Math 210A: Modern Algebra

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Homework 8

Due Thursday night, November 16 (technically **2am** Nov. 17)

Question 1. Let $V = \mathbb{R}^{2n}$ with basis $v_1, \ldots, v_{2n}$.
Find an explicit vector $\omega \in \bigwedge^2 V$ such that $\omega \wedge \omega \wedge \cdots \wedge \omega \in \bigwedge^{2n} V$ is nonzero.

Question 2. Let $R = \mathbb{Z}$ and let M be the R-module $M = \mathbb{Q}/\mathbb{Z}$. Compute $T^*(M)$.

Question 3. Let $R = \mathbb{Z}[\sqrt{-30}]$, and let I be the ideal $I = \{2a + b\sqrt{-30} \mid a, b \in \mathbb{Z}\} = (2, \sqrt{-30}) \subset R$.

Compute $\bigwedge^2 I$ as an abelian group.
(but keep in mind the $\bigwedge^2$ is as an R-module, i.e. it's a quotient of $I \otimes_R I$).

Question 4. Prove that for any R-modules M and N, and any $k \geq 0$, there is an isomorphism

$$\bigwedge^k (M \oplus N) \cong \bigoplus_{a+b=k} (\bigwedge^a M) \otimes (\bigwedge^b N).$$

(If M and N are free, this is pretty easy, because the natural basis for $\bigwedge^k (M \oplus N)$ splits up appropriately; the resulting partition corresponds to the combinatorial identity $\binom{m+n}{k} = \sum_{a+b=k} \binom{m}{a}\binom{n}{b}$. This doesn't help directly for general M and N, but perhaps it at least helps you get straight what's going on.)

NOTE on Q4: If you like, you can prove this just for $k = 3$, i.e. that

$$\begin{aligned} \bigwedge^3 (M \oplus N) \cong \quad & \bigwedge^3 M \\ \oplus \quad & (\bigwedge^2 M) \otimes N \\ \oplus \quad & M \otimes (\bigwedge^2 N) \\ \oplus \quad & \bigwedge^3 N \end{aligned}$$

This is no harder or easier than the general case, but might be simpler notationally.

Question 5. Given an abelian group M and a subgroup $A \subset M$, define the *saturation* of A to be

$$\mathrm{sat}(A) = \{m \in M \mid \exists n \neq 0 \in \mathbb{Z} \text{ s.t. } n \cdot m \in A\}.$$

This $\mathrm{sat}(A)$ is a subgroup of M (you may assume this without proof).

Prove that if M is finitely generated, then for any subgroup $A \subset M$ the saturation $\mathrm{sat}(A)$ is a direct summand of M; that is, there exists a subgroup $N \subset M$ such that $M = \mathrm{sat}(A) \oplus N$.

Question 6. Given k linearly independent vectors $v_1, \ldots, v_k$ in $\mathbb{Z}^n$, there are two definitions of the *discriminant*:

Definition 1: Consider the element $\omega = v_1 \wedge \cdots \wedge v_k \in \bigwedge^k \mathbb{Z}^n$.
Let $\mathrm{disc}_1(v_1, \ldots, v_k)$ be the largest $d \in \mathbb{N}$ such that ω is divisible by d.
(i.e. such that there exists some other $\mu \in \bigwedge^k \mathbb{Z}^n$ such that $\omega = d \cdot \mu$)

Definition 2: Let $K = \langle v_1, \ldots, v_k \rangle$ be the subgroup of $\mathbb{Z}^n$ generated by these elements. Let L be the quotient $L = \mathbb{Z}^n / K$. Let $\mathrm{disc}_2(v_1, \ldots, v_k)$ be the cardinality of the torsion subgroup $\mathrm{Torsion}(L)$.

Prove that $\mathrm{disc}_1(v_1, \ldots, v_k) = \mathrm{disc}_2(v_1, \ldots, v_k)$.

Question 7. Let V be an n-dimensional vector space over $\mathbb{Q}$, and fix $k \geq 1$.

Recall from class that for any endomorphism $T \colon V \to V$, we obtain an endomorphism $T_* \colon V^{\otimes k} \to V^{\otimes k}$ defined on generators by

$$T_*(v_1 \otimes \cdots \otimes v_k) = T(v_1) \otimes \cdots \otimes T(v_k).$$

In this question, we want to *find the endomorphisms of $V^{\otimes k}$ that commute with T_* for all T.*

Recall from class that the permutation group S_k acts on $V^{\otimes k}$ (on the right) by

$$(v_1 \otimes \cdots \otimes v_k) \cdot \sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$$

Let $f_\sigma \in \mathrm{End}_{\mathbb{Q}}(V^{\otimes k})$ be this endomorphism. We can easily check that this commutes with T_*, since

$$\begin{aligned}
\big(T_*(v_1 \otimes \cdots \otimes v_k)\big) \cdot \sigma &= \big(T(v_1) \otimes \cdots \otimes T(v_k)\big) \cdot \sigma \\
&= T(v_{\sigma(1)}) \otimes \cdots \otimes T(v_{\sigma(k)}) \\
&= T_*(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}) \\
&= T_*\big((v_1 \otimes \cdots \otimes v_k) \cdot \sigma\big)
\end{aligned}$$

In other words, $f_\sigma \circ T_* = T_* \circ f_\sigma$.

The same is automatically true for linear combinations: for any $x = \sum a_\sigma \sigma \in \mathbb{Q}[S_k]$, the endomorphism $f_x := \sum a_\sigma f_\sigma$ has the same property that $f_x \circ T_* = T_* \circ f_x$ for all $T \in \mathrm{End}_{\mathbb{Q}}(V)$.

Your task: prove that these are the *only* endomorphisms that commute with all T_*.
That is, prove that if $g \in \mathrm{End}_{\mathbb{Q}}(V^{\otimes k})$ satisfies $g \circ T_* = T_* \circ g$ for all $T \in \mathrm{End}_{\mathbb{Q}}(V)$,
then there exists some $x \in \mathbb{Q}[S_k]$ such that $g = f_x$.