Math 210A: Modern Algebra
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Homework 5

Due Thursday night, October 26 (technically 5am Oct. 27)

Question 1. Let $R = \mathbb{Z}[t]/(t^2 - 1)$. Regard $\mathbb{Z}$ as an R-module by letting t act by the identity. Compute $\operatorname{Tor}_k^R(\mathbb{Z}, \mathbb{Z})$ and $\operatorname{Ext}_R^k(\mathbb{Z}, \mathbb{Z})$ for all $k \geq 0$.

Question 2. Let $R = \mathbb{Z}[\sqrt{-30}]$. Regard $\mathbb{F}_2$ as an R-module by letting $\sqrt{-30}$ act by 0.
Compute $\operatorname{Tor}_k^R(\mathbb{F}_2, \mathbb{F}_2)$ and $\operatorname{Ext}_R^k(\mathbb{F}_2, \mathbb{F}_2)$ for all $k \geq 0$.

Question 3. Let $R = \mathbb{R}[T]$. Let $M = \mathbb{R}^2$, with R-module structure where T acts by $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$. Let $N = \mathbb{R}$ with R-module structure where T acts by 0.
Compute $\operatorname{Tor}_k^R(M, N)$ and $\operatorname{Ext}_R^k(M, N)$ and $\operatorname{Ext}_R^k(N, M)$ for all $k \geq 0$.

Question 4. Let $R = \mathbb{C}[T]$. Given $\lambda \in \mathbb{C}$, let $\mathbb{C}_\lambda$ denote $\mathbb{C}$ regarded as an R-module by letting T act by λ. Compute $\operatorname{Ext}_R^k(\mathbb{C}_\lambda, \mathbb{C}_\mu)$ for all $k \geq 0$, for all $\lambda, \mu \in \mathbb{C}$.

Question 5. Let $R = \mathbb{C}[x, y]$.

(a) Regard $\mathbb{C}$ as an R-module by letting x and y act by 0. Compute $\operatorname{Tor}_k^R(\mathbb{C}, \mathbb{C})$ for all $k \geq 0$.

(b) Let $I \subset R$ be the ideal $I = (x, y)$. We would like to understand $I \otimes_R I$, so:

Give a basis for $I \otimes_R I$ as a complex vector space.
If you can also describe the R-module structure without too much pain, please do.

(cont.)

Recall that R is a PID (principal ideal domain) if R is a domain and every ideal in R is principal (generated by one element).

Remember that you can use earlier questions in later questions.

In the next questions, let R be a PID, and let M and X be R-modules.
NOTE: You **cannot** use the structure theorem for modules over a PID on this homework.

[Note: Q6 was intended to be a helpful intermediate step to help you solve Q7, and some of you did it this way. But for others, Q6′ below might be an easier intermediate step. You can do Q6′ in place of Q6 if you prefer. Or, if you already have a direct proof of Q7, you can just skip Q6/Q6′ entirely.]

Question 6. Prove that if M is torsion-free and finitely generated, then

$$\mathrm{Tor}_k(M,X)=0 \qquad \text{for all } k>0 \qquad \text{and any } X.$$

Question 6′. (replaces Q6) Prove that if M is torsion-free, then

$$\mathrm{Tor}_k(M,X)=0 \qquad \text{for all } k>0 \qquad \text{and any } \textit{finitely generated } X.$$

Question 7. Deduce from Q6, *or* from Q6′, *or* prove directly: for any torsion-free M,

$$\mathrm{Tor}_k(M,X)=0 \qquad \text{for all } k>0 \qquad \text{and any } X.$$

[If you give a self-contained direct proof for Q7, you will automatically get credit for Q6.]

Question 8. Deduce from the previous question that for *any* M,

$$\mathrm{Tor}_k(M,X)=0 \qquad \text{for all } k>1 \qquad \text{and any } X.$$

Do at least one of the following questions. If you've seen one of these questions before, please at least try to do one of the others.

Question 9A. Compute $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z})$.

If M is a $\mathbb{Z}$-module, note that $d|n$ implies $nM \subset dM$, so there is a quotient map $\pi^d_n\colon M/nM \to M/dM$ (it descends from the identity $M \to M$, so in symbols it's just $\overline{m} \mapsto \overline{m}$).

Define $\textit{consist}(M)$ to be the submodule of $\prod_{n\in\mathbb{N}} M/nM$ defined by

$$\textit{consist}(M) \coloneqq \left\{ \ (m_n \in M/nM)_{n\in\mathbb{N}} \ \middle|\ d|n \implies \pi^d_n(m_n)=m_d \ \right\}$$

This makes *consist* an additive functor from $\mathbb{Z}$-modules to $\mathbb{Z}$-modules (you do not have to prove this). (Note that $\mathbb{N}=\{1,2,3,\ldots\}$ here; it does not include 0.)

Question 9B. Is *consist* an exact functor? Prove your answer is correct.

Question 9C. $\textit{consist}(\mathbb{Z})$ has a natural ring structure (for example, it is a subring of $\prod_{n\in\mathbb{N}}\mathbb{Z}/n\mathbb{Z}$); you do not have to prove this.

Describe the commutative ring $\mathbb{Q}\otimes_{\mathbb{Z}} \textit{consist}(\mathbb{Z})$.

(You have some flexibility here in what your "description" should be, but don't just rephrase the definition.)