Math 210A: Modern Algebra

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Homework 6

Due Thursday night, November 2 (technically 5am Nov. 3)

Question 1. Let A and B be two $n \times n$ matrices with entries in a field K.
Let L be a field extension of K, and suppose there exists $C \in \mathrm{GL}_n(L)$ such that $B = CAC^{-1}$.
Prove there exists $D \in \mathrm{GL}_n(K)$ such that $B = DAD^{-1}$.
(that is, turn the argument sketched in class into an actual proof)

Question 2. Let k be an algebraically closed field of characteristic $\neq 2$. Fix two nonzero elements $\lambda, \mu \in k$. Let V and W be 2-dimensional k-vector spaces. Let $\alpha\colon V \to V$ have matrix $\left(\begin{smallmatrix} \lambda & 0 \\ 1 & \lambda \end{smallmatrix}\right)$, and let $\beta\colon W \to W$ have matrix $\left(\begin{smallmatrix} \mu & 0 \\ 1 & \mu \end{smallmatrix}\right)$. Let $\gamma\colon V \otimes W \to V \otimes W$ be $\alpha \otimes \beta$ (defined on elementary tensors by $\gamma(v \otimes w) = \alpha(v) \otimes \beta(w)$).

Find the Jordan decomposition of γ (that is, give a list of blocks and their sizes, and prove your answer is correct). Give a basis for all eigenspaces of γ. What happens if char $k = 2$?

Question 3. Let k be an algebraically closed field of characteristic $\neq 3$. Let V be an n-dimensional k-vector space, and suppose that $T\colon V \to V$ has minimal polynomial $(t - \lambda)^n$ for some nonzero $\lambda \in k$. Find the Jordan decomposition of T^3.

Question 4. Let V be a finite-dimensional nonzero vector space over a field k.

(a) For each monic irreducible $\pi \in k[t]$, define

$$V(\pi) = \left\{ v \in V \;\middle|\; \exists k \in \mathbb{N} \text{ s.t. } (\pi(T))^k(v) = 0 \right\}.$$

(When k is algebraically closed, these are the *generalized eigenspaces* $V_\lambda = V(t - \lambda)$ of T.)

Prove that $V(\pi) \neq 0$ if and only if $\pi | m_T$, and that $V = \bigoplus_{\pi | m_T} V(\pi)$.

An endomorphism $T\colon V \to V$ is *semisimple* if every T-stable subspace of V admits a T-stable complementary subspace: i.e. for every $T(U) \subseteq U$ there exists a decomposition $V = U \oplus W$ with $T(W) \subseteq W$.
(Keep in mind that such a complement is not unique in general; e.g. consider scalar multiplication by 2.)

(b) Use rational canonical form to prove that T is semisimple if and only if m_T has no repeated irreducible factor over k. (Hint: apply (a) to T-stable subspaces of V to reduce to the case when m_T has one monic irreducible factor.) Deduce that

(i) a Jordan block of rank > 1 is never semisimple,

(ii) if T is semisimple then m_T is the "squarefree part" of χ_T, and

(iii) if T is semisimple and $U \subseteq V$ is a T-stable nonzero proper subspace then the induced endomorphisms $T_U\colon U \to U$ and $\overline{T}\colon V/U \to V/U$ are semisimple.

(c) Let V' be another nonzero finite-dimensional k-vector space, and let $T'\colon V' \to V'$ be another endomorphism. Prove that T and T' are semisimple if and only if the endomorphism $T \oplus T'$ of $V \oplus V'$ is semisimple.

Question 5. Let V be a n-dimensional k-vector space with $0 < n < \infty$, and let $T\colon V \to V$ be an endomorphism.

(a) Using rational canonical form and Cayley–Hamilton, prove the following are equivalent:

   (a) $\exists k \geq 1$ such that $T^k = 0$.

   (b) $T^n = 0$.

   (c) There is an ordered basis of V w.r.t. which the matrix for T is upper triangular with 0's on the diagonal.

   (d) $\chi_T = t^n$.

   We call such T *nilpotent*.

(b) We say that T is *unipotent* if $T - 1$ is nilpotent. Formulate characterizations of unipotence analogous to the conditions in (a), and prove that a unipotent T is invertible.

(c) Assume k is algebraically closed. Using Jordan canonical form and generalized eigenspaces, prove that there is a unique expression

$$T = T_{\mathrm{ss}} + T_{\mathrm{n}}$$

where T_{ss} and T_{n} are a pair of *commuting* endomorphisms of V with T_{ss} semisimple and T_{n} nilpotent. (This is the *additive Jordan decomposition* of T.)

Show in general that $\chi_T = \chi_{T_{\mathrm{ss}}}$ (so T is invertible if and only if T_{ss} is invertible)

Show by example with $\dim V = 2$ that uniqueness fails if we drop the "commuting" requirement. (You just need to give the matrix T and the two decompositions $T = T_{\mathrm{ss}} + T_{\mathrm{n}}$ and $T = T'_{\mathrm{ss}} + T'_{\mathrm{n}}$; you do not need to prove these matrices are semisimple/nilpotent, as long as they are.)

(d) Assume k is algebraically closed and $S\colon V \to V$ is invertible. Using the existence and uniqueness of additive Jordan decomposition, prove that there is a unique expression

$$S = S_{\mathrm{ss}} S_{\mathrm{u}}$$

where S_{ss} and S_{u} are *commuting* endomorphisms of V with S_{ss} semisimple and S_{u} unipotent (so S_{ss} is necessarily invertible too). This is the *multiplicative Jordan decomposition* of T.