MATH 210A, FALL 2017
HW 8 SOLUTIONS
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Question 1. Let $V = \mathbf{R}^{2n}$ with basis $v_1, \dots, v_{2n}$.
Find an explicit vector $\omega \in \bigwedge^2 V$ such that $\omega \wedge \omega \wedge \cdots \wedge \omega \in \bigwedge^{2n} V$ is nonzero.

Solution. Relabel the basis $v_1, \dots, v_{2n}$ as $e_1, \dots, e_n, f_1, \dots, f_n$ (i.e. $e_i = v_i$ and $f_i = v_{n+i}$ for $1 \leq i \leq n$).
Define

$$\omega = e_1 \wedge f_1 + \cdots + e_n \wedge f_n$$

We will show by induction on m that

$$\wedge^m (\omega) = m! \sum_{\substack{I \subset [n] \\ |I| = m}} \bigwedge\nolimits_{i \in I}(e_i \wedge f_i) := m! \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq n} (e_{i_1} \wedge f_{i_1}) \wedge (e_{i_2} \wedge f_{i_2}) \wedge \cdots \wedge (e_{i_m} \wedge f_{i_m}) \tag{1}$$

Clearly this formula holds for $m = 1$, by definition of ω. Now we may assume that it holds for some m. We compute:

$$\wedge^{m+1}(\omega) = \omega \wedge \left(\wedge^m(\omega)\right) = m! \sum_{\substack{I \subset [n] \\ |I| = m}} \omega \wedge \left(\bigwedge\nolimits_{i \in I}(e_i \wedge f_i)\right) = m! \sum_{\substack{I \subset [n] \\ |I| = m}} \sum_{j=1}^{n} (e_j \wedge f_j) \wedge \left(\bigwedge\nolimits_{i \in I}(e_i \wedge f_i)\right)$$

Now, note that $(e_j \wedge f_j) \wedge \left(\bigwedge_{i \in I}(e_i \wedge f_i)\right) = 0$ whenever $j \in I$, since the wedge product is alternating. In addition, we have $(e_j \wedge f_j) \wedge (e_i \wedge f_i) = (e_i \wedge f_i) \wedge (e_j \wedge f_j)$, since the permutation swapping these two wedge products is even. Thus, for $j \notin I$, we may rewrite the term $(e_j \wedge f_j) \wedge \left(\bigwedge_{i \in I}(e_i \wedge f_i)\right)$ as $\bigwedge_{i \in I \cup \{j\}}(e_i \wedge f_i)$ (recalling that the latter notation means by definition that the i are taken in increasing order). Thus, we have:

$$\wedge^{m+1}(\omega) = m! \sum_{\substack{I \subset [n] \\ |I| = m}} \sum_{j \notin I} \left(\bigwedge\nolimits_{i \in I \cup \{j\}}(e_i \wedge f_i)\right) = (m+1)! \sum_{\substack{J \subset [n] \\ |J| = m+1}} \bigwedge\nolimits_{j \in J}(e_j \wedge f_j)$$

The last equality is true because there are exactly $m + 1$ ways to each $J \subseteq [n]$ with $|J| = m + 1$ as $J = I \cup \{j\}$ for $j \notin I$ and $|I| = m$. Thus, we have proved our desired formula (1). Taking $m = n$ in (1) shows that $\wedge^n(\omega) = n! \bigwedge_{i \in [n]}(e_i \wedge f_i) = n!\,(-1)^n\,(v_1 \wedge v_2 \wedge \cdots \wedge v_{2n})$. This is a nonzero multiple of the standard basis element for one-dimensional $\mathbf{R}$-vector space $\bigwedge^{2n} V$ with respect to the basis $v_1, \dots, v_{2n}$, so it is nonzero.

Question 2. Let $R = \mathbf{Z}$ and let M be the R-module $M = \mathbf{Q}/\mathbf{Z}$. Compute $T^*(M)$.

Solution. First, we will show that $M \otimes_R M = 0$. Indeed, consider any elementary tensor $\frac{m}{n} \otimes \frac{p}{q}$. Then since $\frac{p}{q} = \frac{pn}{qn}$, we have $\frac{m}{n} \otimes \frac{p}{q} = \frac{m}{n} \otimes \frac{pn}{qn} = \frac{nm}{n} \otimes \frac{p}{qn} = 0$, since $\frac{nm}{n} = m \in \mathbf{Z}$, so it is 0 in $\mathbf{Q}/\mathbf{Z}$. More generally, this argument shows that the tensor product over $\mathbf{Z}$ of any torsion abelian group[1] with any "divisible" abelian group[2] is 0. In this case $\mathbf{Q}/\mathbf{Z}$ is *both* torsion and divisible, so the tensor vanishes. Thus,

$$T^*(M) = \oplus_{n \geq 0} M^{\otimes n} = R \oplus M = \mathbf{Z} \oplus (\mathbf{Q}/\mathbf{Z}).$$

This ring consists of all pairs $(a, \frac{b}{c})$ with $\frac{b}{c} \in \mathbf{Q}/\mathbf{Z}$, and the multiplication is given by $(a, \frac{b}{c}) \cdot (a', \frac{b'}{c'}) = (aa', \frac{a'b}{c} + \frac{ab'}{c})$.

The above is a sufficient answer to the question. But if we liked we could rewrite it in various ways. For example, writing $\epsilon_n = \frac{1}{n} \in \mathbf{Q}/\mathbf{Z}$, we could write this ring alternatively as:

$$T^*(M) = \mathbf{Z}[\epsilon_n \mid n \in \mathbf{Z}_{>0}]/(n\epsilon_n = 0, a\epsilon_{an} = \epsilon_n, \epsilon_n + \epsilon_m = (n+m)\epsilon_{nm}, \epsilon_n \cdot \epsilon_m = 0)$$

More parsimoniously, using the fact that $\mathbf{Q}/\mathbf{Z} = \bigoplus_p \mathbf{Z}[\frac{1}{p}]/\mathbf{Z}$, we could write this as:

$$T^*(M) = \mathbf{Z}[\epsilon_{p^k} \mid p \text{ prime}, k > 0]/(p^k \epsilon_{p^k} = 0, p\epsilon_{p^k} = \epsilon_{p^{k-1}}, \epsilon_{p^k} \cdot \epsilon_{q^\ell} = 0)$$

Question 3. Let $R = \mathbf{Z}[\sqrt{-30}]$, and let I be the ideal $I = \{2a + b\sqrt{-30} \mid a, b \in \mathbf{Z}\} = (2, \sqrt{-30}) \subset R$.
Compute $\bigwedge^2 I$ as an abelian group.
(but keep in mind the $\bigwedge^2$ is as an R-module, i.e. it's a quotient of $I \otimes_R I$).

Solution. First, we will describe $I \otimes_R I$. Since I is generated as an R-module by 2 and $\sqrt{-30}$, $I \otimes_R I$ is generated by $\{2 \otimes \sqrt{-30}, \sqrt{-30} \otimes 2, 2 \otimes 2, \sqrt{-30} \otimes \sqrt{-30}\} =: \{e_1, e_2, e_3, e_4\}$. If $\pi \colon I \otimes_R I \to \bigwedge^2 I$ is the canonical quotient map, we have $\pi(e_3) = \pi(e_4) = 0$ and $\pi(e_2) = -\pi(e_1)$. Thus, $\bigwedge^2 I$ is generated by $v_1 := \pi(e_1) = 2 \wedge \sqrt{-30}$.

We now check directly that $v_1 = 0$. First, we check that $2 \cdot v_1 = 0$. Indeed,

$$2v_1 = 2 \wedge 2\sqrt{-30} = (\sqrt{-30}) \cdot (2 \wedge 2) = (\sqrt{-30}) \cdot 0 = 0.$$

Second, we check that $15 \cdot v_1 = 0$. Indeed,

$$15v_1 = 30 \wedge \sqrt{-30} = (\sqrt{-30}) \cdot (\sqrt{-30} \wedge \sqrt{-30}) = (\sqrt{-30}) \cdot 0 = 0.$$

Therefore $15v_1 - 2v_1 - \cdots - 2v_1 = v_1$ is also equal to 0. This proves that $\bigwedge^2 I = 0$.

For a less direct approach, we could consider the exact sequence

$$0 \to I \to R \to R/I \simeq \mathbf{F}_2 \to 0$$

We get an associated long exact Tor sequence:

$$\cdots \operatorname{Tor}_1^R(R, I) \to \operatorname{Tor}_1^R(R/I, I) \to I \otimes_R I \to I \to I/I^2 \to 0$$

But I is projective, so $\operatorname{Tor}_1^R(R/I, I) = 0$ and thus $I \otimes_R I \simeq \ker(I \to I/I^2) = I^2$. This is the ideal of R given by $(4, 2\sqrt{-30}, -30) = (2)$ (note that 2 is the greatest common divisor of 4 and -30, so 2 is contained

[1] i.e. every element has finite order

[2] i.e. an abelian group A where for any integer n and any $a \in A$ there exists some $b \in A$ with $a = nb$

in this ideal, and conversely every element of this ideal is a multiple of 2). The map $I \otimes_R I \to I^2$ is given by $i_1 \otimes i_2 \mapsto i_1 i_2$. The isomorphism $I \otimes_R I \xrightarrow{\sim} I^2$ maps e_1, e_2 to $2\sqrt{-30}$, e_3 to 4, and e_4 to -30. Thus, this isomorphism sends $8e_3 + e_4$ to the generator 2. Therefore, $8e_3 + e_4$ generates $I \otimes_R I$ as an R-module. But we've seen that $\pi(e_3) = \pi(e_4) = 0$, so $\pi(8e_3 + e_4) = 0$ and therefore $\bigwedge^2 I = 0$.

There is an even less direct way to show that $\bigwedge^2 I = 0$ which might shed some light on what is "really" going on [TC: but this is *not* what I was looking for here]. Recall that I is a finitely presented projective R-module, and in fact it is locally free of rank one. (This follows essentially from the fact that $R_\mathfrak{m}$ is a principal ideal domain for each maximal ideal $\mathfrak{m}$ of R, so $I_\mathfrak{m} \subseteq R_\mathfrak{m}$ is a principal ideal and thus a free $R_\mathfrak{m}$-module). It is not hard to see that the formation of $\bigwedge^2 I$ commutes with localization, i.e. that $(\bigwedge^2 I)_\mathfrak{m} = \bigwedge^2 I_\mathfrak{m}$ (indeed, since localizing at $\mathfrak{m}$ is the same thing as tensoring with $R_\mathfrak{m}$, this is obvious for $I^{\otimes 2}$, and then it is easy to check that this isomorphism preserves the submodule of elements of the form $i \otimes i$). But $I_\mathfrak{m} \simeq R_\mathfrak{m}$, and we know that exterior powers of rank-one free modules are zero. Thus, $(\bigwedge^2 I)_\mathfrak{m} = 0$ for all maximal ideals $\mathfrak{m}$, and thus $\bigwedge^2 I = 0$. This argument is perfectly general and shows that the exterior powers $\bigwedge^k M$ with $k > r$ all vanish when M is locally free of rank r over any ring.

Question 4. Prove that for any R-modules M and N, and any $k \geq 0$, there is an isomorphism

$$\bigwedge^k (M \oplus N) \cong \bigoplus_{a+b=k} (\bigwedge^a M) \otimes (\bigwedge^b N).$$

(If M and N are free, this is pretty easy, because the natural basis for $\bigwedge^k (M \oplus N)$ splits up appropriately; the resulting partition corresponds to the combinatorial identity $\binom{m+n}{k} = \sum_{a+b=k} \binom{m}{a}\binom{n}{b}$. This doesn't help directly for general M and N, but perhaps it at least helps you get straight what's going on.)

NOTE on Q4: If you like, you can prove this just for $k = 3$, i.e. that

$$\bigwedge^3 (M \oplus N) \cong \begin{array}{l} \bigwedge^3 M \\ \oplus\ (\bigwedge^2 M) \otimes N \\ \oplus\ M \otimes (\bigwedge^2 N) \\ \oplus\ \bigwedge^3 N \end{array}$$

This is no harder or easier than the general case, but might be simpler notationally.

Solution. We will define maps in both directions and then verify that they are inverse to each other. First, we define a map from right to left, i.e. a map

$$\Psi\colon \bigoplus_{a+b=k} (\bigwedge^a M) \otimes \left(\bigwedge^b N\right) \to \bigwedge^k (M \oplus N)$$

To do this, via the universal property of the direct sum, this is equivalent to defining maps $\Psi_{a,b}$ for each summand. Thus, we need to define $\Psi_{a,b}\colon (\bigwedge^a M) \otimes \left(\bigwedge^b N\right)$ to $\bigwedge^k (M \oplus N)$. By the universal property of the tensor product and the exterior power, this is the same thing as defining a multilinear map $\psi(m_1, \ldots, m_a, n_1, \ldots, n_b)$ with $m_i \in M, n_i \in N$ taking values in $\bigwedge^k (M \oplus N)$ such that if any two m_i or any two n_i are equal, then $\psi(m_1, \ldots, m_a, n_1, \ldots, n_b) = 0$. We define:

$$\psi(m_1, \ldots, m_a, n_1, \ldots, n_b) = m_1 \wedge \cdots \wedge m_a \wedge n_1 \cdots \wedge n_b$$

Since the wedge product is alternating and multilinear, we see that this definition satisfies the required properties, so we get a well-defined map Ψ.

Going the other direction, to define $\Phi\colon \bigwedge^k(M\oplus N)\to\bigoplus_{a+b=k}\left(\bigwedge^a M\right)\otimes\left(\bigwedge^b N\right)$, the universal property of the direct product (which is naturally isomorphic to the direct sum since there are finitely many summands) implies that it is equivalent to define $\Phi_{a,b}\colon \bigwedge^k(M\oplus N)\to\left(\bigwedge^a M\right)\otimes\left(\bigwedge^b N\right)$. Then, the universal property of the exterior power says that this is equivalent to defining a multilinear map $\varphi((m_1,n_1),\dots,(m_k,n_k))$ to $\left(\bigwedge^a M\right)\otimes\left(\bigwedge^b N\right)$. Using the notation defined in (1), we define:

$$\varphi((m_1,n_1),\dots,(m_k,n_k))=\sum_{\substack{I\sqcup J=[k]\\|I|=a}}\epsilon_{I,J}\left(\bigwedge_{i\in I}m_i\right)\otimes\left(\bigwedge_{j\in J}n_j\right)\tag{2}$$

Here, the $\epsilon_{I,J}$ are ± 1. We will leave these as indeterminates for now, and derive what they have to be in order to make Φ,Ψ well-defined and mutually inverse.

To see that this is alternating, assume that $(m_i,n_i)=(m_j,n_j)$ for some $i<j$. If $I\sqcup J=[k]$ and i,j are either both in I or both in J, then the corresponding term in the sum vanishes due to the fact that the wedge product is alternating and $m_i=m_j$, $n_i=n_j$. For all of the remaining terms, either $i\in I$ or $j\in J$, so we may write:

$$\begin{aligned}\varphi((\mathbf{m},\mathbf{n}))&=\sum_{\substack{I\sqcup J=[k]\\|I|=a\\i\in I,j\in J}}\epsilon_{I,J}\left(\bigwedge_{i\in I}m_i\right)\otimes\left(\bigwedge_{j\in J}n_j\right)+\sum_{\substack{I\sqcup J=[k]\\|I|=a\\i\in J,j\in I}}\epsilon_{I,J}\left(\bigwedge_{i\in Im_i}\right)\otimes\left(\bigwedge_{j\in J}n_j\right)\\&=\sum_{\substack{I\sqcup J=[k]\\|I|=a\\i\in I,j\in J}}\left(\epsilon_{I,J}\left(\bigwedge_{i\in I}m_i\right)\otimes\left(\bigwedge_{j\in J}n_j\right)+\epsilon_{I',J'}\left(\bigwedge_{i\in I'}m_i\right)\otimes\left(\bigwedge_{j\in J'}n_j\right)\right)\end{aligned}$$

Here, for any I,J with $i\in I$, $j\in J$, we define I',J' by swapping i and j, i.e. $I'=(I-\{i\})\cup\{j\}$ and $J'=(J-\{j\})\cup\{i\}$. Since $m_i=m_j$, $\bigwedge_{i\in I'}m_i=(-1)^{N_{I,i,j}}\bigwedge_{i\in I}m_i$, with $N_{I,i,j}$ equal to the number of elements of I which are strictly between i and j. Similarly, $\bigwedge_{j\in J'}n_j=(-1)^{N_{J,i,j}}\bigwedge_{j\in J}n_j$. Note that $N_{I,i,j}+N_{J,i,j}$ is just the number of elements of $[n]$ strictly between i and j, which is $(j-i)-1$.

Thus, we come to our first requirement on $\epsilon_{I,J}$:

$$\epsilon_{I',J'}=(-1)^{(j-i)}\epsilon_{I,J}\tag{3}$$

If the $\epsilon_{I,J}$ satisfy Equation (3), then Φ is well-defined. Now, let's see that Φ and Ψ are mutually inverse. To see that $\Phi\circ\Psi\colon \bigoplus_{a+b=k}\left(\bigwedge^a M\right)\otimes\left(\bigwedge^b N\right)\to\bigoplus_{a+b=k}\left(\bigwedge^a M\right)\otimes\left(\bigwedge^b N\right)$ is equal to the identity, we may check it on each component, i.e. we may check that $\Phi_{a,b}\circ\Psi_{a,b}$ is the identity. Unwinding the definitions, we see that this is the statement that:

$$\varphi((m_1,0),\dots,(m_a,0),(0,n_1),\dots,(0,n_b))=(m_1\wedge\cdots m_a)\otimes(n_1\wedge\cdots\wedge n_b)$$

But we can compute the left-hand side via (2) as:

$$\varphi((m_1,0),\dots,(m_a,0),(0,n_1),\dots,(0,n_b))=\epsilon_{[1,\dots,a],[a+1,\dots,k]}(m_1\wedge\cdots\wedge m_a)\otimes(n_1\wedge\cdots\wedge n_b)$$

Thus, we get our second requirement on $\epsilon_{I,J}$:

$$\epsilon_{[1,\dots,a],[a+1,\dots,k]}=1\tag{4}$$

We've shown that as long as (4) is satisfied, $\Phi \circ \Psi = \mathrm{id}$. Now, we need to check that $\Psi \circ \Phi = \mathrm{id}$. By unwinding the definitions, this says that:

$$(m_1 + n_1) \wedge \cdots \wedge (m_k + n_k) = \sum_{a+b=k} \sum_{\substack{I \sqcup J = [k] \\ |I| = a}} \epsilon_{I,J} \left(\bigwedge_{i \in I} m_i\right) \wedge \left(\bigwedge_{j \in J} n_j\right)$$

The linearity of the wedge product shows that:

$$(m_1 + n_1) \wedge \cdots \wedge (m_k + n_k) = \sum_{a+b=k} \sum_{\substack{I \sqcup J = [k] \\ |I| = a}} \bigwedge_{i \in [n]} \alpha_i$$

Here, $\alpha_i = m_i$ if $i \in I$ and $\alpha_i = n_i$ if $i \in J$. By the skew-symmetry of the wedge product, we see that:

$$\bigwedge_{i \in [n]} \alpha_i = \mathrm{sgn}(\sigma_{I,J}) \left(\bigwedge_{i \in I} m_i\right) \wedge \left(\bigwedge_{j \in J} n_j\right)$$

Here, $\sigma_{I,J}$ is the permutation of $[k]$ which maps $1, \ldots, a$ to $i_1, \ldots, i_a$ (meaning that it maps 1 to i_1, 2 to i_2, etc.) and $a+1, \ldots, k$ to $j_1, \ldots, j_b$, where $i_1 < i_2 < \cdots < i_a$ and $j_1 < j_2 < \cdots < j_b$ are the elements of I, J respectively. Thus, we get our third requirement on $\epsilon_{I,J}$, which serves as a definition for $\epsilon_{I,J}$.

$$\epsilon_{I,J} = \mathrm{sgn}(\sigma_{I,J}) \tag{5}$$

Now, we are reduced to checking that if we define the $\epsilon_{I,J}$ via (5), they satisfy (3) and (4). It is easy to check (4), since $\sigma_{[1,\ldots,a],[a+1,\ldots,k]}$ is the identity permutation. Now, (3) says that:

$$\mathrm{sgn}(\sigma_{I',J'}) = (-1)^{(j-i)} \mathrm{sgn}(\sigma_{I,J})$$

To do this, we will write $\sigma_{I',J'} = \sigma_{i,j}\sigma_{I,J}$ and show that $\mathrm{sgn}(\sigma_{i,j}) = (-1)^{(j-i)}$. We have $\sigma_{i,j} = \sigma_{I',J'}\sigma_{I,J}^{-1}$. Since $\sigma_{I,J}^{-1}$ takes $i_1, \ldots, i_a$ to $1, \ldots, a$ and $\sigma_{I',J'}$ takes $1, \ldots, a$ to $i'_1, \ldots, i'_a$, we see that $\sigma_{i,j}$ takes $i_1, \ldots, i_a$ to $i'_1, \ldots, i'_a$ and likewise for J.

If $p < i$ or $p > j$, then this permutation fixes p: this is because the segments of I (resp. J) and I' (resp. J') below i and above j are the same. Label the elements of I such that:

$$i_1 < \cdots < i = i_\alpha < i_{\alpha+1} < \cdots < i_\beta < j < i_{\beta+1} < \cdots < i_a$$

and similarly for J:

$$j_1 < \cdots < j_{\gamma-1} < i < j_\gamma < \cdots < j_\delta = j < j_{\delta+1} < \cdots < j_b$$

Now, $i'_p = i_p$ for $p < \alpha$ or $p > \beta$, $i'_p = i_{p+1}$ for $\alpha \le p < \beta$, $i'_\beta = j$. Similarly, $j'_p = j_p$ for $p < \gamma$ or $p > \delta$, $j'_\gamma = i$, $j'_p = j_{p-1}$ for $\gamma < p \le \delta$. Thus, $\sigma_{i,j}$ is:

$$(i\ i_{\alpha+1}\ i_{\alpha+2}\ \cdots\ i_{\beta-1}\ i_\beta\ j\ j_{\delta-1}\ j_{\delta-2}\ \cdots\ j_{\gamma+1}\ j_\gamma)$$

This is a cyclic permutation moving all elements $p \in [k]$ with $i \le p \le j$. There are $(j-i)+1$ of these. Now, a cyclic permutation moving ℓ elements is a product of $\ell - 1$ transpositions and thus it has sign $(-1)^{\ell-1}$, so we have $\mathrm{sgn}(\sigma_{i,j}) = (-1)^{j-i}$, as desired. This concludes the proof.

Question 5. Given an abelian group M and a subgroup $A \subset M$, define the *saturation* of A to be

$$\mathrm{sat}(A) = \{m \in M \mid \exists n \neq 0 \in \mathbf{Z} \text{ s.t. } n \cdot m \in A\}.$$

This $\mathrm{sat}(A)$ is a subgroup of M (you may assume this without proof).

Prove that if M is finitely generated, then for any subgroup $A \subset M$ the saturation $\mathrm{sat}(A)$ is a direct summand of M; that is, there exists a subgroup $N \subset M$ such that $M = \mathrm{sat}(A) \oplus N$.

Solution. First, note that if A is a submodule of an abelian group M, then $\mathrm{sat}(A)$ is *saturated*, meaning that if $m \in M$ is such that $nm \in \mathrm{sat}(A)$ for some $n \neq 0 \in \mathbf{Z}$, then $m \in \mathrm{sat}(A)$. Indeed, by the definition of $\mathrm{sat}(A)$, if $nm \in \mathrm{sat}(A)$, then for some $n' \neq 0 \in \mathbf{Z}$, $(n'n)m = n'(nm) \in A$. But since $\mathbf{Z}$ is a domain, $n'n \neq 0$, so this means that $m \in \mathrm{sat}(A)$.

Now, since abelian groups are the same thing as $\mathbf{Z}$-modules, we may apply the structure theorem for finitely generated modules over a PID. In the case that the PID is equal to $\mathbf{Z}$, this says that any finitely generated abelian group L satisfies:

$$L \simeq \mathbf{Z}^r \oplus T$$

Here, T is a finite torsion abelian group. We will apply this to the finitely generated $\mathbf{Z}$-module $M/\mathrm{sat}(A)$: $M/\mathrm{sat}(A) \simeq \mathbf{Z}^r \oplus T$. Now, let $m \in M$ be such that $[m] \in T$. Since T is torsion, for some $n \neq 0 \in \mathbf{Z}$, $n[m] = [nm] = 0$ in M/A. This means that $nm \in \mathrm{sat}(A)$. But because $\mathrm{sat}(A)$ is saturated, this implies $m \in \mathrm{sat}(A)$, i.e. $[m] = 0$. Thus, we've seen that $T = 0$, so $M/\mathrm{sat}(A) \simeq \mathbf{Z}^r$. Thus, we have an exact sequence of abelian groups:

$$0 \to \mathrm{sat}(A) \to M \to \mathbf{Z}^r \to 0$$

However, since $\mathbf{Z}^r$ is a free $\mathbf{Z}$-module, it is projective, so the above exact sequence splits. Let $\sigma\colon \mathbf{Z}^r \to M$ be a splitting. Define $N = \sigma(\mathbf{Z}^r) \simeq \mathbf{Z}^r$ (since σ is injective). Then the map $\mathrm{sat}(A) \oplus N \to M$ induced by the inclusions is an isomorphism (this follows from the definition of a splitting since the composite of the inclusion of N into M and the isomorphism $\mathbf{Z}^r \xrightarrow{\sim} N$ induced by σ is exactly σ).

Question 6. Given k linearly independent vectors $v_1, \ldots, v_k$ in $\mathbf{Z}^n$, there are two definitions of the *discriminant*:

Definition 1: Consider the element $\omega = v_1 \wedge \cdots \wedge v_k \in \bigwedge^k \mathbf{Z}^n$.
Let $\mathrm{disc}_1(v_1, \ldots, v_k)$ be the largest $d \in \mathbf{N}$ such that ω is divisible by d.
(i.e. such that there exists some other $\mu \in \bigwedge^k \mathbf{Z}^n$ such that $\omega = d \cdot \mu$)

Definition 2: Let $K = \langle v_1, \ldots, v_k \rangle$ be the subgroup of $\mathbf{Z}^n$ generated by these elements. Let L be the quotient $L = \mathbf{Z}^n/K$. Let $\mathrm{disc}_2(v_1, \ldots, v_k)$ be the cardinality of the torsion subgroup $\mathrm{Torsion}(L)$.

Prove that $\mathrm{disc}_1(v_1, \ldots, v_k) = \mathrm{disc}_2(v_1, \ldots, v_k)$.

Solution. Let $K_0 = \mathrm{sat}(K)$. Then by Question 5, we have $\mathbf{Z}^n \simeq K_0 \oplus N$ for a subgroup N of $\mathbf{Z}^n$. Since K_0, N are subgroups of the finitely generated free $\mathbf{Z}$-module $\mathbf{Z}^n$, they are both free. Thus, $K_0 \simeq \mathbf{Z}^{m_1}$ and $N \simeq \mathbf{Z}^{m_2}$ for some m_1, m_2. Since $K_0 \oplus N \simeq \mathbf{Z}^n$, we have $m_1 + m_2 = n$. The fact that $v_1, \ldots, v_k$ are linearly independent says exactly that the morphism $\mathbf{Z}^k \to K$ given by mapping the basis vectors to the v_i is injective. Since it is clearly surjective, we see that $K \simeq \mathbf{Z}^k$. Now, since $K \subseteq K_0$, we have $k \leq m_1$. Conversely, let $w_1, \ldots, w_{m_1}$ be a basis for K_0. For each w_i, there is some $\ell_i \neq 0 \in \mathbf{Z}$ such that $\ell_i w_i \in K$. Taking ℓ to be the least common multiple of the ℓ_i, we see that $\ell \cdot K_0 \subseteq K$. But $\ell \cdot K_0 \simeq K_0 \simeq \mathbf{Z}^{m_1}$ since multiplication by ℓ is injective and thus an isomorphism onto its image. Thus, $m_1 \leq k$ so $m_1 = k$ and $m_2 = n - k$.

Since $K \subseteq K_0$, the isomorphism $K_0 \oplus N \xrightarrow{\sim} \mathbf{Z}^n$ gives an isomorphism $K_0/K \oplus N \xrightarrow{\sim} L$. Now, since $K_0 = \operatorname{sat}(K)$, for any $m \in K_0$ we have some $n \neq 0 \in \mathbf{Z}$ such that $nm \in K$, so $n[m] = 0$ in K_0/K. Thus, K_0/K is torsion. Since N is free, we see that the torsion subgroup of L is isomorphic to K_0/K. Thus, $\operatorname{disc}_2(v_1, \ldots, v_k)$ defined as above is the same as $\operatorname{disc}_2(v_1, \ldots, v_k)$ when we regard the v_i as elements of $K_0 \simeq \mathbf{Z}^k$.

Using Question 4 and the isomorphism $K_0 \oplus N \simeq \mathbf{Z}^n$, we will show the analogous statement is true for $\operatorname{disc}_1(v_1, \ldots, v_k)$, and this will allow us to assume that $n = k$. Indeed, Question 4 shows us that:

$$\textstyle\bigwedge^k(\mathbf{Z}^n) \simeq \displaystyle\bigoplus_{a+b=k} (\textstyle\bigwedge^a K_0) \otimes (\textstyle\bigwedge^b N)$$

Now, since $v_i \in K_0$ for each i, $\omega = v_1 \wedge \cdots \wedge v_k$ maps into the summand $\bigwedge^k K_0$ (i.e. the summand with $a = k$ and $b = 0$). This is because the N-component of v_i is 0 for each i, so by looking at the definition (2) of the above isomorphism Φ, we see that $\Phi_{a,b} = 0$ unless $(a, b) = (k, 0)$. Thus, if $\omega = d \cdot \mu$ with $\mu \in \bigwedge^k \mathbf{Z}^n$, $d \in \mathbf{Z}$, then $\mu \in \bigwedge^k K_0$. (This is because $\bigwedge^k(\mathbf{Z}^n)$ is a torsion-free $\mathbf{Z}$-module, so if $d \cdot \mu$ is in a direct summand of this module, then the components of μ in the other direct summands must vanish since their d-multiples do). Thus, we see that $\operatorname{disc}_1(v_1, \ldots, v_k)$ is the same when we regard the v_i as elements of $K_0 \simeq \mathbf{Z}^k$. Thus, we may assume $n = k$.

Now, if $e_1, \ldots, e_k$ is a basis for $\mathbf{Z}^k$, the $\mathbf{Z}$-module $\bigwedge^k(\mathbf{Z}^k)$ is free with generator $e_1 \wedge \cdots \wedge e_k$. Thus, we may write $\omega = Ce_1 \wedge \cdots \wedge e_k$, for some $C \in \mathbf{Z}$. If $\omega = d \cdot \mu$ for some $\mu \in \bigwedge^k(\mathbf{Z}^k)$, we may also write $\mu = C' e_1 \wedge \cdots \wedge e_k$, so $C = dC'$. Thus, we see that $\operatorname{disc}_1(v_1, \ldots, v_k) = |C|$; since the definition of the discriminant does not depend on a choice of basis, neither does $|C|$. Furthermore, if we write the v_j as $v_j = \sum_i a_{ij} e_i$, the matrix $M_{\mathbf{e},\mathbf{v}} = (a_{ij})$ sends e_j to v_j. Then, we have $C = \det M_{\mathbf{e},\mathbf{v}}$, as we can see by expanding ω as:

$$\omega = v_1 \wedge \cdots \wedge v_k = \left(\sum_i a_{i1} e_i\right) \wedge \cdots \wedge \left(\sum_i a_{ik} e_i\right) = \left(\sum_\sigma \operatorname{sgn}(\sigma) \prod_j a_{\sigma(j)j}\right) e_1 \wedge \cdots \wedge e_k$$

This holds since $e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(k)} = \operatorname{sgn}(\sigma) e_1 \wedge \cdots \wedge e_k$, and any terms with one of the e_i repeated are 0. This is one of the definitions of the determinant of the matrix $M_{\mathbf{e},\mathbf{v}} = (a_{ij})$. Note that if we pick another basis $f_1, \ldots, f_k$ of $\mathbf{Z}^k$, we have $M_{\mathbf{f},\mathbf{v}} = M_{\mathbf{e},\mathbf{f}} M_{\mathbf{e},\mathbf{v}} M_{\mathbf{f},\mathbf{e}} = M_{\mathbf{f},\mathbf{e}}^{-1} M_{\mathbf{e},\mathbf{v}} M_{\mathbf{f},\mathbf{e}}$, so $\det M_{\mathbf{e},\mathbf{v}} = \det M_{\mathbf{f},\mathbf{v}}$. Thus, we see that even C does not depend on the choice of basis for $\mathbf{Z}^k$.

Now, we will show that $|C| = \operatorname{disc}(v_1, \ldots, v_k)$ is unchanged by replacing $v_1, \ldots, v_k$ by any set of generators of K, the submodule of $\mathbf{Z}^k$ generated by the v_i. Indeed, $K \simeq \mathbf{Z}^k$ (this follows from linear independence of the v_i, as we've seen above). Thus, a set of generators $w_1, \ldots, w_k$ of $\mathbf{Z}^k$ is actually a basis (this follows from the fact that a surjective endomorphism of free modules is actually an isomorphism). This means that the matrix $M_{\mathbf{v},\mathbf{w}} = (b_{ij})$ taking v_j to $w_j = \sum_i b_{ij} v_i$ is invertible, so its determinant is ± 1. Since $M_{\mathbf{e},\mathbf{w}} = M_{\mathbf{v},\mathbf{w}} M_{\mathbf{e},\mathbf{v}}$, we see that $\operatorname{disc}_1(w_1, \ldots, w_k) = \left|\det M_{\mathbf{e},\mathbf{w}}\right| = \left|\pm \det M_{\mathbf{e},\mathbf{v}}\right| = \operatorname{disc}_1(v_1, \ldots, v_k)$. Note that $\operatorname{disc}_2(v_1, \ldots, v_k)$ by construction only depends on the submodule K generated by $v_1, \ldots, v_k$. Thus, we may freely replace $v_1, \ldots, v_k$ by any other basis for K.

Now, we need to pick some basis $e_1, \ldots, e_k$ of $\mathbf{Z}^k$ and $v_1, \ldots, v_k$ of K where we can compute $\det M_{\mathbf{e},\mathbf{v}}$, and then show that it is equal to $\operatorname{disc}_2(v_1, \ldots, v_k)$. We claim that we may choose these bases such that for each i, $v_i = d_i \cdot e_i$ for some $d_i \in \mathbf{Z}$. This follows from the proof of the structure theorem for finitely generated modules over a PID. Now, we may write down $M_{\mathbf{e},\mathbf{v}}$ as:

$$M_{\mathbf{e},\mathbf{v}} = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix}$$

Thus, $\det M_{\mathbf{e},\mathbf{v}} = \prod_i d_i$, so $\mathrm{disc}_1(v_1, \dots, v_k) = \prod_i |d_i|$. On the other hand, we may compute $\mathbf{Z}^k/K$ as:

$$\mathbf{Z}^k/K \simeq \mathbf{Z}e_1 \oplus \cdots \oplus \mathbf{Z}e_k/(\mathbf{Z}(d_1e_1) \oplus \cdots \oplus \mathbf{Z}(d_ke_k)) \simeq (\mathbf{Z}/d_1\mathbf{Z}) \oplus \cdots \oplus (\mathbf{Z}/d_k\mathbf{Z})$$

This group is torsion and clearly has order $\mathrm{disc}_2(v_1, \dots, v_k) = \prod_i |d_i|$, so we are done.

Question 7. Let V be an n-dimensional vector space over $\mathbf{Q}$, and fix $k \geq 1$.

Recall from class that for any endomorphism $T\colon V \to V$, we obtain an endomorphism $T_*\colon V^{\otimes k} \to V^{\otimes k}$ defined on generators by

$$T_*(v_1 \otimes \cdots \otimes v_k) = T(v_1) \otimes \cdots \otimes T(v_k).$$

In this question, we want to *find the endomorphisms of $V^{\otimes k}$ that commute with T_* for all T.*

Recall from class that the permutation group S_k acts on $V^{\otimes k}$ (on the right) by

$$(v_1 \otimes \cdots \otimes v_k) \cdot \sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$$

Let $f_\sigma \in \mathrm{End}_{\mathbf{Q}}(V^{\otimes k})$ be this endomorphism. We can easily check that this commutes with T_*, since

$$\begin{aligned} (T_*(v_1 \otimes \cdots \otimes v_k)) \cdot \sigma &= \big(T(v_1) \otimes \cdots \otimes T(v_k)\big) \cdot \sigma \\ &= T(v_{\sigma(1)}) \otimes \cdots \otimes T(v_{\sigma(k)}) \\ &= T_*(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}) \\ &= T_*\big((v_1 \otimes \cdots \otimes v_k) \cdot \sigma\big) \end{aligned}$$

In other words, $f_\sigma \circ T_* = T_* \circ f_\sigma$.

The same is automatically true for linear combinations: for any $x = \sum a_\sigma \sigma \in \mathbf{Q}[S_k]$, the endomorphism $f_x := \sum a_\sigma f_\sigma$ has the same property that $f_x \circ T_* = T_* \circ f_x$ for all $T \in \mathrm{End}_{\mathbf{Q}}(V)$.

Your task: prove that these are the *only* endomorphisms that commute with all T_*. That is, prove that if $g \in \mathrm{End}_{\mathbf{Q}}(V^{\otimes k})$ satisfies $g \circ T_* = T_* \circ g$ for all $T \in \mathrm{End}_{\mathbf{Q}}(V)$, then there exists some $x \in \mathbf{Q}[S_k]$ such that $g = f_x$.

Solution. First, we will assume that $n \geq k$. Let $e_1, \dots, e_n$ be a basis for V. Recall that $V^{\otimes k}$ has a basis consisting of all vectors of the form $e_{i_1} \otimes \cdots \otimes e_{i_k}$ for arbitrary $i_1, \dots, i_k$. We want to show that there are constants $a_\sigma \in \mathbf{Q}$ such that $g = \sum_\sigma a_\sigma f_\sigma$. This means exactly that for any $i_1, \dots, i_k$, we have:

$$g\left(e_{i_1} \otimes \cdots \otimes e_{i_k}\right) = \sum_\sigma a_\sigma \Big(e_{i_{\sigma(1)}} \otimes \cdots \otimes e_{i_{\sigma(k)}}\Big) \tag{6}$$

Since $n \geq k$, the element $e_1 \otimes \cdots \otimes e_k$ is in $V^{\otimes k}$. Thus, if Equation (6) holds, we must have:

$$g\left(e_1 \otimes \cdots \otimes e_k\right) = \sum_\sigma a_\sigma \Big(e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(k)}\Big) \tag{7}$$

and that the terms on the right-hand side are linearly independent, so Equation (7) uniquely determines the a_σ. Now, we will show that Equation (7) holds and use this to define the a_σ, then show that this implies that Equation (6) holds.

Let T be the matrix $T = \begin{pmatrix} p_1 & & & \\ & p_2 & & \\ & & \ddots & \\ & & & p_n \end{pmatrix}$ where the $p_i \in \mathbf{Z} \subseteq \mathbf{Q}$ are distinct prime numbers. Then $T_*(e_1 \otimes \cdots \otimes e_k) = \left(\prod_{i=1}^k p_i\right)(e_1 \otimes \cdots \otimes e_k)$. Since g commutes with T_*, g must preserve the $\prod_{i=1}^k p_i$-eigenspace of T_*. Now, we have $T_*(e_{i_1} \otimes \cdots \otimes e_{i_k}) = \prod_{j=1}^k p_{i_j}$. By uniqueness of prime factorization, this number is equal to $\prod_{i=1}^k p_i$ iff $\{i_1, \ldots, i_k\} = \{1, \ldots, k\}$ as subsets of $[n] = \{1, \ldots, n\}$. For $I \subseteq [n]$ with $|I| = k$, define $p_I = \prod_{i \in I} p_i$. We may partition the basis $\{e_{i_1} \otimes \cdots \otimes e_{i_k}\}$ in terms of the sets $\{i_1, \ldots, i_k\}$: the basis can be written as

$$\{e_{i_1} \otimes \cdots \otimes e_{i_k}\} = \bigsqcup_{\substack{I \subseteq [n] \\ |I| = k}} \{e_{i_{\sigma(1)}} \otimes \cdots \otimes e_{i_{\sigma(k)}}\}_{\substack{I = \{i_1, \ldots, i_k\} \\ \sigma \in S_k}}$$

Thus, we may write $g(e_1 \otimes \cdots \otimes e_k)$ uniquely as:

$$g(e_1 \otimes \cdots \otimes e_k) = \sum_{\substack{I \subseteq [n] \\ |I| = k}} \sum_{\sigma} a_\sigma^I e_{i_{\sigma(1)}} \otimes \cdots \otimes e_{i_{\sigma(k)}} \tag{8}$$

Then we have:

$$\begin{aligned} p_{[k]} g(e_1 \otimes \cdots \otimes e_k) &= g(T_*(e_1 \otimes \cdots \otimes e_k)) \\ &= T_*(g(e_1 \otimes \cdots \otimes e_k)) \\ &= T_* \left(\sum_{\substack{I \subseteq [n] \\ |I| = k}} \sum_{\sigma} a_\sigma^I e_{i_{\sigma(1)}} \otimes \cdots \otimes e_{i_{\sigma(k)}} \right) \\ &= \sum_{\substack{I \subseteq [n] \\ |I| = k}} p_I \sum_{\sigma} a_\sigma^I e_{i_{\sigma(1)}} \otimes \cdots \otimes e_{i_{\sigma(k)}} \end{aligned}$$

Since the $\{e_{i_1} \otimes \cdots \otimes e_{i_k}\}$ are linearly independent, we may compare coefficients to see that $a_\sigma^I = 0$ for $I \neq [k]$. Thus, the formula (8) gives us (7).

Now, let $x = \sum_\sigma a_\sigma \sigma \in \mathbf{Q}[S_k]$ with the a_σ defined by (7), so $f_x = \sum_\sigma a_\sigma f_\sigma$. Since this is in the image of the map from $\mathbf{Q}[S_k]$ to $\mathrm{End}(V^{\otimes k})$, we see that f_x commutes with the action of $\mathrm{End}(V)$. Now, we want to show that $g = f_x$. Let $h = g - f_x$. This commutes with the action of $\mathrm{End}(V)$, since g and f_x both do, and by (7), we know that $h(e_1 \otimes \cdots \otimes e_k) = 0$. Now, we want to show that $h = 0$.

It suffices to show that $h(e_{i_1} \otimes \cdots \otimes e_{i_k}) = 0$ for any $i_1, \ldots, i_k$, so fix some such $i_1, \ldots, i_k$. Define an operator $T \in \mathrm{End}(V)$ which sends e_j to e_{i_j} for $j = 1, \ldots, k$ and sends e_j to 0 for $j > k$. This is possible since $e_1, \ldots, e_k$ are linearly independent (so we're really using that $n \geq k$). Then we have:

$$h(e_{i_1} \otimes \cdots \otimes e_{i_k}) = h \circ T_*(e_1 \otimes \cdots \otimes e_k) = T_* \circ h(e_1 \otimes \cdots \otimes e_k) = T_*(0) = 0$$

Thus, $h = 0$, so we have shown the case $n \geq k$.

Now, for the case $n < k$, we need to use a few ideas from representation theory. We will give the proof with $\mathbf{Q}$ replaced by $\mathbf{C}$ (or any algebraically closed field of characteristic 0). It turns out that a similar proof works over $\mathbf{Q}$, but this requires knowing some non-trivial information about the representation theory of symmetric groups.

Our strategy will be to try to embed V in a vector space W of dimension at least k and extend g to a map $\widetilde{g}\colon W^{\otimes k} \to W^{\otimes k}$ which commutes with the action of $\mathrm{End}(W)$ and such that $\widetilde{g}|_{V^{\otimes k}} = g$. Then by the case $n \geq k$, we have $\widetilde{g} = \sum_\sigma a_\sigma f_\sigma$, and we can see directly that this continues to hold when we restrict to $V^{\otimes k}$.

The action of S_k on $V^{\otimes k}$ makes $V^{\otimes k}$ into an S_k-module. Since $\mathbf{Q}$ has characteristic 0, any S_k-module W is a direct sum of irreducible representations of S_k. This is called *Maschke's Theorem*.

Theorem 1 (Maschke's Theorem)**.** If W is a finite-dimensional vector space over a field of characteristic 0 and G is a finite group acting on W, then $W = \oplus_i W_i$ with W_i irreducible representations of G.

Proof. We need to show that if W_1 is an G-invariant subspace of W, then $W = W_1 \oplus W_2$ for some G-invariant subspace W_2. Then we may induct on the dimension of W. In order to construct W_2, we will construct a projection operator $p\colon W \to W$ with $\mathrm{im}\, p = W_1$ and $p|_{W_1} = \mathrm{id}_{W_1}$. Then $W = W_1 \oplus \ker p$. If we arrange that $p(g \cdot w) = g \cdot p(w)$ for all $g \in G, w \in W$, then $\ker p$ is G-invariant and gives the desired splitting. Now, we may chose some projection operator p_0 with image W_1. Let $p(w) = \frac{1}{|G|}\sum_{h\in G} h \cdot p_0(h^{-1} \cdot w)$. Then

$$p(gw) = \frac{1}{|G|}\sum_{h\in G} h \cdot p_0(h^{-1}g \cdot w) = \frac{1}{|G|}\sum_{h\in G} g(g^{-1}h) \cdot p_0((g^{-1}h)^{-1} \cdot w) = gp(w)$$

Moreover, if $w \in W_1$, $h^{-1} \cdot w \in W_1$, so $p_0(h^{-1} \cdot w) = h^{-1} \cdot w$. Thus, $p(w) = \frac{1}{|G|}\sum_{h\in G}(hh^{-1}) \cdot w = w$. Finally, the image of p is contained in W_1 because W_1 is G-stable and the image of p_0 is contained in W_1. □

Thus, we may write $V^{\otimes k} \simeq \bigoplus_\lambda \bigoplus_{i=1}^{n_\lambda} U_{\lambda,i}$. Here, λ ranges through a set indexing all isomorphism classes of irreducible representations U_λ of S_k,[3] and n_λ is the number of times U_λ appears as a direct summand of $V^{\otimes k}$. $U_{\lambda,i}$ just denotes the i-th copy of U_λ. In the case $k = 2$, this is just saying that we can choose a basis for $V^{\otimes 2}$ where every tensor appearing in the basis is either symmetric or anti-symmetric: a symmetric or anti-symmetric tensor is an eigenvector for the action of the non-trivial element of S_2, so it spans a one-dimensional irreducible representation of S_2.

We want to show that if g commutes with the action of $\mathrm{End}(V)$, then in fact g preserves the $U_{\lambda,i}$. In fact, we will prove the following stronger result: [4]

[3]For any finite group, there are only finitely many non-isomorphic irreducible representations over any given field of characteristic 0, and all of these are finite-dimensional. Neither fact is necessary here, however, since we know $V^{\otimes k}$ is a finite-dimensional space and we only need to consider the isomorphism classes appearing as direct summands of this representation. In addition, we know that for the group S_k, the irreducible representations of S_k are indexed by partitions λ of $[n]$. The representation associated to a partition λ is determined by studying the "Young tableaux of shape λ". This is an interesting (and accessible) fact at the intersection of combinatorics and representation theory.

[4] In fact, this result implies the full statement of the problem due to the *double centralizer theorem*. Let W be a representation of a group G over a field k which is a direct sum of irreducible representations (a "semi-simple" representation). Define $C_G \subseteq \mathrm{End}(W)$ to be the centralizer of G, i.e. the set of endomorphisms which commute with the action of G. Then this theorem says that if $h \in \mathrm{End}(W)$ commutes with every element of C_G, actually h is the image of an element of $k[G]$. We can take $W = V^{\otimes k}$ and $G = S_k$. Then Lemma 2 says that C_G is the image of $\mathrm{End}(V)$ in $\mathrm{End}(W)$, so the double centralizer theorem says that any element of $\mathrm{End}(W)$ which commutes with the action of $\mathrm{End}(V)$ is actually of the form f_x for $x \in \mathbf{Q}[S_k]$.

Lemma 2. If $h \colon V^{\otimes k} \to V^{\otimes k}$ commutes with f_σ for every $\sigma \in S_k$, then h is a k-linear combination of T_* for various $T \in \mathrm{End}(V)$.

Proof. First of all, we may write any such h as a sum of elements of the form $h_1 \otimes \cdots \otimes h_k$, which we may furthermore assume are linearly independent from each other:

$$h = \sum_i h_1^i \otimes \cdots \otimes h_k^i \tag{9}$$

Indeed, we have a natural map $\big(\mathrm{End}(V)\big)^{\otimes k} \to \mathrm{End}(V^{\otimes k})$ induced by the multilinear map sending $(h_1, \ldots, h_k)$ to $h_1 \otimes \cdots \otimes h_k$. The space on the left has a basis consisting of all tensors of the form $(e_{i_1 j_1}) \otimes \cdots \otimes (e_{i_k j_k})$ for elementary matrices $e_{i_\ell j_\ell}$. The image of an element of this basis is the endomorphism of $V^{\otimes k}$ which sends $e_{i_1} \otimes \cdots e_{i_k}$ to $e_{j_1} \otimes \cdots \otimes e_{j_k}$ and is zero on any other element of the basis $\{e_{p_1} \otimes \cdots \otimes e_{p_k}\}$ of $V^{\otimes k}$. Thus, the image of these basis elements are linearly independent in $\mathrm{End}(V^{\otimes k})$.

Now, since h commutes with f_σ for all σ, we have $f_\sigma \circ h \circ f_{\sigma^{-1}} = h$. Note that for any $v_1 \otimes \cdots v_k$, we have:

$$\begin{aligned}
f_\sigma \circ h_1^i \otimes \cdots \otimes h_k^i \circ f_{\sigma^{-1}}\,(v_1 \otimes \cdots \otimes v_k) &= f_\sigma \circ (h_1^i \otimes \cdots \otimes h_k^i)(v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}) && (10)\\
&= f_\sigma \left(h_1^i(v_{\sigma^{-1}(1)})) \otimes \cdots \otimes (h_k^i(v_{\sigma^{-1}(k)}) \right) && (11)\\
&= h_{\sigma(1)}^i(v_1) \otimes \cdots \otimes h_{\sigma(k)}^i(v_k) && (12)\\
&= (h_{\sigma(1)}^i \otimes \cdots \otimes h_{\sigma(k)}^i)(v_1 \otimes \cdots \otimes v_k) && (13)
\end{aligned}$$

We expand this identity using (9) and (10):

$$\sum_i h_1^i \otimes \cdots \otimes h_k^i = \sum_i h_{\sigma(1)}^i \otimes \cdots \otimes h_{\sigma(k)}^i$$

By linear independence we may conclude that for each i, we have:

$$h_1^i \otimes \cdots \otimes h_k^i = h_{\sigma(1)}^i \otimes \cdots \otimes h_{\sigma(k)}^i$$

This implies that:

$$h_1^i \otimes \cdots \otimes h_k^i = \frac{1}{k!} \sum_{\sigma \in S_k} h_{\sigma(1)}^i \otimes \cdots \otimes h_{\sigma(k)}^i \tag{14}$$

Now, for $J \subseteq [k]$, let T_J^i be the endomorphism $\sum_{j \in J} h_j^i$. We will use the following fact:

Lemma 3.

$$\sum_{\sigma \in S_k} h_{\sigma(1)}^i \otimes \cdots \otimes h_{\sigma(k)}^i = \sum_{J \subseteq [k]} (-1)^{k-|J|} (T_J^i)_* \tag{15}$$

We will give the proof only in the case $k = 2$, but the general case is a messy but straightforward induction (think "inclusion-exclusion"). For $k = 2$, the formula is the obvious statement that:

$$h_1^i \otimes h_2^i + h_2^i \otimes h_1^i = (h_1^i + h_2^i) \otimes (h_1^i + h_2^i) - (h_1^i \otimes h_1^i + h_2^i \otimes h_2^i)$$

Thus, we see that h^i is a k-linear combination of T_* for various $T \in \mathrm{End}(V)$. $\square$

Now, the projection operator $p_{\lambda,i}$ from $V^{\otimes k}$ to $U_{\lambda,i}$ is S_k-invariant, so by Lemma 2, it is a linear combination of T_* for various T. Since g commutes with all such T_*, it commutes with this projection operator. Thus, for $u \in U_{\lambda,i}$, $g(u) = g \circ p_{\lambda,i}(u) = p_{\lambda,i} \circ g(u) \in U_{\lambda,i}$, so g preserves $U_{\lambda,i}$. Let $g_{\lambda,i} = g|_{U_{\lambda,i}}$. Now, for each $i < j$ choose a map $\alpha_{i,j} \colon U_{\lambda,i} \xrightarrow{\sim} U_{\lambda,j}$ which is an S_k-equivariant isomorphism. We can extend this to an endomorphism $h_{i,j}$ of $V^{\otimes k}$ by requiring $h|_{U_{\mu,j}} = 0$ whenever $(\mu, j) \neq (\lambda, i)$. This is S_k-invariant, so it commutes with g as above. Thus, for $u \in U_{i,\lambda}$, we have $\alpha_{i,j} \circ g_{\lambda,i}(u) = h_{i,j} \circ g(u) = g \circ h_{i,j}(u) = g_{\lambda,j} \circ \alpha_{i,j}(u)$. Thus, for each i, j, we have:

$$g_{\lambda,j} = \alpha_{i,j} \circ g_{\lambda,1} \circ \alpha_{i,j}^{-1} \tag{16}$$

Now, we will use this to construct the extension $\widetilde{g}$. Write $W^{\otimes k} = \bigoplus_\lambda \bigoplus_{i=1}^{n'_\lambda} U_{\lambda,i}$. Here, $n'_\lambda \geq n_\lambda$ for each λ, and we may assume that for $i \leq n_\lambda$, $U_{\lambda,i} \subseteq V^{\otimes k}$ and that it is the same $U_{\lambda,i}$ from the decomposition of $V^{\otimes k}$. For each λ, choose isomorphisms of S_k-modules $\alpha_{i,j} \colon U_{\lambda,i} \xrightarrow{\sim} U_{\lambda,j}$ such that for $i, j \leq n_\lambda$, these are the same $\alpha_{i,j}$ considered above. Now, we define $\widetilde{g}|_{U_{\lambda,i}}$ to be 0 whenever $n_\lambda = 0$ and $\alpha_{1,i} \circ g_{\lambda,1} \circ \alpha_{1,i}^{-1}$ whenever $n_\lambda \neq 0$. By the previous paragraph, for $i \leq n_\lambda$, $\widetilde{g}|_{U_{\lambda,i}} = g_{\lambda,i} = g|_{U_{\lambda,i}}$.

Now, we must show that $\widetilde{g}$ commutes with $\mathrm{End}(W)$. Let $T \in \mathrm{End}(W)$. Note that T_* commutes with $\mathbf{Q}[S_k]$, so $(T_*)|_{U_{\lambda,i}}$ maps into $\bigoplus_{j=1}^{n'_\lambda} U_{\lambda,j}$. Then, for each j, we get a map $T_{\lambda,i,j} \colon U_{\lambda,i} \to U_{\lambda,j}$ by projecting to the j-th term of this direct sum. Thus, $\alpha_{i,j}^{-1} \circ T_{\lambda,i,j}$ is an S_k-equivariant endomorphism of the irreducible representation $U_{\lambda,i}$. We claim that it must be a scalar [5]. This is:

Lemma 4 (Schur's Lemma)**.** Any endomorphism ρ of an irreducible representation U of a group G over an algebraically closed field which commutes with the G-action is a scalar.

Proof. Since the field is algebraically closed, any endomorphism of U must have an eigenvector v, so $\rho(v) = \lambda \cdot v$ for some scalar λ. But then the nonzero eigenspace $\ker(\rho - \lambda)$ is G-invariant: if $\rho(v) = \lambda \cdot v$, then $\rho(g(v)) = g(\rho(v)) = g(\lambda \cdot v) = \lambda \cdot g(v)$. Thus, since U is irreducible, this eigenspace is equal to U. □

Thus, $T_{\lambda,i,j} = C_{\lambda,i,j}\alpha_{i,j}$ for $C_{\lambda,i,j} \in \mathbf{C}$. It suffices to show that g commutes with $T_{\lambda,i,j}$ for all λ, i, j. But this is immediate from the definition of $\widetilde{g}|_{U_{\lambda,i}}$ via Equation (16).

[5] this is where we are using that we are working over $\mathbf{C}$ instead of over $\mathbf{Q}$: with more specific analysis of the representation theory of S_k, we may prove this "by hand" over $\mathbf{Q}$.