Math 210A: Modern Algebra

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Homework 7

Due Thursday night, November 9 (technically **2am** Nov. 10)

Question 1. Recall that in class we used the free resolution from HW4 Q4(g) to compute for $G = \mathbb{Z}/2 = \{1, s\}$ that

$$H^k(\mathbb{Z}/2;\, M) = \begin{cases} M^G & k = 0 \\ \frac{\{m \in M \mid sm+m=0\}}{\{sn-n \mid n \in M\}} & k = 1, 3, 5, \ldots \\ \frac{\{m \in M \mid sm=m\}}{\{sn+n \mid n \in M\}} & k = 2, 4, 6, \ldots \end{cases}$$

For $G = \mathbb{Z}/n = \{1, s, \ldots, s^{n-1}\}$, find a similar description of $H^k(\mathbb{Z}/n;\, M)$ for a $\mathbb{Z}G$-module M.
(Hint: find a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module; note that $\mathbb{Z}G \cong \mathbb{Z}[s]/(s^n - 1)$.
The resolution will again be 2-periodic just like for $\mathbb{Z}[s]/(s^2 - 1)$.

Question 2. Let G be a group.

(a) Prove that $H^0(G; \mathbb{Z}G) \cong \mathbb{Z}$ if G is finite, and $H^0(G; \mathbb{Z}G) = 0$ if G is infinite.

(b) Prove that $H^1(G; \mathbb{Z}G) \neq 0$ if $G = \mathbb{Z} = \{\ldots, t^{-1}, 1, t, \ldots\}$.

(c) (Hard, very optional) Can you find another group for which $H^1(G; \mathbb{Z}G) \neq 0$?

Question 3. Let L/K be a finite Galois extension with Galois group $G = \mathrm{Gal}(L/K)$. The unit group $L^\times$ is an abelian group with an action of G, so we may consider the group cohomology $H^k(G; L^\times)$. A theorem of Noether states that $H^1(G; L^\times) = 0$; you may assume this without proof.

(a) Use Noether's theorem to prove that if $\mathrm{Gal}(L/K)$ is generated by a single element s, then every element $\ell \in L$ with norm 1 has[1] the form $s(z)/z$ for some $z \in L$.

(b) Use part (a) to give a parametrization in two rational parameters of the rational points on the unit circle:

$$S^1(\mathbb{Q}) = \{(x \in \mathbb{Q}, y \in \mathbb{Q}) \mid x^2 + y^2 = 1\}.$$

That is, give two rational functions $x(a, b) \in \mathbb{Q}(a, b)$ and $y(a, b) \in \mathbb{Q}(a, b)$ such that the resulting function $f\colon \mathbb{Q}^2 \to \mathbb{Q}^2$ given by $(a, b) \mapsto \big(x(a, b), y(a, b)\big)$ has image $S^1(\mathbb{Q})$.

NOTE: f might not not actually be a function $\mathbb{Q}^2 \to \mathbb{Q}^2$ because a rational function like $x(a, b)$ might take the value ∞ sometimes. So formally this should say
"the resulting function $f\colon \{\text{subset of } \mathbb{Q}^2 \text{ where both } x \text{ and } y \text{ are not } \infty\} \to \mathbb{Q}^2$ has image $S^1(\mathbb{Q})$."

(cont.)

[1] Recall that for a Galois extension L/K the norm $N_K^L\colon L \to K$ is given by $N_K^L(\ell) = \prod_{g \in \mathrm{Gal}(L/K)} g \cdot \ell$.

Given a chain complex $C_\bullet = \to \cdots C_2 \to C_1 \to C_0 \to 0$ and a chain map $f\colon C_\bullet \to C_\bullet$:

We call f an *involution* if $f \circ f = \mathrm{id}$.

We call f a *weak involution* if there is a *homotopy* $f \circ f \sim \mathrm{id}$.

Question 4. Give an example of a chain complex $C_\bullet$ and a weak involution $f\colon C_\bullet \to C_\bullet$ that is not an involution.

Question 5. (Optional, replaces Q4) Give an example of a chain complex $C_\bullet$ and a weak involution $f\colon C_\bullet \to C_\bullet$ that is not *homotopic* to an involution.
(That is, there does not exist any involution $g\colon C_\bullet \to C_\bullet$ with $g \circ g = \mathrm{id}$ and $f \sim g$.)

Question 6. (Stupid hard, worth 0 points; only respect and admiration)
OK to discuss with classmates and even turn in joint solution, but *not* to discuss with others outside class. Can be turned in any time before the last day of class.

Give an example of a chain complex $C_\bullet$ and a weak involution $f\colon C_\bullet \to C_\bullet$ that cannot be made homotopic to an involution *even if we replace $C_\bullet$ by a homotopy equivalent complex.*

More precisely, note that given a homotopy equivalence $\varphi\colon C_\bullet \to D_\bullet$ with homotopy inverse $\psi\colon D_\bullet \to C_\bullet$, then the map $\varphi \circ f \circ \psi\colon D_\bullet \to D_\bullet$ will be a weak involution.

Say that f is "fixable" if for some such φ and ψ this weak involution $\varphi \circ f \circ \psi$ is homotopic to an involution $g\colon D_\bullet \to D_\bullet$.

Give an example of a chain complex $C_\bullet$ and weak involution $f\colon C_\bullet \to C_\bullet$ that is not "fixable".