CAAP Math C, Mr. Church, Midterm Exam
Monday, July 28
`http://www.math.uchicago.edu/~tchurch/`

Write your name on the front of your test book. **You should write all answers in the blue book provided. Do not write your answers on this sheet.** Please label your answers clearly in your test book. You have 80 minutes to complete this test.

You should also have been handed a separate sheet listing some definitions. You should look over it before you start the test.

1. (80 points) We define two operations on the set of **positive real numbers** by $a \Box b = \frac{a+b}{2}$ and $a \triangle b = \frac{1}{\frac{1}{a}+\frac{1}{b}}$. For each question below, if the answer is "No", then give a counterexample. (For Question 1, it is not necessary to answer the parts in order.)
   (a) (5 points) Is $\Box$ commutative? Is $\Box$ associative?

   (b) (10 points) Is $\triangle$ commutative? If your answer is "Yes", prove it.

   (c) (10 points) Is $\triangle$ associative? If your answer is "Yes", prove it.

Let $\mathbb{Q}_+ = \{x \in \mathbb{Q} | x > 0\}$ be the set of positive rationals, and recall that $\mathbb{N} = \{n \in \mathbb{Z} | n > 0\}$ is the set of positive integers.

   (d) (10 points) Is $\mathbb{N}$, the set of natural numbers, closed under the operation $\Box$? Is $\mathbb{Q}_+$, the set of positive rationals, closed under $\Box$? If your answer to either part is "Yes", then prove it.

   (e) (10 points) Is $\mathbb{N}$ closed under $\triangle$? Is $\mathbb{Q}_+$ closed under $\triangle$? If your answer to either part is "Yes", then prove it.

   (f) (10 points) Is there an identity for the operation $\Box$? Is there an identity for the operation $\triangle$?

   (g) (10 points) Prove that your answers for part (f) are correct.

Let $D = \left\{ \frac{p}{2^n} \in \mathbb{Q} | p > 0 \text{ and } n \geq 0 \right\}$ be the set of positive rational numbers that can be written with denominator a power of 2.

   (h) (15 points) Is D closed under $\Box$? Is D closed under $\triangle$? If your answer to either part is "Yes", then prove it.

2. (70 points) For this question, we say that an integer n is *Type 0* if $3|n$; that n is *Type 1* if $3|(n-1)$; and that n is *Type 2* if $3|(n-2)$. For example, the integer 3 is Type 0, the integer 7 is Type 1, and the integer -10 is Type 2. Every integer is either Type 0, 1, or 2. (For this question, it will help you to solve the parts in order.)

   (a) (40 points) Let a be *any* Type 0 integer, let b be any Type 1 integer, and let c be any Type 2 integer. Prove that:

      i. $a+a$ is Type 0, $a+b$ is Type 1 and $a+c$ is Type 2.
      ii. $b+b$ is Type 2, $b+c$ is Type 0, and $c+c$ is Type 1.
      iii. For any integer n, an is Type 0.
      iv. b^2 is Type 1, bc is Type 2, and c^2 is Type 1.

   Note that it is **not enough** to show that these are true by plugging in numbers; you need to prove that they are true for *all* integers.

For easy reference, the corresponding addition and multiplication tables are:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

For parts (b) and (c), you may assume everything from part (a), even if you were not able to prove it.

   (b) (15 points) Prove that for any integer n, the number $(n-1)n(n+1)$ is divisible by 3; in our notation from class, $3|(n-1)n(n+1)$.

   (c) (15 points) For which numbers n does n^3 have the same Type as n? Prove that your answer is correct. (That is, n and n^3 are both Type 0, or both Type 1, or both Type 2.)