ such that $s \cdot g = \frac{f}{S}$.
(Note: you don't have to take R to be some crazy ring for this.)

(cont)

[1] in class I called this $S^{-1}f$ instead of $\frac{f}{S}$, but that will get too hard to write.

Question 9. Given elements $r_1, \ldots, r_k$ in a commutative ring R, prove the following are equivalent.

(A) These elements generate the unit ideal: $(r_1, \ldots, r_k) = R$;
in other words, there exist $a_1, \ldots, a_k \in R$ such that $a_1 r_1 + \cdots + a_k r_k = 1$.

(B) An R-module M is 0 $\iff$ the $R[\frac{1}{r_i}]$-module $M[\frac{1}{r_i}]$ is 0 for all $i = 1, \ldots, k$.

Question 10. Let R be a commutative ring, and let M be an R-module.
Prove that if M is **finitely presented**, the following are equivalent.

(A) M is projective. (see Q2)

(B) M is *locally free*, meaning there exist $r_1, \ldots, r_k$ in R with $(r_1, \ldots, r_k) = R$ such that $M[\frac{1}{r_i}]$ is a free $R[\frac{1}{r_i}]$-module for all $i = 1, \ldots, n$.

(C) M_P is a free R_P-module for all prime ideals P.

(D) $M_\mathfrak{m}$ is a free $R_\mathfrak{m}$-module for all maximal ideals $\mathfrak{m}$.

Question 11. (Hard) Extend the equivalence in Q10 to include the following equivalent condition (still under the assumption that M is finitely presented):

(E) Every linear dependence in M is trivial, in the sense below.

A linear dependence in M is a list of module elements $m_1, \ldots, m_n \in M$ and ring elements $r_1, \ldots, r_n \in R$ such that $r_1 m_1 + \cdots + r_n m_n = 0$ in M.

A *trivial* linear dependence is, colloquially, something like

$$\begin{aligned} &(10v_1 - 3v_2) \\ +2\cdot&(-3v_1 + v_2) \\ +&(-4v_1 + v_2) \\ =&(10 - 6 - 4)v_1 + (-3 + 2 + 1)v_2 \\ =&0v_1 + 0v_2 = 0. \end{aligned}$$

Formally, a linear dependence is *trivial* if there exist module elements $v^1, \ldots, v^k \in M$ and ring elements $a_i^j \in R$ such that

$$a_i^1 v^1 + a_i^2 v^2 + \cdots + a_i^k v^k = m_i \qquad \text{for all } i$$

$$r_1 a_1^j + r_2 a_2^j + \cdots + r_n a_n^j = 0 \qquad \text{for all } j$$