Math 210A: Modern Algebra

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Homework 4

Due Thursday night, October 19 (technically 5am Oct. 20)

Question 1. Consider the situation of the snake lemma, where each row is exact:

$$\begin{array}{ccccccccc} & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & & \end{array}$$

(a) Construct a connecting homomorphism $d\colon \ker\gamma \to \operatorname{coker}\alpha$.

(b*) Check that this yields a complex $\ker\alpha \to \ker\beta \to \ker\gamma \xrightarrow{d} \operatorname{coker}\alpha \to \operatorname{coker}\beta \to \operatorname{coker}\gamma$.

(c*) Check that this sequence is exact at $\ker\beta$ and $\operatorname{coker}\beta$.

(d) Check that this sequence is exact at $\ker\gamma$ and $\operatorname{coker}\alpha$.

(e*) Check that if f is injective, then $0 \to \ker\alpha \to \ker\beta$ is exact at $\ker\alpha$ also;
and that if g' is surjective, then $\operatorname{coker}\beta \to \operatorname{coker}\gamma \to 0$ is exact at $\operatorname{coker}\gamma$ also.

Note you only need to write up (a) and (d).

A *free resolution* of an R-module M is a complex

$$\cdots \to F_3 \to F_2 \to F_1 \to F_0 \to M \to 0$$

which is exact everywhere and where each F_i is free.
(Similarly, a *projective resolution* of M is an exact sequence $\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$ where each P_i is projective, and so on.)

Question 2. Prove that every R-module M has a free resolution

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0.$$

Question 3. Let M and N be R-modules, and suppose you have free resolutions
$\cdots \to F_2 \xrightarrow{d} F_1 \xrightarrow{d} F_0 \to M \xrightarrow{d} 0$ and $\cdots \to G_2 \xrightarrow{d} G_1 \xrightarrow{d} G_0 \xrightarrow{d} N \to 0$.
Given a homomorphism $f\colon M \to N$, prove there exist maps $f_i\colon F_i \to G_i$ making a commutative diagram

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & F_2 & \xrightarrow{d} & F_1 & \xrightarrow{d} & F_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \\ \cdots & \longrightarrow & G_2 & \xrightarrow{d} & G_1 & \xrightarrow{d} & G_0 & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

(To think about, and write up if you find a good answer:) In what sense are the maps f_i unique?

Question 4. Compute an explicit free resolution for M in the following situations:

(a) $R = \mathbb{Z}, \quad M = \mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z}$

(b) $R = \mathbb{Z}, \quad M = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$

(c) $R = \mathbb{R}[T], \quad M = \mathbb{R}^2$, with R-module structure where T acts by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

(d) $R = \mathbb{R}[x, y], \quad M = \mathbb{R}$, with R-module structure where x and y act by 0.

(e) $R = \mathbb{Z}[\sqrt{-30}], \quad M = \mathbb{F}_2$, with R-module structure where $\sqrt{-30}$ acts by 0.

(f) $R = \mathbb{R}[x, y], \quad M = \mathbb{R}[x, y]/I$
where I is the ideal of all polynomials with no constant, linear, or quadratic term.
(in other words, M consists of at-most-quadratic polynomials in x and y)

(g) $R = \mathbb{Z}[t]/(t^2 - 1), \quad M = \mathbb{Z}$, with R-module structure where t acts by the identity.

(h) $R =$ a ring of your choice, $\quad M =$ an interesting module of your choice.

Question 5. Consider the map $f\colon M \to N$ from $M = \mathbb{Z} \oplus \mathbb{Z}/12\mathbb{Z}$ to $N = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ sending $(a \in \mathbb{Z}, b \in \mathbb{Z}/12\mathbb{Z})$ to $(\overline{a} \in \mathbb{Z}/3\mathbb{Z}, \overline{b} \in \mathbb{Z}/4\mathbb{Z})$. If

$$\cdots \to F_1 \to F_0 \to M \to 0 \qquad \text{and} \qquad \cdots \to G_1 \to G_0 \to N \to 0$$

are the free resolutions of M and N that you constructed in Q4(a) and Q4(b), describe explicitly the maps $f_i\colon F_i \to G_i$ as in Q3.