MATH 210A, FALL 2017
HW 4 SOLUTIONS

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Question 1. Consider the situation of the snake lemma, where each row is exact:

$$\begin{array}{ccccccc} & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & & \end{array}$$

(a) Construct a connecting homomorphism $d\colon \ker\gamma \to \operatorname{coker}\alpha$.

(b*) Check that this yields a complex $\ker\alpha \to \ker\beta \to \ker\gamma \xrightarrow{d} \operatorname{coker}\alpha \to \operatorname{coker}\beta \to \operatorname{coker}\gamma$.

(c*) Check that this sequence is exact at $\ker\beta$ and $\operatorname{coker}\beta$.

(d) Check that this sequence is exact at $\ker\gamma$ and $\operatorname{coker}\alpha$.

(e*) Check that if f is injective, then $0 \to \ker\alpha \to \ker\beta$ is exact at $\ker\alpha$ also;
and that if g' is surjective, then $\operatorname{coker}\beta \to \operatorname{coker}\gamma \to 0$ is exact at $\operatorname{coker}\gamma$ also.

Note you only need to write up (a) and (d).

Solution. [Comment from Prof. Church: yes, writing out the proof of this kind of diagram-chase argument can be painful — that's why we only do it once, so the rest of the time we can just quote the snake lemma! If you prefer, you can watch Jill Clayburgh give a complete proof[1] for (a) in the 1980 film It's My Turn.]

(a) Let $c \in \ker\gamma$. Since $g\colon B \to C$ is surjective, we can lift c to some $b \in B$ such that $g(b) = c$. Since $\gamma(c) = \gamma(g(b)) = 0$ and $\gamma \circ g = g' \circ \beta$ (by commutativity of the diagram), it follows that $g'(\beta(b)) = 0$. In other words, $\beta(b) \in \ker g'$. But by exactness of the diagram we know that $\ker g' = \operatorname{im} f'$, so there is some $a' \in A'$ such that $f'(a') = \beta(b)$. Note that since f' is injective, a' is determined uniquely by $\beta(b)$. We will define d by setting $d(c) = [a']$, where $[a']$ means the image of a' in $\operatorname{coker}\alpha$.

First, we have to check that this makes sense: to construct a', we had to make a non-canonical choice of b with $g(b) = c$. Now, let $b' \in B$ be some other choice, so $g(b') = g(b) = c$. This means that $g(b - b') = 0$, so $b - b' \in \ker g = \operatorname{im} f$, so there is some (not necessarily unique, since we don't know that the top row is exact on the left) $a \in A$ such that $b' = b + f(a)$. This means that

$$\beta(b') = \beta(b) + \beta(f(a)) = f'(a) + f'(\alpha(a)) = f'(a' + \alpha(a))$$

Now, $[a'] = [a' + \alpha(a)]$ in $\operatorname{coker}\alpha = A'/\operatorname{im}\alpha$, so any two choices of b with $g(b) = c$ give the same element of $\operatorname{coker}\alpha$, so d is well-defined.

Now, we also need to check that d is a homomorphism. If we have $c = r \cdot c_1 + c_2$ and pick some b_1, b_2 with $g(b_1) = c_1, g(b_2) = c_2$, then we have $g(r \cdot b_1 + b_2) = c$. Then, $\beta(r \cdot b_1 + b_2) = r \cdot \beta(b_1) + \beta(b_2)$, so if $f'(a'_1) = \beta(b_1)$ and $f'(a'_2) = \beta(b_2)$, then $f'(r \cdot a'_1 + a'_2) = \beta(r \cdot b_1 + b_2)$. Thus, $d(c) = r \cdot a'_1 + a'_2 = r \cdot d(c_1) + d(c_2)$.

[1] `https://youtu.be/etbcKWEKnvg`. To think about, but not write up: the gender dynamic in this classroom.

(b)

(c)

(d) The sequence $\ker\alpha \to \ker\beta \to \ker\gamma \to \operatorname{coker}\alpha \to \operatorname{coker}\beta \to \operatorname{coker}\gamma$ is exact at $\ker\gamma$ iff $\ker d = \operatorname{im}(\ker\beta \to \ker\gamma)$. In other words, we need to show that if $c \in \ker\gamma$ is such that $d(c) = 0$, then there is some $b \in \ker\beta$ such that $g(b) = c$.

To show this, let's trace through the definition of d again. Let $b \in B$ be such that $g(b) = c$. Then let $a' \in A$ be such that $\beta(b) = f'(a')$; then $d(c) = [a'] \in \operatorname{coker}\alpha$. If $d(c) = [a'] = 0$, then $a' \in \operatorname{im}(\alpha)$, so $a' = \alpha(a)$ for some $a \in A$. Then $\beta(b) = f'(\alpha(a)) = \beta(f(a))$. Thus, $\beta(b - f(a)) = 0$. But $g(b - f(a)) = g(b) - g(f(a)) = g(b)$, since $\operatorname{im} f = \ker g$. Thus, $b' = b - f(a) \in \ker\beta$ is such that $g(b') = c$. This confirms that $\ker d \subset \operatorname{im}(\ker\beta \to \ker\gamma)$.

To see that the sequence is exact at $\operatorname{coker}\alpha$, we need to show that if $[a'] \in \operatorname{coker}\alpha$ is such that $[f'(a')] = 0 \in \operatorname{coker}\beta$, then there is some $c \in \ker\gamma$ such that $d(c) = [a']$. The condition that $[f'(a')] = 0 \in \operatorname{coker}\beta$ means exactly that there is some $b \in B$ such that $\beta(b) = f'(a')$. Now, let $c = g(b)$. By the definition of $d(c)$, since $g(b) = c$, if $f'(a') = \beta(b)$, then $d(c) = [a']$. Thus, $[a'] \in \operatorname{im} d$.

A *free resolution* of an R-module M is a complex

$$\cdots \to F_3 \to F_2 \to F_1 \to F_0 \to M \to 0$$

which is exact everywhere and where each F_i is free.
(Similarly, a *projective resolution* of M is an exact sequence $\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$ where each P_i is projective, and so on.)

Question 2. Prove that every R-module M has a free resolution

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0.$$

Solution. Choose F_0 to be some free module with a map $F_0 \to M \to 0$. In other words, pick any (not necessarily finite) generating set $\{m_i\}_{i\in I}$ of M and let F_0 be the free module on generators $\{e_i\}_{i\in I}$, and define the map $F_0 \to M$ by $e_i \mapsto m_i$. This is surjective because the m_i are generators for M. Now, we can define all of the $F_i \to F_{i-1}$ by recursion. Assume we have a complex $F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$ which is exact everywhere except at F_n. Then we need to construct some F_{n+1} and a map $\alpha : F_{n+1} \to F_n$ such that $\operatorname{im}\alpha$ is equal to $K_n = \ker(F_n \to F_{n-1})$. But K_n is some R-module, and we can pick a free module F_{n+1} on a set of generators for K_n just as we did above to get a map $F_{n+1} \twoheadrightarrow K_n$. Then composing this map with the inclusion $K_n \hookrightarrow F_n$, we get a map $\alpha : F_{n+1} \to F_n$ such that $\operatorname{im}\alpha = K_n$. We can continue on this way forever (note that we are not asked to show that the resolution ever terminates, and in some cases it can't!).

Question 3. Let M and N be R-modules, and suppose you have free resolutions

$$\cdots \to F_2 \xrightarrow{d} F_1 \xrightarrow{d} F_0 \to M \xrightarrow{d} 0 \qquad \text{and} \qquad \cdots \to G_2 \xrightarrow{d} G_1 \xrightarrow{d} G_0 \xrightarrow{d} N \to 0.$$

Given a homomorphism $f\colon M \to N$, prove there exist maps $f_i\colon F_i \to G_i$ making a commutative diagram

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & F_2 & \xrightarrow{d} & F_1 & \xrightarrow{d} & F_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \\
\cdots & \longrightarrow & G_2 & \xrightarrow{d} & G_1 & \xrightarrow{d} & G_0 & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

(To think about, and write up if you find a good answer:) In what sense are the maps f_i unique?

Solution. We'll construct the f_i inductively, starting with f_0. We have the following diagram:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & F_2 & \xrightarrow{d} & F_1 & \xrightarrow{d} & F_0 & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 & & & & & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \\
\cdots & \longrightarrow & G_2 & \xrightarrow{d} & G_1 & \xrightarrow{d} & G_0 & \xrightarrow{\pi'} & N & \longrightarrow & 0
\end{array}$$

We want to fill in the dotted arrow. Since F_0 is free, to define f_0, we just need to figure out where to map each generator of F_0. Call these $\{e_i \mid i \in I\}$ for some set I. We want $\pi' \circ f_0 = f \circ \pi$, so consider the elements $n_i = f(\pi(e_i)) \in N$. Since π' is surjective, for each i, there is some g_i such that $\pi'(g_i) = n_i$. Then we can define $F_0 \to G_0$ by sending e_i to g_i. Then $\pi' \circ f_0(e_i) = \pi'(g_i) = n_i = (f \circ \pi)(e_i)$, so $\pi' \circ f_0 = f \circ \pi$.

Now assume that we've defined f_i for $i = 0, \ldots, n$, and we want to define f_{n+1}. In other words, we have the following diagram, where we want to fill in the dotted arrow:

$$\begin{array}{ccccccccccccccc}
\cdots & \longrightarrow & F_{n+1} & \xrightarrow{d_{n+1}} & F_n & \xrightarrow{d_n} & F_{n-1} & \longrightarrow & \cdots & \longrightarrow & F_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \\
\cdots & \longrightarrow & G_{n+1} & \xrightarrow{d_{n+1}} & G_n & \xrightarrow{d_n} & G_{n-1} & \longrightarrow & \cdots & \longrightarrow & G_0 & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

In other words, we want to define f_{n+1} such that $d_{n+1} \circ f_{n+1} = f_n \circ d_{n+1}$. Let the generators of F_{n+1} be $\{e_i \mid i \in I\}$, and let $g_i = f_n(d(e_i))$. If we can lift the g_i to elements $g_i' \in G_{n+1}$ such that $d_{n+1}(g_i') = g_i$, then defining $f_{n+1}(e_i) = g_i'$ ensures that $d_{n+1} \circ f_{n+1} = f_n \circ d_{n+1}$. Because the bottom row of the diagram is an exact sequence, $\operatorname{im} d_{n+1} = \ker d_n$, so we need to show that $g_i \in \ker d_n$, that is, that $d_n(g_i) = 0$. But by commutativity of the diagram, we have $d_n(g_i) = d_n(f_n(d_{n+1}(e_i))) = f_{n-1}(d_n(d_{n+1}(e_i))) = 0$, since $d_n \circ d_{n+1} = 0$. So now we can define f_{n+1} by sending e_i to g_i'.

The question of uniqueness will be discussed in class.

Note: the same statement is true for projective resolutions, and the proof is fairly similar. See if you can fill in the details.

Question 4. Compute an explicit free resolution for M in the following situations:

Solution. (a) $R = \mathbf{Z}, \quad M = \mathbf{Z} \oplus \mathbf{Z}/12\mathbf{Z}$.

M is generated by $(1,0)$ and $(0,1)$, so sending a basis for $\mathbf{Z}^2$ to these generators gives the first map $d_0\colon \mathbf{Z}^2 \to M$, which is surjective. Explicitly, this sends (a,b) to $(a, b+12\mathbf{Z})$. The kernel of this map consists of pairs (a,b) such that $(a, b+12\mathbf{Z}) = (0,0)$. Thus, the kernel is exactly the set of pairs $(0,12n)$ for $n \in \mathbf{Z}$. We get a map $d_1\colon \mathbf{Z} \to \mathbf{Z}^2$ by sending n to $(0,12n)$, and this exactly parametrizes $\ker d_1$. Since multiplication by 12 is injective in the domain $\mathbf{Z}$, this map is injective. So we have a resolution:

$$0 \longrightarrow \mathbf{Z} \xrightarrow[\binom{0}{12}]{d_1} \mathbf{Z}^2 \xrightarrow{d_0} M \longrightarrow 0$$

Note: Once we have a 0 in a free resolution, we can always just continue with 0 from then on:

$$\cdots \to 0 \to 0 \to 0 \to \mathbf{Z} \xrightarrow{d_1} \mathbf{Z}^2 \xrightarrow{d_0} M \to 0$$

So in the cases below where we have a "finite" free resolution (one with a 0 in it), we'll usually leave off the infinite string of 0's.

(b) $R = \mathbf{Z}, \quad M = \mathbf{Z}/3\mathbf{Z} \oplus \mathbf{Z}/4\mathbf{Z}$

M is generated by $(1,0)$ and $(0,1)$ so we have a surjection $d_0\colon \mathbf{Z}^2 \to M$ sending (a,b) to $(a+3\mathbf{Z}, b+4\mathbf{Z})$. The kernel of this map consists of pairs (a,b) with $a \in 3\mathbf{Z}$, $b \in 4\mathbf{Z}$. Thus, we can define $d_1\colon \mathbf{Z}^2 \to \mathbf{Z}^2$ by sending (m,n) to $(3m,4n)$, so that $\operatorname{im} d_1 = \ker d_0$. Since every element of $3\mathbf{Z}$ can be written as $3m$ for a unique m, and likewise for $4\mathbf{Z}$, it follows that d_1 is injective, so this gives our resolution:

$$0 \longrightarrow \mathbf{Z}^2 \xrightarrow[\left(\begin{smallmatrix}3 & 0\\ 0 & 4\end{smallmatrix}\right)]{d_1} \mathbf{Z}^2 \xrightarrow{d_0} M \longrightarrow 0$$

For a different approach, we could use the Chinese remainder theorem to cut down the ranks a bit: since 3 and 4 are coprime, by CRT we have $\mathbf{Z}/12\mathbf{Z} \simeq \mathbf{Z}/3\mathbf{Z} \oplus \mathbf{Z}/4\mathbf{Z}$ as $\mathbf{Z}$-modules. In other words, the map $d_0\colon \mathbb{Z} \to \mathbf{Z}/3\mathbf{Z} \oplus \mathbf{Z}/4\mathbf{Z}$ defined by $a \mapsto (a+3\mathbf{Z}, a+4\mathbf{Z})$ is surjective, and the kernel is $3\mathbf{Z} \cap 4\mathbf{Z} = 12\mathbf{Z}$. Thus we can take a resolution:

$$0 \longrightarrow \mathbf{Z} \xrightarrow[(\,12\,)]{d_1} \mathbf{Z} \xrightarrow{d_0} M \longrightarrow 0$$

where $d_1\colon n \mapsto 12n$.

(c) $R = \mathbf{R}[T], \quad M = \mathbf{R}^2$, with R-module structure where T acts by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

Let $e_1 = (1, 0)$, $e_2 = (0, 1)$ be the standard basis for $\mathbf{R}^2$ as a $\mathbf{R}$-module. Since e_1 and e_2 generate M as a $\mathbf{R}$-module, they certainly generate M as an $R = \mathbf{R}[T]$-module. However, note that by the definition of the T-action, we have $T \cdot e_2 = 2e_1 + e_2$, so:

$$\frac{1}{2}(T - 1) \cdot e_2 = \frac{1}{2}(T \cdot e_2 - e_2) = \frac{1}{2}(2e_1 + e_2 - e_2) = e_1$$

Thus, e_2 actually generates M as an R-module (i.e. we can write $(x, y) \in \mathbf{R}^2$ as $\left(\frac{x}{2}(T - 1) + y\right) \cdot e_2 = xe_1 + ye_2$). This means that we can define a surjective map $d_0 \colon R \to M \to 0$ sending $p(T)$ to $p(T) \cdot e_2$.

Any quotient of R is isomorphic to R/I for some ideal I. In this case, the kernel of d_0 is the ideal I of R consisting of all polynomials $p(T)$ such that $p(T) \cdot e_2 = 0$, so let us find this ideal I. Since e_2 and Te_2 are linearly independent, no constant or linear polynomial belongs to I. But by the same token, since $\mathbf{R}^2$ is 2-dimensional, $T^2 e_2$ must be a linear combination of e_2 and Te_2. Direct computation shows that

$$T^2 \cdot e_2 = T(2e_1 + e_2) = 2e_1 + (2e_1 + e_2) = 4e_1 + e_2 = 2T(e_2) - e_2 = (2T - 1) \cdot e_2.$$

Therefore the polynomial $f(T) := T^2 - 2T + 1$ belongs to I. Since $R/(f)$ is 2-dimensional over $\mathbf{R}$, as is $M \cong R/I$, we conclude that $I = (f)$. [You should think about this if you haven't seen if before.] Therefore we can define $d_1 \colon R \to R$ by $d_1(p) = f \cdot p$. Since R is a domain, this is injective, and we obtain the free resolution

$$0 \longrightarrow R \xrightarrow[(T^2-2T+1)]{d_1} R \xrightarrow{d_0} M \longrightarrow 0$$

(d) $R = \mathbf{R}[x, y], \quad M = \mathbf{R}$, with R-module structure where x and y act by 0.

Since $1 \in \mathbf{R}$ generates M as a $\mathbf{R}$-module, it certainly generates M as a $R = \mathbf{R}[x, y]$-module, so we can define a surjective homomorphism $d_0 \colon R \to M$ by sending $p(x, y)$ to $p(x, y) \cdot 1$. Let $p(x, y) = a_0 + xp_1(x, y) + yp_2(x, y)$ for some $p_1, p_2 \in R, a_0 \in \mathbf{R}$. Then

$$p(x, y) \cdot 1 = a_0 \cdot 1 + p_1(x, y) \cdot (x \cdot 1) + p_2(x, y) \cdot (y \cdot 1) = a_0$$

So the kernel of d_0 is the ideal $I \subseteq \mathbf{R}[x, y]$ consisting exactly of those polynomials whose constant coefficient a_0 is 0. Every term of such a polynomial has positive degree in either x or y, so if $p \in \ker d_0$, we have $p = xp_1 + yp_2$ for $p_1, p_2 \in R$. In other words, the ideal $\ker d_0$ is generated as an R-module by x and y (though it is not a principal ideal, since no polynomial of positive degree divides both x and y).

So now, we can define a map $d_1 \colon R^2 \to R$ by sending (p, q) to $xp + yq$, and by what we've just shown, the following sequence is exact:

$$R^2 \xrightarrow[(x \;\; y)]{d_1} R \xrightarrow{d_0} M \longrightarrow 0$$

Now, what is the kernel of d_1? This consists of all $(p,q) \in R^2$ such that $xp + yq = 0$. Let (p,q) be some such pair. Rearranging this equation, we have $xp = -yq$. Consider this polynomial $f = xp = -yq$. Since $f(x,y) = x \cdot p(x,y)$, every term of f must be divisible by x. Similarly, since $f(x,y) = -y \cdot q(x,y)$ every term of f must be divisible by y. Therefore every term of f must be divisible[2] by xy. In other words, we can write $f(x,y) = xy \cdot g(x,y)$ for some unique g. Note that $p = y \cdot g$ and $q = -x \cdot g$.

We have found that whenever we have a pair (p,q) with $xp + yq = 0$, there exists a unique g such that $(p,q) = (yg, -xg)$. In other words, the map $d_2\colon R \to R^2$ defined by $g \mapsto (yg, -xg)$ surjects to $\ker d_1$. This map is injective since R is a domain, so we have our resolution:

$$0 \longrightarrow R \xrightarrow[\begin{pmatrix} y \\ -x \end{pmatrix}]{d_2} R^2 \xrightarrow[\begin{pmatrix} x & y \end{pmatrix}]{d_1} R \xrightarrow{d_0} M \longrightarrow 0$$

[2] In general you might use that $\mathbf{R}[x,y]$ is a *unique factorization domain* for an argument like this, but for x and y we can just do it directly.

(e) $R = \mathbf{Z}[\sqrt{-30}]$, $\quad M = \mathbf{F}_2$, with R-module structure where $\sqrt{-30}$ acts by 0.

M is generated by 1, so we can take a surjection $d_0\colon R \to M$ sending $a+b\sqrt{-30}$ to $(a+b\sqrt{-30})\cdot 1 = a \pmod 2$. This identifies M with the quotient R/I, where $I = \ker d_0$ consists of those $a+b\sqrt{-30}$ where a is even. So we have a short exact sequence

$$0 \to I \hookrightarrow R \xrightarrow{d_0} M \to 0.$$

To continue this, we must find generators for I. We claim that I is generated by the elements 2 and $\sqrt{-30}$; indeed, if b is even then (since a is assumed even) $a+b\sqrt{-30}$ is already a multiple of 2, and if b is odd then by subtracting $\sqrt{-30}$ we can reduce to the previous case. (This shows that their $\mathbb{Z}$-linear combinations generate; but we'd have even their R-linear combinations, if we needed them!)

So if we define $d_1\colon R^2 \to R$ by $(x,y) \mapsto 2\cdot x + \sqrt{-30}\cdot y$, we have $\operatorname{im} d_1 = \ker d_0 = I$. This gives a partial resolution

$$R^2 \xrightarrow[\begin{pmatrix} 2 & \sqrt{-30}\end{pmatrix}]{d_1} R \xrightarrow{d_0} M \longrightarrow 0$$

To continue we must compute $\ker d_1$. We will do this

Suppose that $(x,y) \in \ker d_1$. Write $x = c + d\sqrt{-30}$ and $y = e + f\sqrt{-30}$. Then

$$d_1(x,y) = 2\cdot(c+d\sqrt{-30}) + \sqrt{-30}(e+f\sqrt{-30}) = (2c-30f) + (2d+e)\sqrt{-30}.$$

Therefore $(x,y) \in \ker d_1 \iff c = 15f$ and $e = -2d$. At this point, let us make a detour to observe that *the kernel of d_1 is isomorphic to I itself.* Indeed, this kernel consists of $(x,y) = (15f + d\sqrt{-30}, -2d + f\sqrt{-30})$. Note that y automatically belongs to I, and given $y \in I$ there is a unique x such that $(x,y) \in \ker d_1$. In other words,

$$\ker d_1 = \left\{ \left(\tfrac{-\sqrt{-30}}{2} y, y\right) \,\middle|\, y \in I \right\} \simeq I$$

So the inclusion of $\ker d_1$ into R^2 can be viewed as a map $I \to R^2$ giving a short exact sequence

$$0 \to I \to R^2 \to I \to 0.$$

Thus we should be able to just repeat the same computations over and over, so we expect a *periodic* free resolution.

Returning to $\ker d_1$, it is clear that $\ker d_1$ is generated by the elements $(-\sqrt{-30}, 2)$ and $(15, \sqrt{-30})$ (since even their $\mathbb{Z}$-linear combinations span). These generators yield a map $d_2\colon R^2 \to R^2$ giving a partial resolution

$$R^2 \xrightarrow[\begin{pmatrix} -\sqrt{-30} & 15 \\ 2 & \sqrt{-30}\end{pmatrix}]{d_2} R^2 \xrightarrow[\begin{pmatrix} 2 & \sqrt{-30}\end{pmatrix}]{d_1} R \xrightarrow{d_0} M \longrightarrow 0$$

But we can check that $\ker d_2$ is actually *equal to* $\ker d_1$. Indeed, the second coordinate of d_2 is exactly the same as d_1; and the first coordinate is $\frac{-\sqrt{-30}}{2}$ times the second, so they vanish at the same time. Therefore we can keep using the same map over and over, yielding a free resolution:

$$\cdots \longrightarrow R^2 \xrightarrow[\begin{pmatrix} -\sqrt{-30} & 15 \\ 2 & \sqrt{-30}\end{pmatrix}]{d_k} R^2 \longrightarrow \cdots \longrightarrow R^2 \xrightarrow[\begin{pmatrix} -\sqrt{-30} & 15 \\ 2 & \sqrt{-30}\end{pmatrix}]{d_2} R^2 \xrightarrow[\begin{pmatrix} 2 & \sqrt{-30}\end{pmatrix}]{d_1} R \xrightarrow{d_0} M \longrightarrow 0$$

[Note from TC: this is a good example of why it's nice sometimes to be able to use *projective* resolutions. The ideal $I = (2, \sqrt{-30})$ is projective, as I mentioned in class; so if we are just looking for a projective resolution we could just take

$$0 \to I \to R \to M \to 0$$

and not need to go any further.]

(f) $R = \mathbf{R}[x, y], \quad M = \mathbf{R}[x, y]/I$
where I is the ideal of all polynomials with no constant, linear, or quadratic term.
(in other words, M consists of at-most-quadratic polynomials in x and y)

We already have M written as $\mathbf{R}[x, y]/I = R/I$, so we have an exact sequence:

$$0 \longrightarrow I \longrightarrow R \xrightarrow{d_0} M \longrightarrow 0$$

If $p(x, y) \in I$, then every term of $p(x, y)$ has degree at least 3. Thus, each term is divisible by some monomial of degree 3. There are four of these: $\alpha = x^3$, $\beta = x^2y$, $\gamma = xy^2$, $\delta = y^3$. These 4 elements generate the ideal I, and thus define a surjection $R^4 \to I$ sending

$$(p_1, p_2, p_3, p_4) \mapsto p_1\alpha + p_2\beta + p_3\gamma + p_4\delta = p_1x^3 + p_2x^2y + p_3xy^2 + p_4y^3.$$

Composing this with $I \hookrightarrow R$, we have $d_1\colon R^4 \to R$ with $\operatorname{im} d_1 = I = \ker d_0$:

$$R^4 \xrightarrow[\begin{pmatrix} x^3 & x^2y & xy^2 & y^3 \end{pmatrix}]{d_1} R \xrightarrow{d_0} M \longrightarrow 0$$

To continue, we need to find relations among the elements α, β, γ, and δ. A few relations jump right out at us: $y\alpha = x\beta$ (both equal x^3y), $y\beta = x\gamma$ (both equal x^2y^2), and $y\gamma = x\delta$ (both equal xy^3). These three relations lead us to consider the map $d_2\colon R^3 \to R^4$ sending the basis to the elements $(y, -x, 0, 0)$, $(0, y, -x, 0)$, and $(0, 0, y, -x)$.

Since these were relations, we know that $d_1 \circ d_2 = 0$, or in other words $\operatorname{im} d_2 \subset \ker d_1$. Now, one natural way to proceed would be to prove that these relations generate all relations between $\alpha, \beta, \gamma, \delta$; in other words, that $\operatorname{im} d_2 = \ker d_1$. This would work fine, but for variety, we will take a different approach.

We have a complex

$$R^3 \xrightarrow[\begin{pmatrix} y & & \\ -x & y & \\ & -x & y \\ & & -x \end{pmatrix}]{d_2} R^4 \xrightarrow[\begin{pmatrix} x^3 & x^2y & xy^2 & y^3 \end{pmatrix}]{d_1} I \longrightarrow 0$$

but keep in mind that we do *not* yet know it is exact at R^4. Instead, let us show that d_2 is injective. This is surprisingly easy. Suppose $(f, g, h) \in R^3$ belongs to $\ker d_2$. Applying d_2, we have

$$d_2(f, g, h) = (yf, \;\; yg - xf, \;\; yh - xg, \;\; , -xh).$$

If this is 0, then examining the first coordinate shows that $yf = 0$, and thus $f = 0$ (since R is a domain). Given this, examining the second coordinate shows that $yg = 0$, and thus $g = 0$; and then the third coordinate shows that $yh = 0$ and thus $h = 0$. Therefore d_2 is injective, and so we know this complex is exact *except* possibly at R^4.

$$0 \longrightarrow R^3 \xrightarrow[\begin{pmatrix} y & & \\ -x & y & \\ & -x & y \\ & & -x \end{pmatrix}]{d_2} R^4 \xrightarrow[\begin{pmatrix} x^3 & x^2y & xy^2 & y^3 \end{pmatrix}]{d_1} I \longrightarrow 0$$

It remains to show that $\operatorname{im} d_2$ is all of $\ker d_1$. We can do this by showing that they have the "same dimension" in a certain sense. As a vector space, we can split $R = \mathrm{R}[x, y]$ as a direct sum $R = \bigoplus_{n \geq 0} R_n$ where $R_n = \langle x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n \rangle$ is the "degree n" part of R. note that $\dim_{\mathrm{R}} R_n = n + 1$.

Because every element in the matrix for d_2 is a pure linear polynomial, the map d_2 is "homogeneous", in the sense that $d_2(R_n^3) \subset R_{n+1}^4$. Similarly, since every element in the matrix for d_1 is a pure cubic polynomial, the map d_1 is homogeneous: $d_1(R_m^4) \subset R_{m+3}$. Therefore we can split this complex up as a direct sum over $k \geq 0$ of the complexes

$$0 \longrightarrow R_{k-4}^3 \xrightarrow{d_2^{(k)}} R_{k-3}^4 \xrightarrow{d_1^{(k)}} I_k \longrightarrow 0$$

In particular, $\operatorname{im} d_2^{(k)}$ is a subspace of $\ker d_1^{(k)}$. Let us look at the dimensions here for large k first (we'll check small k afterwards). For large k we have $\dim R_{k-3}^4 = 4((k-3)+1) = 4(k-2) = 4k - 8$, and $\dim I_k = \dim R_k = k + 1$. Since $d_1^{(k)}$ is surjective, we conclude that

$$\dim \ker d_1^{(k)} = \dim R_{k-3}^4 - \dim I_k = 3k - 9.$$

But at the same time $\dim R_{k-4}^3 = 3((k-4)+1) = 3(k-3) = 3k - 9$. Since $d_2^{(k)}$ is injective, we find that $\dim \operatorname{im} d_2^{(k)} = 3k - 9 = \dim \ker d_1^{(k)}$. It follows that $\operatorname{im} d_2^{(k)} = \ker d_1^{(k)}$, at least for large k.

The computation above holds as long as $k - 4 \geq 0$, i.e. when $k \geq 4$. For $k = 3$ the complex is just

$$0 \to 0 \to R_0^4 \xrightarrow{d_1^{(3)}} I_3 \to 0$$

Since $\dim I_3 = \dim R_3 = 4$, and $d_1^{(3)}$ is surjective, it is an isomorphism, so this is still exact. Finally, for $k < 3$ all three terms here vanish. We conclude that $\operatorname{im} d_2^{(k)} = \ker d_1^{(k)}$ for all k, and thus $\operatorname{im} d_2 = \ker d_1$. Therefore our free resolution is

$$0 \longrightarrow R^3 \xrightarrow[\begin{pmatrix} y & & \\ -x & y & \\ & -x & y \\ & & -x \end{pmatrix}]{d_2} R^4 \xrightarrow[\begin{pmatrix} x^3 & x^2y & xy^2 & y^3 \end{pmatrix}]{d_1} R \xrightarrow{d_0} M \longrightarrow 0.$$

(g) $R = \mathbf{Z}[t]/(t^2-1)$, $\quad M = \mathbf{Z}$, with R-module structure where t acts by the identity.

M is generated by 1, so we have a surjection $\pi\colon R \to M$ sending r to $r \cdot 1$. We can write an element $r \in R$ uniquely as $a + bt$: if $p(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_0 \in \mathbf{Z}[t]$ for $n \geq 2$, then in R, $p(t) = p(t) - t^{n-2}(t^2-1)$, and this has degree $n-1$. We can repeat this process until we reach a representative with degree 1. This representation is unique since $(a+bt) - (a'+b't)$ is always linear, so it is never divisible by t^2-1 in $\mathbf{Z}[t]$. Then we have $\pi(a+bt) = (a+bt)\cdot 1 = a+b$. This is zero iff $a = -b$, i.e. if $r = b(t-1)$ for some $b \in \mathbf{Z}$. Conversely, if $r = r'(t-1)$ for some $r' \in R$, we have $r \cdot 1 = r' \cdot ((t-1)\cdot 1) = r' \cdot 0 = 0$. Thus, $\ker \pi = (t-1)R$, so we can take $f_0\colon R \to R$ to be multiplication by $(t-1)$, and $\operatorname{im} f_0 = \ker \pi$.

Since R is not a domain, we can't automatically conclude that f_0 is injective as we have before. In fact, f_0 is not injective, since $(t+1) \neq 0$ in R, but $f_0(t+1) = (t-1)(t+1) = t^2-1 = 0$ in R. Thus, $(t+1)R \subseteq \ker f_0$. Conversely, let $r \in \ker f_0$. We can write r as $r = a+bt$ for $a, b \in \mathbf{Z}$. Then $0 = f_0(r) = (t-1)(a+bt) = bt^2 + (a-b)t - a = b(t^2-1) + (a-b)t - (a-b) = (a-b)(t-1)$. Since the representation of an element of R as $a+bt$ is *unique*, this implies that $a - b = 0$, so we have $r = a(1+t)$. Thus, we see that $\ker f_0 = (t+1)R$. This allows us to define $f_1\colon R \to R$ to be multiplication by $(t+1)$, and $\operatorname{im} f_1 = \ker f_0$.

Now, $\ker f_1$ consists of those elements $a+bt$ such that $(t+1)(a+bt) = bt^2 + (a+b)t + a = (a+b)(t-1) = 0$. As above, this is true iff $a+b=0$, so $\ker f_1$ consists of elements of the form $a(t-1)$, i.e. $\ker f_1 = (t-1)R$. Note that this is the same as $\ker \pi$, so we can define $f_2\colon R \to R$ to be equal to f_0, and repeat off to infinity. What we end up with is an infinite resolution:

$$\cdots \longrightarrow R \xrightarrow{f_4} R \xrightarrow{f_3} R \xrightarrow{f_2} R \xrightarrow{f_1} R \xrightarrow{f_0} R \xrightarrow{\pi} M \longrightarrow 0$$

with f_n equal to multiplication by $(t+1)$ when n is odd and f_n equal to multiplication by $(t-1)$ when n is even.

Question 5. Consider the map $f\colon M \to N$ from $M = \mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z}$ to $N = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ sending $(a \in \mathbb{Z}, b \in \mathbb{Z}/12\mathbb{Z})$ to $(\overline{a} \in \mathbb{Z}/3\mathbb{Z}, \overline{b} \in \mathbb{Z}/4\mathbb{Z})$. If

$$\cdots \to F_1 \to F_0 \to M \to 0 \qquad \text{and} \qquad \cdots \to G_1 \to G_0 \to N \to 0$$

are the free resolutions of M and N that you constructed in Q4(a) and Q4(b), describe explicitly the maps $f_i\colon F_i \to G_i$ as in Q3.

Solution. Copying down the free resolutions written above, we have:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z} & \xrightarrow{d_1} & \mathbf{Z}^2 & \xrightarrow{d_0} & M & \longrightarrow & 0 \\ & & \downarrow f_1 & & \downarrow f_0 & & \downarrow f & & \\ 0 & \longrightarrow & \mathbf{Z} & \xrightarrow{d_1'} & \mathbf{Z} & \xrightarrow{d_0'} & N & \longrightarrow & 0 \end{array}$$

and our job is to describe the vertical arrows. Recall that $d_0\colon \mathbf{Z}^2 \to M$ is defined by $(a,b) \mapsto (a, b+12\mathbf{Z})$ and $d_0'\colon \mathbf{Z} \to N$ is defined by $n \mapsto (n+3\mathbf{Z}, n+4\mathbf{Z})$. Now, $f(d_0(1,0)) = f(1,0) = (1+3\mathbf{Z}, 0)$. Following the proof of Question 3, we define $f_0(1,0)$ by choosing some $n \in \mathbf{Z}$ such that $d_0'(n) = (1+3\mathbf{Z}, 0)$. Choosing

$n = 4$ works, since 4 is 1 mod 3 and 0 mod 4. Note that the possible choices for $f_0(1, 0)$ are exactly $4 + 12\mathbf{Z}$. Similarly, to define $f_0(0, 1)$, we need to pick some $n \in \mathbf{Z}$ such that $d'_0(n) = (0, 1 + 4\mathbf{Z})$. $n = 9$ works, since this is 0 mod 3 and 1 mod 4 (and again, the possible choices are $9 + 12\mathbf{Z}$).

So $f_0 \colon \mathbf{Z}^2 \to \mathbf{Z}$ sends (a, b) to $4a + 9b$. Now, f_1 is uniquely determined by $f_1(1)$. This is defined to by choosing some $n \in \mathbf{Z}$ such that $d'_1(n) = f_0(d_1(1))$. But recall that $d_1 \colon \mathbf{Z} \to \mathbf{Z}^2$ is the map sending n to $(0, 12n)$, so $f_0(d_1(1)) = f_0(0, 12) = 108$. Now, d'_1 is defined to be multiplication by 12, so we must choose $f_1(1) = 9$. Thus, f_1 is multiplication by 9. Note that if we had chosen f_0 differently, by replacing 9 with $9 + 12k$ for some $k \in \mathbf{Z}$, f_1 would have to become multiplication by $9 + 12k$.