# Math 210A: Modern Algebra

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# Homework 9

Due Thursday night, November 30 (technically **2am** Dec. 1)

Let $V$ be a finite-dimensional vector space over a field $\mathbb{F}$, and let $\omega\colon V \times V \to \mathbb{F}$ be an alternating form. An *$\omega$-symplectic basis* is an ordered basis $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ for $V$ with the property that

$$\omega(a_i, b_i) = 1 \quad \text{for all } i$$

$$\omega(a_i, a_j) = \omega(a_i, b_j) = \omega(b_i, a_j) = \omega(b_i, b_j) = 0 \text{ if } i \neq j$$

**Question 1.** Suppose that $\omega$ is a nondegenerate alternating form over an arbitrary[1] field $\mathbb{F}$. Prove there exists an $\omega$-symplectic basis.

**Question 2.** Let $V$ be a $2n$-dimensional vector space over $\mathbb{F}$. Recall that $V^\vee$ denotes the dual vector space $V^\vee = \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$.

Let $\omega\colon V \times V \to \mathbb{F}$ be an alternating form. We can view $\omega$ as an element of $\bigwedge^2(V^\vee)$. (make sure you understand how this correspondence works)

Is it true that $\omega$ is nondegenerate as a bilinear form if and only if $\omega \wedge \cdots \wedge \omega \in \bigwedge^{2n}(V^\vee)$ is nonzero?

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**Question 3.** Let $\mathbb{F}_q$ be a finite field of order $q$ and characteristic $p \neq 2$, and let $V$ be a 2-dimensional vector space over $\mathbb{F}_q$. Let us say a "*quasi-definite*[2] form" is a symmetric bilinear form $\omega\colon V \times V \to \mathbb{F}_q$ with the property that $\omega(v, v) \neq 0$ for all $v \neq 0 \in V$.

(a) How many different isomorphism classes of quasi-definite forms are there?
Please begin your answer by giving the number of isomorphism classes, and then giving one clear representative of each isomorphism class (and then prove your answer is correct, of course). Note that the answer[3] may depend on properties of $q$ or $\mathbb{F}_q$.

(b) (Optional) Same question, but when $q = 2^k$.

[1] Note we do not need to assume anything about which elements are squares, nor anything about char $\mathbb{F}$.

[2] "quasi-definite" isn't an official term; I made it up because it's kind of like positive-definite, except of course "positive" doesn't mean anything in $\mathbb{F}_q$

[3] but of course the number is finite, because there are only finitely many *set functions* $V \times V \to \mathbb{F}_q$

We know that every ideal in $\mathbb{R}[x]$ is principal (generated by one element). How about $\mathbb{Z}[x]$?

**Question 4.** Let $R = \mathbb{Z}[x]$, and consider an ideal $I \subset \mathbb{Z}[x]$. Prove that $I$ is generated by finitely many elements. Is there an upper bound on how many generators we need? (i.e. is every ideal gen by 2 elements? or by 5 elements? etc.)

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**Question 5.** Let $V$ be a finite-dimensional vector space over $\mathbb{R}$, and let $T: V \to V$ be a linear transformation. Let $V_{\mathbb{C}} \coloneqq V \otimes_{\mathbb{R}} \mathbb{C}$. By functoriality we have a map $T_{\mathbb{C}}: V_{\mathbb{C}} \to V_{\mathbb{C}}$, defined by $T_{\mathbb{C}}(v \otimes z) = T(v) \otimes z$, called the *complexification* of $T$; this is a $\mathbb{C}$-linear transformation.

*Without* using the structure theorem for PIDs or rational canonical form, prove that the minimal polynomial $m_{T_{\mathbb{C}}} \in \mathbb{C}[t]$ is equal to the minimal polynomial $m_T \in \mathbb{R}[t]$ (just prove it directly!); in particular, $m_{T_{\mathbb{C}}}$ has coefficients in $\mathbb{R}$.

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Let $V$ and $W$ be finite-dimensional vector spaces over $\mathbb{R}$, and let $\omega_V: V \times V \to \mathbb{R}$ and $\omega_W: W \times W \to \mathbb{R}$ be positive definite symmetric forms. Given a linear transformation $T: V \to W$, the *adjoint* $T^*: W \to V$ is defined as follows (first verbosely, but see (ADJ) below for a self-contained definition).

Recall that the nondegeneracy of $\omega_V$ means that $\omega$ induces an isomorphism $V \to V^\vee$. Concretely, this means that for every linear map $\lambda: V \to \mathbb{R}$ there is a unique vector $v$ such that $\omega_V(v, x) = \lambda(x)$. For a given $w \in W$, the function $\lambda_w: V \to \mathbb{R}$ given by $\lambda_w: v \mapsto \omega_W(T(v), w)$ is a linear map from $V$ to $\mathbb{R}$. We define $T^*(w) \in V$ to be the vector corresponding as above to $\lambda_w$. In other words, $T^*$ is defined by the identity

$$\omega_W(T(v), w) = \omega_V(v, T^*(w)) \quad \text{for all } v \in V \text{ and all } w \in W \tag{ADJ}$$

It is very straightforward to verify from this definition the following properties:

(i) $T^*$ is an $\mathbb{R}$-linear transformation;
(ii) $(T_1 + T_2)^* = T_1^* + T_2^*$ and $(cT)^* = c(T^*)$ for $c \in \mathbb{R}$;
(iii) $(S \circ T)^* = T^* \circ S^*$.

**Question 6.** Let $V$ be a finite-dimensional vector space over $\mathbb{R}$, and let $\omega_V: V \times V \to \mathbb{R}$ be a positive definite symmetric form. An endomorphism $T: V \to V$ is called *self-adjoint* if $T^* = T$.

Suppose that $T: V \to V$ is self-adjoint. Prove that the minimal polynomial $m_T \in \mathbb{R}[t]$ splits completely (i.e. $m_T(t) = (t - \lambda_1)(t - \lambda_2) \cdots (t - \lambda_k)$ for some[4] $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$). (For the different linear maps or bilinear forms you use in the proof, be *very* careful to make clear what the domain and codomain are, and to what extent they are linear/bilinear; this is the key point of the problem.)

[Note you cannot use the spectral theorem (since we used Q6 to prove the spectral theorem).]

[4] as I mentioned in class, you do not need to prove here that the $\lambda_1, \ldots, \lambda_k$ are distinct (even though that turns out to be true); you just need to prove $m_T(t)$ factors over $\mathbb{R}$ into linear terms